A martingale fixed-point approach to optimal reserves exploitation

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Similarity in stochastic control problems

- mining
- electricity generation
- emission trading
- asset liquidation
- pension funds

Same problem structure

Given:

- at time $t = 0, 1 \ldots, T - 1$,
- ξ_t, activity to be scheduled
- $C_t(\xi_t)$, costs known at t
- $E_t(\xi_t)$, resource consumption known at t
- at T, the effect of resource consumption is uncertain

Problem:

- at each time t determine optimal ξ_t
  (balance between $C_t(\xi_t)$ and $E_t(\xi_t)$)

Example: Mining

- activity: mining strategy (different technologies)
- costs: income from mining
- resource consumption: level at the end

Example: Hydro-generation

- activity: energy generation (different units)
- costs: income from generation
- resource consumption: water level at the end

Example: Emission trading

- activity: production
- costs: income from business
- resource consumption: greenhouse gas level

Example: Order liquidation

- activity: order placement (order and size and type)
- costs: revenue from sold asset
- resource consumption: asset position at the end

Example: Pension fund:

- activity: fund withdrawal
- costs: payout (fees for large withdraw)
- resource consumption: funds remaining

Difficulty with these problems:

the space Ξ of all activities ξ_t is complex

Observation: There is a related problem, which is easier, whose control is one-dimensional.

Idea: If there was a price A_t for a right to receive one unit of resources at the end, then an optimal activity should minimize

$$\xi_t \mapsto C_t(\xi_t) + A_t E_t(\xi_t)$$

Surprising:

The (virtual) resource price starts driving the activity!

Indeed

In emission trading central planer attempts to find the cheapest emission savings, and makes emission certificates tradable.

Certificate price becomes driver: It is optimal to apply all emission abatement measures which are cheaper than the certificate price

(Market finds an optimal way,...)

We try to recover this principle for a class of control problems

Consider Ξ-valued $(\mathcal{F}_t)_{t=0}^{T}$-adapted activity policies $\xi = (\xi_t)_{t=0}^{T-1}$.

We aim at minimizing the burden

$$B(\xi) = \mathbb{E}\Big(\sum_{t=0}^{T-1} C_t(\xi_t) + H\big(\sum_{t=0}^{T-1} E_t(\xi_t) - \Gamma\big)\Big)$$

of the policy $\xi = (\xi_t)_{t=0}^{T}$ in a non-Markovian situation, where

$$C_t, E_t : \Xi \times \Omega \to \mathbb{R}, \text{ are } \mathcal{B}(\Xi) \otimes \mathcal{F}_t\text{–measurable}$$

with

- Γ random (unknown future resource level)
- H convex (monetary value of resource remaining)

Solution to the *optimal control problem*

$$\left.\begin{array}{c}\text{determine a policy } \xi^* = (\xi^*_t)_{t=0}^{T-1} \text{ which}\\ \text{minimizes the burden } B(\xi^*) \leq B(\xi)\\ \text{over all policies } \xi = (\xi_t)_{t=0}^{T-1}.\end{array}\right\} \text{(OCP)}$$

can be given in terms of *martingale fixed point problem*

$$\left.\begin{array}{c}\text{determine a martingale } A^* = (A^*_t)_{t=0}^{T} \text{ whose}\\ \text{terminal value is } A^*_T = \nabla H(\sum_{t=0}^{T-1} E_t(\mathcal{X}_t(A^*_t)) - \Gamma)\end{array}\right\} \text{(MFP)}$$

under standing assumption

$$\left.\begin{array}{c}\text{for } a \in \mathbb{R}\text{, the minimization problem}\\ \inf_{x \in \Xi} (C_t(x) + aE_t(x))\\ \text{possesses a unique solution } \mathcal{X}_t(a)\end{array}\right\}$$

Lagrange parameter a?

Connection (OCP) $\longleftarrow$ (MFP)

Under some technical assumptions, a solution $(A_t^*)_{t=0}^{T}$ to (MFP) yields a solution $(\xi_t^* = \mathcal{X}_t(A_t^*))_{t=0}^{T-1}$ to (OCP).

Technical assumption:

$$\left.\begin{array}{c}\text{function } H \text{ is convex whose derivative } \nabla H \\ \text{exists and is continuous at (Lebesgue) almost each point} \\ \text{and is bounded } \infty < \underline{h} = \inf \nabla H,\ \infty > \overline{h} = \sup \nabla H.\end{array}\right\}$$

Technical assumption:

$$\left.\begin{array}{c}\text{the conditioned distribution } \mathbb{P}_{T-1}(\Gamma \in dy) \\ \text{possess no point masses almost surely.}\end{array}\right\}$$

Solution to (MFP)

Introduce the quantity

$$e_t(a) := E_t(\mathcal{X}_t(a)), \qquad a \in \mathbb{R}$$

where

$$\mathcal{X}_t(a) \text{ is the unique solution to } \inf_{x \in X} (C_t(x) + aE_t(x))$$

Existence of solution to (MFP)

If $a \mapsto e_t(a)$ is non-increasing and continuous (almost surely) then there exists a martingale $(A^*_t)_{t=0}^T$ which satisfies

$$A^*_T = \nabla H\Big(\sum_{t=0}^{T-1} e_t(A^*_t) - \Gamma\Big)$$

Idea for the existence of solution to (MFP)

at time t the entire situation shall be uniquely determined by the estimated (resource consumption) level

$$G_t = \sum_{s=0}^{t-1} e_t(A_s^*) - \underbrace{\mathbb{E}_t(\Gamma)}_{\Gamma_t}, \quad t = 0, \dots, T.$$

Idea for the existence

try out

$$A_t^*(\omega) = h_t(G_t(\omega))(\omega), \quad \omega \in \Omega, \;\; t = 0, \ldots, T,$$

assuming hypothetic functionals

$$h_t : \mathbb{R} \times \Omega \to \mathbb{R}, \;\; \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t\text{-measurable, for } t = 0, \ldots T$$

Resource: running level and price

the recursion

$$
\begin{aligned}
A_t^* &:= h_t(G_t), \\
G_{t+1} &:= G_t + e_t(A_t^*) - \underbrace{(\Gamma_{t+1} - \Gamma_t)}_{\varepsilon_{t+1}}, \\
&\quad \text{started at } G_0 := \Gamma_0.
\end{aligned}
$$

generates a solution $(A_t^*)_{t=0}^T$ to (MFP) if, for all $t = 0, \dots T-1$

$$
\underbrace{A_t^*}_{h_t(G_t)} = \mathbb{E}_t(\underbrace{A_{t+1}}_{h_{t+1}(G_{t+1})})
$$

for which we shall fulfill

$$
\underbrace{h_t(g)}_{a^*} = \mathbb{E}_t^{\mathrm{Q}}(h_{t+1}(g + e_t(\underbrace{h_t(g)}_{a^*}) - \varepsilon_{t+1})), \quad \text{for all } g \in \mathbb{R}.
$$

Finding functionals

we obtain $h_t(g) := a^*$ recursively, for $t = T-1, \dots, 0$ as fixed points

$$a^* = \mathbb{E}_t^{\mathrm{Q}}(h_{t+1}(g + e_t(a^*) - \varepsilon_{t+1})), \quad \text{for all } g \in \mathbb{R}.$$

It turns out that there exist functionals

$$h_t : \mathbb{R} \times \Omega \to [\underline{h}, \overline{h}], \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t\text{-measurable}, \, t = 0, \dots T \tag{1}$$

which fulfill for all $g \in \mathbb{R}$

$$\begin{aligned} h_T(g) &= \nabla H(g), \\ h_t(g) &= \mathbb{E}_t(h_{t+1}(g + e_t(h_t(g)) - \varepsilon_{t+1})), \quad t = 0, \dots, T-1. \end{aligned}$$

Putting together

given the functionals $(h_t)_{t=0}^T$, the recursion

$$\begin{aligned} A_t^* &:= h_t(G_t), \\ G_{t+1} &:= G_t + e_t(A_t^*) - \underbrace{(\Gamma_{t+1} - \Gamma_t)}_{\varepsilon_{t+1}}, \\ &\text{started at } G_0 := \Gamma_0. \end{aligned}$$

generates a solution $(A_t^*)_{t=0}^T$ to (MFP)

Outlook

a solution is most interesting in the intermediate case, when (OCP) is driven by the resource price

$$\inf_{(A_t)_{t=0}^{T-1}} \mathbb{E}\left(\sum_{t=0}^{T-1} \underbrace{C_t(\mathcal{X}_t(A_t))}_{c_t(A_t)} + H\Big(\sum_{t=0}^{T-1} \underbrace{E_t(\mathcal{X}_t(A_t))}_{e_t(A_t)} - \Gamma\Big).\right)$$

Outlook

the optimal control problem

$$\inf_{(A_t)_{t=0}^{T-1}} \mathbb{E}\left(\sum_{t=0}^{T-1} c_t(A_t) + H(\sum_{t=0}^{T-1} e_t(A_t) - \Gamma)\right)$$

boils down to Markovian situation, if there exists a Markov state process $(Z_t)_{t=0}^{T}$ such that

$$\mathcal{F}_t\text{-conditioned distribution of } (\varepsilon_s)_{s=t}^{T} \text{ depends on } Z_t \text{ only}$$

and the randomness in $c_t(\cdot), e_t(\cdot)$ enters through Z_t only.

Outlook

Under appropriate additional assumptions the value function is convex, and the problem can be treated using

Convex Switching Systems (CSS) technique

- Optimal stochastic switching under convexity assumptions *SIAM Journal on Control and Optimization, 52(1), 2014*
- Algorithms for optimal control of stochastic switching systems *TPA, forthcoming* (to appear)
- Using convex switching techniques for partially observable decision processes (IEEE-TAC, forthmocoming)
- R package: rcss (with Jeremy Yee, on CRAN soon)

Thank you!