Brownian trading excursions

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Stochastic heat equation with multiplicative noise

- Consider the SPDE, for g smooth and positive,

$$dv(t,x) = \left(\frac{1}{2}\partial_x^2 v\,(t,x) + g\right)\,dt + \partial_x v(t,x)\,dW_t,$$
$$v(0,x) = 0.$$

- Existence and uniqueness (in a weak form) have already been proved in da Prato & Zabczyk (1992).
- Explicit solution? By economic intuition!

Some key references

1. P. Biane, J. Pitman and M. Yor (2001). Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. Bull. Amer. Math. Soc. 38, p. 435-465
2. R. Mansuy and M. Yor (2008). Aspects of Brownian Motion. Springer
3. C. G. J. Jacobi, (1829), Fundamenta Nova Theoriae Functionum. Reprinted by Cambridge University Press 2012
4. D. Revuz and M. Yor (2004). Continuous Martingales and Brownian Motion. 3rd edition, Springer (in particular Ch. XII: Excursion theory).
5. L. C. G. Rogers (1981). Williams' characterization of the Brownian excursion law: proof and applications (Brownian motion). LNM 850 XV: 16, 227–250

Limit order book model

- We work in business time and model the mid price process W as a Brownian motion; for a discussion of various price concepts in the context of the limit order book, see Delattre, Robert & Rosenbaum (2013). A more realistic model in real time would result from subordinating B.
- During an infinitesimal time interval dt, it is assumed that new limit orders are created at every level $W_t + u$ with volume density $g(u)\,du$, for some integrable function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$.
- We can write the volume field in terms of order arrivals A_t^u as

$$V_t^u = \int_{\ell_u(t)}^{t} dA_s^u,$$

where $\ell_u(t)$ denotes the *last exit time* from level u before time t, and

$$dA_t^u = g\left(u - W_t\right)\,dt.$$

Basic model

- In our basic model, we consider the special case $dA_t^u = \delta_{u-(W_t+\mu)}\, dt$ for some fixed displacement $\mu > 0$, and where δ is the Dirac delta at level 0. This is just an informal specification; however, we can make it formal by setting

$$A_t^u = L_t^{u-\mu},$$

  so the total volume of orders at level u (before any order gets executed) equals Brownian local time $L_t^{u-\mu}$ at level $u-\mu$; for a random walk this is just the number of times the walk hits this level. The total orders at some level get executed once the mid price hits this level.

- We will focus just on the ask side (sell orders), i.e. we assume $g(x) = 0$ for $x < 0$. Exogenous cancellations can be included in our model without much effort, but for the purpose of this talk will be excluded.

SPDE for the order volume

- The relative volume random field $v(t,x) := V(t, x + W_t)$ is the unique weak solution of the SPDE

$$dv(t,x) = \left(\frac{1}{2}\partial_x^2 v(t,x) + g\right) dt + \partial_x v(t,x)\, dW_t,$$
$$v(0,x) = 0.$$

- We can express the solution in terms of local time as

$$v(t,x) = \int_{\mathbb{R}} \left(L_t^y - L_{\ell_x(t)}^y\right) g(x-y)\, dy.$$

where $\ell_x(t)$ is the *last exit time* from level x before time t. A time reversed variant involving a first passage time is also possible.

Sobolev spaces

- Let $H := L^2\left(\left(\mathbb{R}_+, \mathcal{B}, \lambda\right), \mathbb{R}\right)$ and consider the spaces (all are dense in H)

$$H^1 := \left\{ f \in H : f \text{ is weakly differentiable and } f' \in H \right\},$$
$$H^2 := \left\{ f \in H^1 : f' \in H^1 \right\}.$$

- For any $f \in H^1$, define the norm $\|f\|_1^2 := |f|^2 + |f'|^2$ and denote the dual space by H^{-1}. It contains for instance the point evaluations

$$\delta_\mu : H^1 \to \mathbb{R},\ f \mapsto f\left(\mu\right).$$

- The weak derivative operator is denoted by

$$D : H^1 \to H,\ f \mapsto f'.$$

Weak solution

- We are looking for a weak solution $v \in H^{-1}$ with given $v(0) = g_0 \in H^{-1}$, $g \in H^{-1}$ such that

$$dv(t) = \left(\frac{1}{2}D^2 v(t) + g\right) dt + Dv(t)\, dW_t.$$

- Special case: Let $A(t,x) := L_t^{x-\mu} - L_{\ell_x(t)}^{x-\mu}$ and $a(t,x) := A\left(t, x + W_t\right)$. Then a is the weak solution of the SPDE with $a(0,x) = a(t,0) = 0$ and $g = \delta_\mu \in H^{-1}$ in the sense that for all $\phi \in H_0^2$, we have

$$d\left\langle \phi, a(t)\right\rangle = \left(\frac{1}{2}\left\langle D_0^2\phi, a(t)\right\rangle + \phi\left(\mu\right)\right) dt + \left\langle D\phi, a(t)\right\rangle\, dW_t.$$

SPDE with exogenous cancellations

- In this extended model it is assumed that orders get cancelled if the price process moves further than $\nu > \mu$ away from the order level. We then get for the relative order book volume

$$v(t,x) = V(t, x + W_t) = L_t^{x+W_t-\mu} - L_{\ell_{x+W_t}(t) \vee \ell_{x+W_t-\nu}}^{x+W_t-\mu},$$

  which satisfies the SPDE

$$dv(t,x) = \left(\frac{1}{2}\partial_x^2 v(t,x) + \delta_\mu \right) dt + \partial_x v(t,x)\, dW_t,$$
$$v(t,0) = 0 = v(t,\nu).$$

Execution mechanisms

- As a special case, we consider $g(x) = \delta_{x+\mu}$ for some fixed displacement $\mu > 0$, so the total volume of orders at level u equals the local time at level $u - \mu$. We now assume that the Brownian motion W, $W_0 = 0$, has just surpassed the level μ:

1. Type I execution: an order is triggered whenever the running maximum of B is increasing.
2. Type II execution: like type I, but after a downward excursion straddling at least a size of μ. If the excursion lasts less than time ε we call it a '*flash crash*'.

- We take record if there is no order execution in a time period lasting longer than ε, i.e. when a downward excursion takes place with duration at least of ε.

Trading times

- We say that a random time $\tau > 0$ is a *trading time* if $V(\tau_-, W_\tau) > 0$ and $V(\tau, W_\tau) = 0$.
- Let $\Upsilon(\tau)$ denote the supremum of trading times before trading time τ; if there was no trading time before τ we set $\Upsilon(\tau) = 0$. Similarly, let $\Xi(\tau)$ denote the infimum of trading times after trading time τ.
- We say that a *type I* trade occurs if either $\Upsilon(\tau) = \tau$ and $\tau = \Xi(\tau)$ (*type Ia* trade) or $\Upsilon(\tau) = \tau$ but $\tau \neq \Xi(\tau)$ (*type Ib* trade) or $\Upsilon(\tau) \neq \tau$, but $\sup_{\Upsilon(\tau) \leq t \leq \tau} \left(W_{\Upsilon(\tau)} - W_t \right) < \mu$ (*type Ic* trade).
- Otherwise we call it a *type II* trade.

Order avalanches

- Orders in the LOB get executed via avalanches. In other words, limit orders may accumulate on some levels, and when the price process crosses those values, we will see a sudden decrease of the number of orders in the LOB.
- Let $T_\varepsilon^{\text{start}}$ denote the time when there is the first order execution after a time of at least ε since the last execution, and $T_\varepsilon^{\text{end}}$ similarly the last execution time before a downward excursion lasting at least ε takes place.
- An ε-*avalanche* is defined as the stopped process $\{W_t : T_\varepsilon^{\text{start}} \leq t \leq L^\varepsilon\}$ where the *avalanche length* L^ε is the difference between the last and the first execution time,

$$L^\varepsilon := T_\varepsilon^{\text{end}} - T_\varepsilon^{\text{start}}.$$

- We are interested into the distribution of the avalanche length.

Laplace transform for avalanche length with no flash crash

- Let us first assume that there is no downward excursion straddling at least a size of μ, and lasting less than time ε.
- Gauthier (2002) as well as Dassios and Lim (2009) derive the Laplace transform of the avalanche length L^ε in the context of Parisian options as

$$E\left[e^{-\lambda L^\varepsilon}\right] = \frac{1}{\sqrt{\lambda\varepsilon\pi}\,\mathrm{erf}\left(\sqrt{\lambda\varepsilon}\right) + e^{-\lambda\varepsilon}}.$$

- The same formula can be inferred (Laurent de Dok du Wit, Diploma Thesis 2012) from the Lévy measure of the subordinator consisting of Brownian passage times.

Hyperbolic function table for intertrading times

- **Hyperbolic function table.** Assume τ is a type Ib trade. Let T denote the time to the next trade. Moreover, let σ be any trading time, and denote by U the time to the next type II trade. Then we get for the Laplace transforms wrt. the BES_3-law, denoting $\widetilde{\lambda} = \sqrt{2\lambda}$, that

$$E\left[e^{-\lambda T}\right] = 1 - \frac{\widetilde{\lambda}}{2}\tanh(\mu\widetilde{\lambda}); \tag{1}$$

$$E\left[e^{-\lambda T}\mathbf{1}_{H<\mu}\right] = 1 - \frac{\widetilde{\lambda}}{2}\coth(\mu\widetilde{\lambda}); \tag{2}$$

$$E\left[e^{-\lambda T}\mathbf{1}_{H\geq\mu}\right] = \widetilde{\lambda}\,\mathrm{csc\,h}(2\mu\widetilde{\lambda}); \tag{3}$$

$$E\left[e^{-\lambda U}\mathbf{1}_{H\geq\mu}\right] = \frac{1}{2}\mathrm{sec\,h}^2(\mu\widetilde{\lambda}). \tag{4}$$

- Regarding entry (**??**) of the hyperbolic table, we are interested in the time U to the next Type II trade, conditioned that we have no Type I trade before. This means that we condition on that there will be, starting from some fixed level $x = W(\tau)$, a downward excursion with height of at least μ. Denote the first hitting time of the level $x - \mu$ by T_μ. The process $x - W$ on $[\tau, T_\mu]$ has then the law of a three-dimensional Bessel process BES_3. By Biane, Pitman & Yor (2001) the Laplace transform of T_μ is $\mu\sqrt{2\lambda}/\sinh(\mu\sqrt{2\lambda})$. The significance of T_μ is that after this time, it is certain that there will be a Type II trade before the next Type I trade.
- As T_μ is a stopping time for the filtration generated by W, after T_μ by the strong Markov property W has the law of a Brownian motion. The Type II trade will get triggered once $W_t - \min_{t \geq T_\mu} W_t$ equals μ, for $t \geq T_\mu$. We have that $W - \min W$ has the same law as $|W|$. The Laplace transform of the modulus of Brownian motion equals $1/\cosh\left(\mu\sqrt{2\lambda}\right)$. Using the doubling formula for the hyperbolic sine, it results that

$$E\left[e^{-\lambda U}\right] = \mu\sqrt{2\lambda}\operatorname{csch}\left(2\mu\sqrt{2\lambda}\right).$$

Laplace transform for full avalanche length, including flash crashes

- We can show that the Laplace transform of the full avalanche length A^ε is given as

$$E\left[e^{-\lambda A^\varepsilon}\right] = \frac{\int_\varepsilon^\infty h(x)\,dx}{\int_0^\varepsilon \left(1 - e^{-\lambda x}\right) h(x)\,dx + \int_\varepsilon^\infty h(x)\,dx}$$

where

$$2h(x) = \frac{x^{-3/2}}{\sqrt{2\pi}} + 2\sum_{k\geq 1}\left(\frac{x^{-3/2}}{\sqrt{2\pi}} - 2\sqrt{\frac{2}{\pi}}\frac{k^2y^2}{x^{5/2}}\right)e^{-2k^2y^2/x}$$

and

$$\int_0^\infty (1 - e^{-\lambda x})h(x)dx = \frac{1}{2}\sqrt{2\lambda}\tanh\left(\mu\sqrt{2\lambda}\right).$$

A short primer on excursion theory

- Denote by $(U, \mathcal{U})$ the measurable space of Brownian excursions, and by $(e_t, t > 0)$ the excursion process. We enhance it by the zero-excursion (which is set equal to δ) on the set where the local time at zero is strictly increasing, and denote the resulting space by $U_\delta = U \cup \{\delta\}$, equipped with the σ-algebra $\mathcal{U}_\delta = \sigma(\mathcal{U}, \{\delta\})$.
- For a measurable subset Γ of $\mathcal{U}_\delta$, one sets

$$N_t^\Gamma(\omega) = \sum_{0<u\leq t} \mathbf{1}_\Gamma(e_u(\omega)).$$

- The *Ito measure* n is the σ-finite measure defined on $\mathcal{U}$ by

$$n(\Gamma) := E\left[N_1^\Gamma\right]$$

  and extended to $\mathcal{U}_\delta$ by $n(\delta) = 0$.
- It turns out that the excursion process is a Poisson Point Process, and hence the Ito measure is its characteristic measure.

Ito's description of n

- Denoting by R the excursion length, the density of R under n^+ (the Ito measure restricted to positive excursions) is
$$\frac{1}{2\sqrt{2\pi r^3}}.$$
- Moreover, under n^+ and conditionally on $R = r$, the coordinate process w has the law π_r of the Bessel Bridge of dimension 3 over $[0, r]$.
- Hence if Γ is a measurable subset of U^+, then
$$n_+(\Gamma) = \int_0^\infty \pi_r(\Gamma \cap \{R = r\}) \frac{dr}{2\sqrt{2\pi r^3}}.$$

Moment generating function of normalized excursion height

- In fact, the mgf of the height N of the normalized Brownian excursion can be determined as

$$\frac{1}{2}E\left[\left(\frac{\pi}{2}N\right)^s\right] = \xi\left(2s\right), \quad s > 1,$$

where ξ denotes the Riemann Xi function which is connected to the Riemann zeta function ζ by

$$\xi\left(s\right) = \frac{s\left(s-1\right)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta\left(s\right).$$

- In particular, the Xi function has no zeroes outside of the critical strip, and satisfies the reflection principle

$$\xi\left(1-s\right) = \xi\left(s\right).$$

Joint law of excursion length and height

- Our next result gives a transform of the joint distribution of the excursion length R and its height H. This transform is a bivariate Laplace-Mellin transform and states that for $\lambda > 0$, $s > 1$

$$\int_U e^{-\lambda R(u)} H(u)^{s-1} \, n^+(du) = \frac{1}{\sqrt{8\pi}} \lambda^{1-\frac{s}{2}} \Gamma\left(\frac{s}{2} - 1\right) \xi\left(\frac{s}{2}\right).$$

- Moreover, we get for the joint law, i.e. the distribution of a flash crash under the lower Ito measure,

$$n^-(R < \varepsilon; H > \mu) = \frac{1}{\mu} \sum_{n=1}^{\infty} \left(e^{-\frac{\pi^2 n^2 \varepsilon}{2\mu^2}} - 1\right).$$

Thank You for Your attention!