Random Periodic Processes, Periodic Measures and Ergodicity

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I. Why are we bothered to study random periodicity?

Three reasons to study the random periodicity:

(1) Mathematics:

(a) The study of periodic solution is a critical problem to understand dynamical systems and has been central to this subject since Poincaré's pioneering work. There are many examples of differential equations with periodic solutions. So such a natural question to ask is the periodicity for the stochastic counterpart.

The difficulty is normally the periodicity is broken immediately by noise. This is true even for the fixed point case.

Example 1-break of fixed point and stationary solution

Simplest ever with a nontrivial stationary solution:
As a random perturbation to the deterministic equation

$$\frac{dy}{dt} = -y, \quad y(0) = x,$$

we consider the Ornstein-Uhlenbeck process

$$dy = -ydt + dB_t, \quad y(0) = x.$$

Variation of constant formula gives the following solution:

$$\Phi_t^\omega x = x\mathrm{e}^{-t} + \int_0^t \mathrm{e}^{-(t-s)} dB_s(\omega).$$

It is easy to check that

$$Y^*(\omega) = \int_{-\infty}^0 \mathrm{e}^s dB_s(\omega).$$

satisfies

$$\Phi_t^\omega Y^*(\omega) = Y^*(\theta_t\omega) = \int_{-\infty}^{t} e^{-(t-s)} dB_s(\omega),$$

where $(\theta_t\omega)(s) = B(t+s) - B(t)$. Moreover, for any $x \in R^1$, as $t \to \infty$

$$\begin{aligned} & |\Phi_t^\omega x - Y^*(\theta_t\omega)| \\ = \quad & e^{-t}|x - \int_{-\infty}^{0} e^s dB_s(\omega)| \\ \to \quad & 0. \end{aligned}$$

Remark 1

The fixed point of the deterministic system $x = 0$ breaks immediately as soon as noise presents. Somehow Y^* keeps important property of the fixed point as an equilibrium in the stochastic counterpart.

(b) Ergodic theory: Existing ergodic theory is based

invariant measures and stationary processes

Can we move away from this assumption and establish an ergodic theory in the random periodic regime with

periodic measures and random periodic processes?

However, many basic assumptions and key proofs in the ergodic theory break down without stationary assumption such as

coupling method, t_0-*irreducibility*, t_0-*regularity etc.*

Ergodic theory under a periodic regime in general did not exist. In fact it was not clear how to define a random periodic process till recently.

(2) Physics and applied science: It is of great interests to physicists and applied scientists to try to understand what happens to periodic solutions in random environments. In fact physicists had been interested in this question for SDEs. For example see

Weiss and Knoblock, A stochastic return map for stochastic differential equations, Journal of Statistical Physics, Vol. 58 (1990), 863-883.

Physicists have not made much real progress even in the non-rigorous level on this problem ahead of mathematicians due to the lack of rigorous mathematical concept and efficient methodologies.

There are some recent works in climate dynamics (Chekroun, Simonnet and Ghil, Physica D (2011)), biology (Borisyk, Preprint, recent work).

(3) Real world random periodicity phenomena: Periodic phenomena exist in many real world problems. But many real world systems are very often subject to the influence of randomness.

Periodicity and randomness often mix together in many phenomena.

They may be best described by random periodic motion rather than a periodic motion. For instance

–the maximum daily temperature in any fixed place

–Many economic problems, goods prices

Rigorous mathematical study of random periodicity would be useful to understand such kind of real world problems and provide new insight to understand structure of random phenomena, data etc.

II. Generation of stochastic dynamical systems-a natural set up

It is natural to regard SDEs/SPDEs as stochastic dynamical systems/flows. The stochastic flow idea went back to from late 70's with the work on stochastic flows/RDS generated by SDEs by
Elworthy,
Kunita,
Baxandale,
Bismut,
Meyer,
Arnold,
et al

and by SPDEs later by:
Flandoli (AP, 1996),
Mohammed-Zhang-Zhao (Memiors AMS 2008).

Consider a stochastic differential equation on the state space X

$$\begin{aligned} du_t &= b(u_t)dt + \sigma(u_t)dB_t \\ x_0 &= x \end{aligned} \qquad (1)$$

driven by a standard BM on a canonical Wiener space. Now set a map

$$\Phi(t, \omega)x = u_t^x(\omega), \omega \in \Omega, x \in X,$$

and $\theta : R \times \Omega \to \Omega$ by

$$(\theta_s \omega)(t) = B(t+s) - B(s).$$

Definition 2

A measurable random dynamical system on the measurable space $(X, \mathcal{B}(X))$ *over a metric dynamical system* $(\Omega, \mathcal{F}, P, (\theta_s)_{s\in R})$ *is a mapping:*

$$\Phi : R^+ \times \Omega \times X \to X, \ \ (t, \omega, x) \mapsto \Phi(t, \omega, x),$$

with the following properties:
(i) Measurability: Φ *is* $(\mathcal{B}(R^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$*-measurable.*
(ii) Cocycle property:

$$\Phi(0, \omega) = id_X \ \ for \ all \ \omega \in \Omega,$$

$$\Phi(t + s, \omega) = \Phi(t, \theta_s \omega) \circ \Phi(s, \omega) \ \ for \ all \ s, t \in R^+, \ \ \omega \in \Omega \ .$$

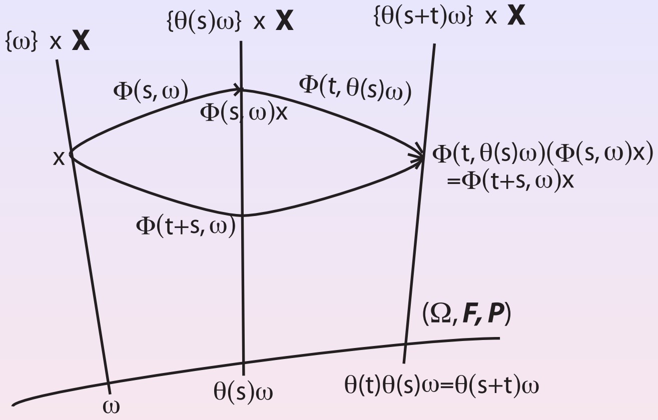

Figure : Random dynamical systems

Illustrating example-random periodic solution

Consider

$$\frac{dx(t)}{dt} = x(t) - y(t) - x(t)(x^2(t) + y^2(t)),$$
$$\frac{dy(t)}{dt} = x(t) + y(t) - y(t)(x^2(t) + y^2(t)).$$

It is well-known that above equation has a limit cycle

$$x^2(t) + y^2(t) = 1.$$

Consider a random perturbation (special only for computation purpose of this example)

$$dx = (x - y - x(x^2 + y^2))dt + x \circ dB(t),$$
$$dy = (x + y - y(x^2 + y^2))dt + y \circ dB(t).$$

Here $W(t)$ is a one-dimensional Brownian motion. Set

$$\rho^*(\omega) = (2 \int_{-\infty}^{0} e^{2s+2B(s)} ds)^{-\frac{1}{2}}$$

and

$$Y(t, \omega) = (\rho^*(\theta_t \omega) \cos(t), \rho^*(\theta_t \omega) \sin(t)).$$

Then one can check that

$$
\begin{aligned}
\Phi(t,\omega)Y(0,\omega) &= Y(t,\omega), \\
Y(t+2\pi,\omega) &= Y(t,\theta_{2\pi}\omega)
\end{aligned}
$$

and for any $U_0 = (x_0, 0) \in R^2, x_0 \neq 0$,

$$
\Phi(t+2k\pi, \theta_{-2k\pi}\omega)U_0 \to Y(t,\omega) \text{ for any } t \in [0,2\pi], \text{ as } k \to \infty.
$$

Ref of the example: Zhao and Zheng (2009).

Definition 3

(Z. and Zheng JDE (2009), Feng, Z. and Zhou JDE (2011), Feng and Z. JFA (2012)) A random periodic path of period τ *is an* $\mathcal{F}$*-measurable function* $Y : R \times \Omega \to X$ *such that for a.e.* $\omega \in \Omega$,

$$Y(\tau + s, \omega) = Y(s, \theta_\tau \omega), \quad \Phi(t, \theta_s \omega) Y(s, \omega) = Y(t + s, \omega), \tag{2}$$

for all $t \in R^+, s \in R$. *It is a stationary solution if* $Y(t, \theta_{-t}\omega) = Y(0, \omega)$ $=: Y_0(\omega)$ *for all* $t \in R^+$, *i.e.*

$$\Phi(t, \omega, Y_0(\omega)) = Y_0(\theta_t \omega), \; t \in R^+ a.s.$$

Stationary solutions: Sinai (1991, 1996), Schmalfuss (2001), E, Khanin, Mazel and Sinai (AM 2000), Mattingly (CMP, 1999), Caraballo, Kloeden and Schmalfuss (2004), Q. Zhang and Zhao (JFA 2007, JDE 2010, SPA 2013), Liu and Zhao (SD 2009).

Example 4

(Feng,Z.,Zhou JDE (2011), Feng,Wu,Z. Preprint (2015a)) Consider the following stochastic differential equations on R^d

$$dx = (Ax + f(x))dt + \sum_{k=1}^{M} B_k x \circ dB_t^k + \sum_{k=1}^{M} \gamma_k dB_t^k. \qquad (3)$$

Theorem 5

Assume A *is hyperbolic and the function* $f \in C^3$ *is uniformly bounded with bounded first order derivatives. If the deterministic system* $\frac{dx}{dt} = Ax + f(x)$ *has a periodic solution* z *with period* $\tau > 0$ *and* $z(t)$ *is* C^3 *in* t*, then Eqn. (3) has a random periodic solution of period* τ*.*

SPDEs (Feng and Zhao JFA (2012), Feng, Wu and Zhao Preprint (2015b)). Random mapping (Lian and Zhao, DElworthy volume (2012)).

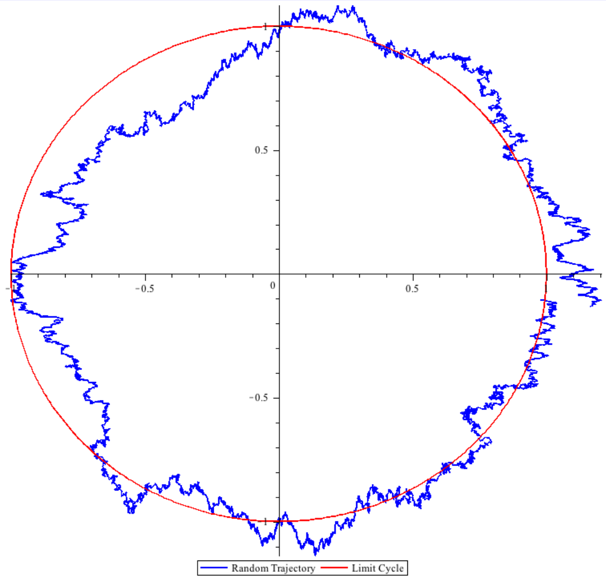

Figure : Random trajectory subject to mutiplicative noise

Example 6

Consider the following well known example of discrete time Markov chain with three states $\{1, 2, 3\}$ *and transition probability matrix*

$$P = \begin{pmatrix} 0, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 0, & 0 \\ 1, & 0, & 0 \end{pmatrix}.$$

Recall that in the theory of Markov chain the period $d(i)$ *of state* i *is the greatest common divisor of* $\{n : P_{ii}^n > 0\}$*. It is easy to see that* $d(1) = d(2) = d(3) = 2$*.*

Random dynamical systems: set $\omega(\cdot) = \cdots j_{-2}j_{-1} : j_0 : j_1 j_2 \cdots$ as a sign sequence where $j_n = 0$ or 1 with equal probabilities for all $n = 0, \pm 1, \pm 2, \cdots$. Here $\omega(0) = j_0$. Define Ω to be a set containing all possible ω above and

$$\theta\omega = \cdots j_{-2}j_{-1}j_0 : j_1 : j_2 j_3 \cdots , \tag{4}$$

and

$$\theta^n = \theta\theta\cdots\theta. \tag{5}$$

Set

$$\Phi(\omega)1 = \begin{cases} 2 \text{ when } \omega(0) = 0, \\ 3 \text{ when } \omega(0) = 1, \end{cases}$$

and

$$\Phi(\omega)\{2,3\} = 1 \ \text{ for any } \omega \in \Omega.$$

It is easy to that

$$\Phi_n(\omega) = \Phi(\theta^{n-1}\omega)\Phi(\theta^{n-2}\omega)\cdots\Phi(\omega), \tag{6}$$

defines a cocycle. Let

$$Y(0,\omega) = 1 \text{ for all } \omega, Y(1,\omega) = \begin{cases} 2 & \text{when } \omega(0) = 0, \\ 3 & \text{when } \omega(0) = 1. \end{cases}$$

Then $Y(k+2n,\omega) = Y(k,\theta^{2n}\omega), k = 0,1$, defines a random periodic path of the random dynamical system Φ_n.

The random periodic path definition
–completely different from the definition of a periodic state in the Markov chain theory
– equivalent in the Markov chain case.

Consider a Markovian cocycle random dynamical system Φ on a filtered dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in R}, (\mathcal{F}_s^t)_{t\geq s})$, i.e. for any $s, t, u \in R, s \leq t$, $\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$ and for any $t \in R^+$, $\Phi(t, \cdot)$ is measurable with respect to $\mathcal{F}_0^t$.

Assume the random periodic path $Y(s)$ is adapted i.e. for each $s \in R$, $Y(s, \cdot)$ is measurable with respect to $\mathcal{F}_{-\infty}^s := \vee_{r\leq s}\mathcal{F}_r^s$. Define for any $B \in \mathcal{B}$

$$\rho_s(B) = E\delta_{Y(s,\omega)}(B) = P\{\omega : Y(s, \omega) \in B\},$$

and

$$P(t, x, B) = P\{\omega : \Phi(t, \omega)x \in B\},\ \ t \in R^+.$$

For any measure ρ on $\mathcal{B}$, define $P^*(t) : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$(P_t^*\rho)(B) = \int_H P(t, x, B)\rho(dx), \ t \in R^+.$$

Theorem 7

Above defined $\rho_s, s \in R$, is a periodic measure of Φ on $(H, \mathcal{B})$ i.e.

$$\rho_{\tau+s} = \rho_s, \ \ P_t^*\rho_s = \rho_{t+s}, \ t \in R^+ \tag{7}$$

and $\bar{\rho} = \frac{1}{\tau}\int_{[0,\tau)} \rho_s ds$ is an invariant measure i.e.

$$P^*(t)\bar{\rho} = \bar{\rho}.$$

Remark 8

Note ρ_s is an invariant measure of $P(k\tau)$, $k \in \mathbb{N}$.

IV. Poincare decomposition and ergodic theorems under random periodicity

Definition 9

The sets $L_s \subset X, \ s \geq 0$, are called the Poincaré sections *of $P(t, \cdot, \cdot)$, $t \geq 0$, if $L_{s+\tau} = L_s$ and for ρ_s-almost all $x \in L_s$, $t \geq 0$,*

$$P(t, x, L_{s+t}) = 1. \quad (8)$$

Remark: Consider when periodic measure exists.
(i) Let $L_s = \mathrm{supp}(\rho_s)$. Note any $t \geq 0$

$$\int_{L_s} P(t, x, L_{t+s})\rho_s(dx) = \rho_{t+s}(L_{t+s}) = 1. \tag{9}$$

This, together with the fact that $0 \leq P(t, x, L_{t+s}) \leq 1$, implies that

$$P(t, x, L_{t+s}) = 1 \text{ for } \rho_s - \text{almost all } x \in L_s, \text{ for any } t \geq 0. \tag{10}$$

So Poincaré sections (nontrivial) exist automatically.
(ii) The choice of Poincaré sections is not unique. More conditions are needed to make for uniqueness
(iii) Define

$$L = \bigcup \{L_s : 0 \leq s < \tau\}. \tag{11}$$

Then $\bar{\rho}(L) = 1$.

Recall a Markovian semigroup $P(t), t \geq 0$ is said to be *stochastically continuous* if

$$\lim_{t \to 0} P(t, x, B(x, \gamma)) = 1, \text{ for all } x \in X, \ \gamma > 0.$$

Denote by $B_b(X)$, the space of all bounded Borel measurable functions on X, and $C_b(X)$ the space of all bounded continuous functions on X. For any $\phi \in B_b(X)$, define

$$P(t)\phi(x) = \int_X P(t, x, dy)\phi(y), \text{ for } t \geq 0.$$

Recall that $P(t), t \geq 0$,
–*Feller* on a subset Γ if for any $\phi \in C_b(X)$, we have $P(t)\phi(x)|_{x \in \Gamma} \in C_b(\Gamma)$.
–*strong Feller* at a time $t_0 > 0$ on Γ of X if for any $\phi \in B_b(X)$, we have $P(t_0)\phi(x)|_{x \in \Gamma} \in C_b(\Gamma)$.

Define now for any $\Gamma \in \mathcal{B}(X)$,

$$R_N(x, \Gamma) := \frac{1}{N} \sum_{k=1}^{N} P(k\tau, x, \Gamma),$$

and

$$(R_N^* \nu)(\Gamma) := \int_X R_N(x, \Gamma) \nu(dx),$$

for a measure $\nu \in \mathcal{P}(X)$. Note if ν has a support in L_0, then

$$(R_N^* \nu)(L_0) = \frac{1}{N} \sum_{k=1}^{N} \int_X P(k\tau, x, L_0) \nu(dx) = 1.$$

So $\mathrm{supp}(R_N^* \nu) \subset L_0$.

Theorem 10

Assume $L_s, s \in \mathbb{R}$ are Poincaré sections of Markovian semigroup $P(t), t \geq 0$ and $P(t)$ is a Feller semigroup on L_0. If for some $\nu \in \mathcal{P}(X)$ with its support in L_0 and a subsequence N_i with $N_i \to \infty$ as $i \to \infty$ such that

$$R^*_{N_i}\nu \to \rho_0,$$

weakly as $i \to \infty$. Define, if $s \geq 0$

$$\rho_s(\Gamma) = \int_{L_0} P(s, x, \Gamma)\rho_0(dx),$$

and if $s < 0$,

$$\rho_s(\Gamma) = \rho_{s+k\tau}(\Gamma),$$

where k is the smallest integer such that $s + k\tau \geq 0$. Then $\rho_s, s \in \mathbb{R}$, is a periodic measure with $\mathrm{supp}(\rho_s) \subset L_s$.

Denote $X^{\mathbb{R}}$ the space of all X-valued functions defined on $\mathbb{R}$, $\mathcal{B}(X^{\mathbb{R}})$ is the smallest σ-field containing all cylinderical sets in $X^{\mathbb{R}}$.

By the Kolmogorov extension theorem, there exists a unique probability measure $P^{\bar{\rho}}$ on $(\Omega^*, \mathcal{F}^*) = (X^{\mathbb{R}}, \mathcal{B}(X^{\mathbb{R}}))$ generated from the invariant measure on X.

For any $\omega^* \in \Omega^*$, denote its canonical process by $W_{\omega^*}(t) = \omega^*(t)$, which is a Markovian process. Define

$$\theta^* \omega^*(s) = W^*(t + s). \tag{12}$$

Define a linear transformation $U_t : \mathcal{H}_{\mathbb{C}}^{\bar{\rho}} \to \mathcal{H}_{\mathbb{C}}^{\bar{\rho}}$, where $\mathcal{H}_{\mathbb{C}}^{\bar{\rho}} = L_{\mathbb{C}}^2(\Omega^*, \mathcal{F}^*, P^{\bar{\rho}})$ defined by

$$U_t\xi(\omega^*) = \xi(\theta_t^*\omega^*),\ \xi \in \mathcal{H}^{\bar{\rho}},\ \omega^* \in \Omega^*, t \in \mathbb{R}, \tag{13}$$

then $(\Omega^*, \mathcal{F}^*, \theta_t^*, P^{\bar{\rho}})$ is a continuous metric dynamical system. This U_t is called the *induced linear transformation* of the canonical dynamical system from $\mathcal{H}_{\mathbb{C}}^{\bar{\rho}}$ to $\mathcal{H}_{\mathbb{C}}^{\bar{\rho}}$. The invariant measure $\bar{\rho}$ is called ergodic if $(\Omega^*, \mathcal{F}^*, \theta_t^*, P^{\bar{\rho}})$ is ergodic i.e.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T < U_t\xi, \eta > dt =< \xi, 1 >< \eta, 1 >, \quad \text{for any } \xi, \eta \in \mathcal{H}_{\mathbb{C}}^{\bar{\rho}}.$$

Recall an important result that $\bar{\rho}$ as the invariant measure of the stochastically continuous semigroup P_t is ergodic is equivalent to if a set $\Gamma \in \mathcal{B}(X)$ satisfies for all $t > 0$,

$$P_t I_\Gamma = I_\Gamma, \ \bar{\rho} - a.e.$$

then either $\bar{\rho}(\Gamma) = 0$ or $\bar{\rho}(\Gamma) = 1$

Condition A: *The Markovian cocycle* $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ *has a periodic measure* $\rho : \mathbb{R} \to \mathcal{P}(X)$ *and for any* $B \in \mathcal{B}(X)$, *we have when* $k \to \infty$,

$$\int_X \left|\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} P(s,y,B)ds - \bar{\rho}(B)\right|\bar{\rho}(dy) \to 0, \tag{14}$$

where $\bar{\rho} = \frac{1}{\tau}\int_0^\tau \rho_s ds$.

Lemma 11

Assume the Markovian semigroup $P(t)$ *is stochastically continuous and Condition A holds. Then the invariant measure* $\bar{\rho}$ *is ergodic. Moreover,* L *defined by (11) is the unique set (up to a* $\bar{\rho}$*-measure 0 set) with positive* $\bar{\rho}$ *measure satisfying* $P(t,x,L) = I_L(x)$.

Definition 12

(The $k_0\tau$-irreducibility condition on a Poincaré section L_s): *There exists $k_0 \in \mathbb{N} \setminus \{0\}$, such that for an arbitrary nonempty relatively open set $\Gamma \subset L_s$, we have*

$$P(k_0\tau, x, \Gamma) > 0, \text{ for any } x \in L_s. \tag{15}$$

Theorem 13

Assume $P(t)$ is stochastically continuous and has a τ-periodic measure $\rho_s, s \in \mathbb{R}$. Let the Poincaré sections $L_s, s \in \mathbb{R}$ satisfy the $k_0\tau$-irreducible and strong Feller on each L_s for all $t \geq t_0$ with some $t_0 > 0$.

Then Condition A is satisfied and $\bar{\rho}$ is ergodic. Moreover, $\Gamma = L$ is the unique invariant set satisfying $P(t, x, \Gamma) = I_\Gamma(x)$ for any $t \geq 0$.

Theorem 14

Under the same conditions as of Theorem 13, one of the three cases happens:

(i) $L_s \cap L_t = \emptyset$ *for any* $s \neq t$, $s, t \in [0, \tau)$.
iff τ *is the smallest number such that (7) holds.*

(ii) There exist $s, \tilde{s} \in [0, \tau)$, $s < \tilde{s}$ *such that* $L_s \cap L_{\tilde{s}} \neq \emptyset$ *and* $L_s \cap L_r = \emptyset$ *for any* $r \in (s, \tilde{s})$.
iff there exists $k \in \mathbb{N} \setminus \{0\}$ *such that* $\tau = k(\tilde{s} - s)$ *and* $\tilde{\tau} = \tilde{s} - s$ *is the smallest real number* τ *such that (7) holds.*

(iii) There exists $s \in [0, \tau)$ *and a sequence* $s_k \to s$, $s_k > s$ *such that* $L_s \cap L_{s_k} \neq \emptyset$.
iff $\rho_s = \rho_t$ *for any* $s, t \in \mathbb{R}$. *So* $\bar{\rho} = \rho_s$ *is an invariant measure for* $P(t), t \geq 0$.

Optional for the talk:

In fact, we have proved a lot more result: the following four statements equivalent

- (i),(ii), $L_s \cap L_t = \emptyset$ for any $s \neq t$, $s, t \in [0, \tau)$
- τ is the smallest number such that (7) holds
- the angle variable is nontrivial and $\lambda = \frac{2\pi}{\tau}$
- the operator $\mathcal{A}$ has two simple and no other eigenvalues, 0 and $\frac{2\pi}{\tau}$

Here angle variable

$$U_t\alpha = \alpha_0 + \lambda t,$$

and $i\mathcal{A}$ is the infinitesimal generator of U_t.

The flowing statements are equivalent:-

- (iii) There exist $s \in [0, \tau)$ and a sequence $s_k \to s$, $s_k > s$ such that $L_s \cap L_{s_k} \neq \emptyset$
- $\rho_s = \rho_t$ for any $s, t \in \mathbb{R}$. So $\bar{\rho} = \rho_s$ is an invariant measure for $P(t), t \geq 0$
- the angle variable is a constant
- the operator $\mathcal{A}$ has one simple and no other eigenvalues, 0 (Koopman-von Neumann).

Theorem 15

(SLLN, Feng and Z. (2015)) Let Y be an adapted random periodic process with a weaker condition A (replacing $\bar{\rho}(dy)$ by $\rho_0(dy)$), then as $T \to \infty$,

$$\frac{1}{T}\int_0^T I_B(Y(s,\omega))ds \to \bar{\rho}(B)\ a.s.$$

SLLN was known for iid random variables (Kolmogorov) and stationary processes (Birkhoff).
This is a new type of processes satisfying the SLLN.

V. Construct a random periodic path from a periodic measure

Set $I_\tau = [0, \tau)$, $\mathcal{B}_{I_\tau} = \{\emptyset, I_\tau\}$, $\hat{\Omega} = I_\tau \times \Omega \times X$, $\hat{\mathcal{F}} = \mathcal{B}_{I_\tau} \otimes \mathcal{F} \otimes \mathcal{B}(X)$ and $\hat{\mu}$

$$\hat{\mu}(I_\tau \times A) = \frac{1}{\tau}\int_0^\tau \mu_s(A)ds, \quad \hat{\mu}(\emptyset \times A) = 0, \text{ for any } A \in \mathcal{F} \otimes \mathcal{B}(X).$$

Define the skew product $\hat{\Theta} : \mathbb{R}^+ \times \hat{\Omega} \to \hat{\Omega}$ as

$$\hat{\Theta}(t)\hat{\omega} = (s + t \; mod \; \tau, \theta(t)\omega, \Phi(t, \omega)x).$$

Theorem 16

The measure $\hat{\mu}$ *is a probability measure,* $\hat{\Theta}(t) : \hat{\Omega} \to \hat{\Omega}$ *is measure* $\hat{\mu}$ *preserving, and* $\hat{\Theta}(t_1)\hat{\Theta}(t_2) = \hat{\Theta}(t_1 + t_2)$*. Moreover* $\hat{\Phi}$ *defined by*

$$\hat{\Phi}(t, \hat{\omega})x = \Phi(t, \omega)x, \tag{16}$$

is a random dynamical system on the enlarged probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, \hat{\Theta}(t))$ *and has a random periodic path* $\hat{Y}(t, \hat{\omega})$.

To make the ergodic theorem and SLLN working if we only assume the existence of a periodic measure, one needs the following:

Lemma 17

Assume ρ_s *is a periodic measure with respect to the semigroup transition probability of a Markovian random dynamical system* Φ*. Then for any* $B \in \mathcal{B}_X$

$$\hat{\mu}\{\hat{\omega} : \hat{Y}(t, \hat{\omega}) \in B\} = \rho_t(B),$$

and

$$\hat{P}(t, y, B) = \hat{\mu}\{\hat{\omega} : \hat{\Phi}(t, \hat{\omega})y \in B\} = P(t, y, B).$$

Thus $\rho_\cdot$ *is a periodic measure with respect to* $\hat{\mu}$ *as well.*

Denote $\Delta := \{(t,s) \in \mathbb{R}^2, s \le t\}$. Consider a stochastic semi-flow $u : \Delta \times \Omega \times X \to X$, which satisfies the following standard condition

$$u(t,r,\omega) = u(t,s,\omega) \circ u(s,r,\omega), \quad \text{for all } r \le s \le t, \; r,s,t \in \mathbb{R}. \tag{17}$$

Definition 18

We call u is a τ-periodic stochastic semi-flow if it satisfies an additional periodicity property: there exists a constant $\tau > 0$ such that

$$u(t+\tau, s+\tau, \omega) = u(t,s,\theta(\tau)\omega). \tag{18}$$

A random periodic path of period τ is an $\mathcal{F}$-measurable map $Y : \mathbb{R} \times \Omega \to X$

$$u(t,s,\omega)Y(s,\omega) = Y(t+s,\omega), \quad Y(s+\tau,\omega) = Y(s,\theta(\tau)\omega). \tag{19}$$

Lemma 19

We lift the τ-periodic stochastic semi-flow $u : \Delta \times \Omega \times X \to X$ to a random dynamical system on a cylinder $\tilde{X} := [0, \tau) \times X$ by the following:

$$\tilde{\Phi}(t, \omega)(s, x) = (t + s \; mod \; \tau, u(t + s, s, \theta(-s)\omega)x). \tag{20}$$

Then $\tilde{\Phi} : R^+ \times \Omega \times \tilde{X} \to \tilde{X}$ is a cocycle on $\tilde{X}$ over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(s))_{s \in \mathbb{R}})$.
Moreover, assume $Y : \mathbb{R} \times \Omega \to X$ is a random periodic solution of the semi-flow u with period $\tau > 0$. Then $\tilde{Y} : \mathbb{R} \times \Omega \to \tilde{X}$ defined by

$$\tilde{Y}(s, \omega) := (s \; mod \; \tau, Y(s, \omega)), \tag{21}$$

is a random periodic solution of the cocycle $\tilde{\Phi}$ on $\tilde{X}$.