On the Chaotic Representation Property of Certain Families of Martingales*

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1. Introduction and Motivation

K. Itô (1951):
Multiple Wiener integral, *J. Math. Soc. Japan* **3**, 157–169

Theorem (Chaotic Representation Property (CRP))

Every square integrable functional of a normal random measure can be expanded as an orthogonal sum of multiple integrals.

Related works: N. Wiener (1938) (*“The Homogeneous Chaos“*)
R.H. Cameron & W.T. Martin (1947), S. Kakutani (1950)

- The relation between orthogonal multiple integrals and *Hermite polynomials* was established.
- K. Itô (1951) also pointed out that multiple integrals of a normal random measure induced by a Wiener process W can be regarded as *iterated stochastic integrals* with respect to W.

Introduction and Motivation

Second important paper by K. Itô (1956): *Spectral type of the shift transformation of differential processes with stationary increments.* Trans. Amer. Math. Soc. **81** (1956), 253–263

- Generalization of the chaotic representation property (CRP) to *orthogonal* random measures, defined as a sum of a normal random measure and a *compensated Poisson random measure*.
- These random measures are associated with processes with independent increments (Lévy processes).
- *Iterated integrals* and their relationship to *multiple integrals* have not been considered there.

Some more recent works on the CRP:

M. Emery (1989), R. Léandre and P.-A. Meyer (1989),
P.-A. Meyer (1990, 1992), D. Nualart and W. Schoutens (2000)

Why the CRP is important?

- The CRP implies the PRP (predictable representation property)
- The PRP is indispensable for the theory of BSDEs
- The CRP is a basic ingredient of:

  Malliavin Calculus,

  Quantum Probability Theory
- PRP in Mathematical Finance:

  Complete Financial Markets,

  Completion of Financial Markets

Objectives

The objectives of the present talk are:

- To investigate the CRP for, as general as posssible, families $\mathscr{X} = \{X^{(\alpha)} : \alpha \in \Lambda\}$ of square integrable martingales;
- to deduce suitable sufficient conditions for the CRP of $\mathscr{X}$;
- to discuss applications of our main results to concrete families $\mathscr{X}$.

2. Iterated Integrals and CRP

- $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F}$ satisfying the usual conditions.
- $\mathscr{H}^2_T := \mathscr{H}^2_T(\mathbb{F})$ is the Hilbert space of square integrable (càdlàg) $\mathbb{F}$-martingales X on $[0, T]$, $X_0 = 0$, with norm $\|X\|_{\mathscr{H}^2_T} = \mathbb{E}[X^2_T]^{\frac{1}{2}}$.
- As usual, $\langle X, Y\rangle$ denotes the predictable covariation of $X,\ Y \in \mathscr{H}^2_T$.

Let $\mathscr{X} := \{X^{(\alpha)},\ \alpha \in \Lambda\} \subseteq \mathscr{H}^2_T(\mathbb{F})$ be a parametrized family of square integrable martingales.

Standing Hypothesis: $\langle X^{(\alpha)}, X^{(\beta)}\rangle$ is deterministic, $\alpha, \beta \in \Lambda$.

Notation: $m^{(\alpha)}$ *finite* measure on $[0, T]$ induced by $\langle X^{(\alpha)}, X^{(\alpha)}\rangle$.

$F := F_1 \otimes F_2 \cdots \otimes F_n$ denotes the *tensor product* of $F_1, \ldots, F_n$ with F_k measurable and bounded on $[0, T]$, $k = 1, \ldots, n$.

We say that F is an *elementary function* of order n.

Definition (Elementary iterated integral)

The *elementary* iterated integral of F with respect to $(X^{(\alpha_1)}, X^{(\alpha_2)}, \dots, X^{(\alpha_n)})$ is defined inductively by setting $J_0 \equiv 1$ and for all $1 \le m \le n,\ t \in [0, T]$,

$$J_m^{(\alpha_1,\dots,\alpha_m)}(F_1 \otimes \cdots \otimes F_m)_t := \int_0^t J_{m-1}^{(\alpha_1,\dots,\alpha_{m-1})}(F_1 \otimes \cdots \otimes F_{m-1})_{u-}\, F_m(u)\, \mathrm{d}X_u^{(\alpha_m)}\,.$$

For $0 \le t \le T$, we introduce the set

$$M_t^{(n)} := \{(t_1, \dots, t_n) :\ 0 \le t_1 \le t_2 \dots \le t_n \le t\}\,.$$

By linearity and continuity, the mapping $F \mapsto J_n^{(\alpha_1,\dots,\alpha_n)}(F)_t$ can be extended to an isometry of $L^2(M_t^{(n)}, \bigotimes_{k=1}^n m^{(\alpha_k)})$ into $L^2(\mathbb{P})$. We denote this extension by $J_n^{(\alpha_1,\dots,\alpha_n)}(\cdot)_t,\ t \in [0, T]$.

Definition (n-fold iterated integral)

For $F \in L^2(M_T^{(n)}, \bigotimes_{k=1}^{n} m^{(\alpha_k)})$, we call the $\mathscr{H}_T^2$-martingale $J_n^{(\alpha_1,\dots,\alpha_n)}(F) := (J_n^{(\alpha_1,\dots,\alpha_n)}(F)_t)_{t\in[0,T]}$ the n-fold iterated stochastic integral of F with respect to $(X^{(\alpha_1)}, X^{(\alpha_2)}, \dots, X^{(\alpha_n)})$.

Notation: $\mathscr{J}_n^{(\alpha_1,\dots,\alpha_n)}$ is the closed subspace of $\mathscr{H}_T^2$ of n-fold iterated stochastic integrals relative to $(X^{(\alpha_1)}, X^{(\alpha_2)}, \dots, X^{(\alpha_n)})$.

We put $\mathscr{J}_0 = \mathbb{R}$ and for all $n \geq 1$

$$\mathscr{J}_n = \text{cl}\Big(\text{span}\Big(\bigcup_{(\alpha_1,\dots,\alpha_n)\in\Lambda^n} \mathscr{J}_n^{(\alpha_1,\dots,\alpha_n)}\Big)\Big)_{\mathscr{H}_T^2},$$

$$\mathscr{J} = \text{cl}\Big(\text{span}\Big(\bigcup_{n\geq 0} \mathscr{J}_n\Big)\Big)_{\mathscr{H}_T^2}.$$

We call $\mathscr{J}$ the space of iterated integrals *generated by* $\mathscr{X}$.

$\mathscr{I}_T$, $\mathscr{I}_{n,T}^{(\alpha_1,\dots,\alpha_n)}$, $\mathscr{I}_{n,T}$ are defined as the subspaces of $L^2(\mathbb{P})$ of terminal variables of iterated integrals from $\mathscr{I}$, $\mathscr{I}_n^{(\alpha_1,\dots,\alpha_n)}$, $\mathscr{I}_n$.

Definition (Chaotic Representation Property)

$\mathscr{X} = \{X^{(\alpha)},\ \alpha \in \Lambda\}$ possesses the *chaotic representation property* (CRP) on the Hilbert space $L^2(\mathbb{P}) = L^2(\Omega, \mathscr{F}, \mathbb{P})$ if $\mathscr{I}_T = L^2(\mathbb{P})$ (or, equivalently, $\mathscr{I} = \mathscr{H}_T^2$).

Proposition

(i) *It holds* $\mathscr{I}_T = \bigoplus_{n=0}^{\infty} \mathscr{I}_{n,T}$.
(ii) $\mathscr{X}$ *possesses the CRP if and only if* $L^2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathscr{I}_{n,T}$.
(iii) *If* $\mathscr{X} := \{X^{(n)},\ n \geq 1\} \subseteq \mathscr{H}_T^2$ *is countable and its elements are mutually orthogonal, then the CRP holds if and only if*

$$L^2(\mathbb{P}) = \mathbb{R} \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\dots,j_n)\in\mathbb{N}^n} \mathscr{I}_{n,T}^{(j_1,\dots,j_n)} \right).$$

- Consider $\mathscr{X} := \{X^{(\alpha)} : \alpha \in \Lambda\} \subseteq \mathscr{H}_T^2$.
- $\mathbb{F}^{\mathscr{X}}$ denotes the natural filtration of $\mathscr{X}$.
- We will be working on the probability space $(\Omega, \mathcal{F}_T^{\mathscr{X}}, \mathbf{P})$.
- We consider the set of monomials $\mathscr{K} = \mathscr{K}(\mathscr{X})$:

$$\mathscr{K} := \left\{ \prod_{i=1}^{m} X_{t_i}^{(\alpha_i)},\ \alpha_i \in I,\ i = 1, \ldots, m,\ t_i \in \mathbb{R},\ m \in \mathbb{N} \right\}.$$

$\mathscr{K}$ generates $\mathcal{F}_T^{\mathscr{X}}$. Its linar hull is just the space $\mathscr{P} = \mathscr{P}(\mathscr{X})$ of polynomials in $X_t^{(\alpha)},\ \alpha \in \Lambda,\ t \in [0, T]$.

We shall need that $\mathscr{P}$ is dense in $L^2(\mathbf{P}) := L^2(\Omega, \mathcal{F}_T^{\mathscr{X}}, \mathbf{P})$ or equivalently, that $\mathscr{K}$ is *total* in $L^2(\mathbf{P})$.

Lemma (Denseness of Polynomials)

If for every $\alpha \in \Lambda$ and $t \in [0, T]$ there exists $c_\alpha(t) > 0$ such that $\mathbb{E}\left\{\exp\left(c_\alpha(t)|X_t^{(\alpha)}|\right)\right\} < +\infty$ then $\mathscr{P}(\mathscr{X})$ is dense in $L^2(\mathbf{P})$.

Compensated Covariation

Recall that the covariation $[X, Y]$ of $X, Y \in \mathscr{H}_T^2$ is defined by

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{0 \le u \le t} \Delta X_u \Delta Y_u \,,$$

X^c, Y^c are the continuous martingale parts of X, Y.

For arbitrary $X, Y \in \mathscr{H}_T^2$ we have:

- $[X, Y]$ is of integrable variation, hence its compensator exists.
- The compensator of $[X, Y]$ is $\langle X, Y \rangle$.
- The process $[X, Y] - \langle X, Y \rangle$ is a martingale (called the *compensated-covariation process* of X and Y).

Definition (Compensated-Covariation Stability)

We say that the family $\mathscr{X} = \{X^{(\alpha)} : \ \alpha \in \Lambda\}$ is compensated-covariation stable if for all $\alpha, \beta \in \Lambda$

$$X^{(\alpha,\beta)} := [X^{(\alpha)}, X^{(\beta)}] - \langle X^{(\alpha)}, X^{(\beta)} \rangle \in \mathscr{X} \,.$$

Theorem (Chaotic Representation Property)

Let $\mathscr{X} = \{X^{(\alpha)} : \alpha \in \Lambda\}$ be a family of square integrable martingales with respect to $\mathbb{F}^{\mathscr{X}}$ on $(\Omega, \mathcal{F}_T^{\mathscr{X}}, \mathbf{P})$. Suppose that the following conditions are satisfied:

(A) *$\mathscr{X}$ is compensated-covariation stable.*

(B) *The predictable covariation $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ is deterministic for all $\alpha, \beta \in \Lambda$.*

(C) *The set $\mathscr{P} = \mathscr{P}(\mathscr{X})$ of polynomials is dense in $L^2(\mathbf{P})$.*

Then $\mathscr{X}$ possesses the CRP with respect to $\mathbb{F}^{\mathscr{X}}$.

Continuous Gaussian Families

Theorem

Every Gaussian family $\mathscr{X}$ of continuous martingales possesses the CRP w.r.t. $\mathbb{F}^{\mathscr{X}}$.

- $\mathscr{X} \cup \{0\}$ is compensated-covariation stable.
- $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ are deterministic.
- The monomials $\mathscr{K}(\mathscr{X})$ are total in $L^2(\mathbf{P})$ because of the lemma above: All the $X_t^{(\alpha)}$ have exponential moments of arbitrary order). □

If $\mathscr{X} = \{X^{(n)} : n \geq 1\}$ is countable and uncorrelated then

$$L^2(\Omega, \mathscr{F}^{\mathscr{X}}, \mathbb{P}) = \mathscr{J}_T(\mathscr{X}) = \mathbb{R} \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \mathscr{J}_{n,T}^{(j_1,\ldots,j_n)} \right).$$

Theorem (Independent Poisson Families)

Let $\mathscr{X} = \{X^{(\alpha)} : \alpha \in \Lambda\}$ be an independent family of compensated Poisson processes with continous intensity functions a_α. Then $\mathscr{X}$ possesses the CRP w.r.t. $\mathbb{F}^{\mathscr{X}}$.

- $\mathscr{X} \cup \{0\}$ is compensated-covariation stable:

$$X^{(\alpha,\alpha)} = [X^{(\alpha)}, X^{(\alpha)}] - \langle X^{(\alpha)}, X^{(\alpha)} \rangle = \sum_{0<s\leq \cdot} (\Delta X_s^{(\alpha)})^2 - a_\alpha(\cdot) = X^{(\alpha)}$$

and, for $\alpha \neq \beta$, $[X^{(\alpha)}, X^{(\beta)}] = 0$, $\langle X^{(\alpha)}, X^{(\beta)} \rangle = 0$ and $X^{(\alpha,\beta)} = 0$.

- $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ are deterministic.
- $X_t^{(\alpha)}$ has exponential moments of arbitrary order.

If $\Lambda = \mathbb{N}$ then

$$L^2(\Omega, \mathscr{F}^{\mathscr{X}}, \mathbb{P}) = \mathscr{J}_T(\mathscr{X}) = \mathbb{R} \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \mathscr{J}_{n,T}^{(j_1,\ldots,j_n)} \right).$$

Let $(L, \mathbb{F}^L)$ be a Lévy Process on $[0, T]$:

- L is càdlàg and $L_0 = 0$.
- L has homogeneous and independent increments.
- L is stochastically continuous.

The jump measure M of L on $(E, \mathscr{B}(E))$, $E := [0, T] \times \mathbb{R}$, is given by:

$$\mathrm{M}(\omega, A) := \sum_{s \geq 0} 1_{\{\Delta L_s(\omega) \neq 0\}} 1_A(s, \Delta L_s(\omega)), \ \omega \in \Omega, \ A \in \mathscr{B}(E) \, .$$

(β, σ^2, ν) characteristics of L, ν Lévy measure, $m := \lambda_+ \otimes \nu$

The compensated jump measure $\overline{\mathrm{M}}$ of L is defined by:

$$\overline{\mathrm{M}}(A) := \mathrm{M}(A) - \mathrm{m}(A), \ A \in \mathscr{B}(E), \ m(A) < \infty \, .$$

Introduce the measure $\mu := \sigma^2 \delta_0 + \nu$ on R.

Let W^{σ} be the Gaussian part of L. For all $f \in L^2(\mu)$ we put

$$X_t^{(f)} = f(0)\mathrm{W}_t^{\sigma} + \overline{\mathrm{M}}(1_{[0,t]}1_{\mathbb{R}\setminus\{0\}}f), \ t \in [0,T],$$

where the second summand is the stochastic integral w.r.t. the elementary orthogonal random measure $\overline{\mathrm{M}}$.

Properties:

- If $f = 1_{\{0\}}$ then $X^{(f)} = \mathrm{W}^{\sigma}$.
- $X^{(f)}$ is a Lévy process and $\mathbb{E}[(X_t^{(f)})^2] = t\,\mu(f^2) < +\infty$.
- $X^{(f)} \in \mathscr{H}_T^2(\mathbb{F}^L)$ and $\langle X^{(f)}, X^{(g)}\rangle_t = t\mu(fg)$, $f, g \in L^2(\mu)$.
- $\Delta X^{(f)} = f(\Delta L)1_{\{\Delta L \neq 0\}}$ a.s. and $X^{(f)}$ is locally bounded if f is bounded.
- $X^{(f)} = 0$ a.s. if and only if $f = 0$ μ-a.e.
- $X^{(f)}$ and $X^{(g)}$ are orthogonal if and only if $f, g \in L^2(\mu)$ are orthogonal.

Consider systems $\mathscr{C}$ of bounded real functions with the following properties:

- $\mathscr{C} \subseteq L^1(\mu) \cap L^2(\mu)$;
- $\mathscr{C}$ is total in $L^2(\mu)$;
- $\mathscr{C}$ is stable under multiplication and $1_{\mathbb{R}\setminus\{0\}} f \in \mathscr{C}$ if $f \in \mathscr{C}$;

Example:

$\mathscr{C} := \{f = c1_{\{0\}} + 1_{(a,b]},\ a, b \in \mathbb{R} : a < b,\ 0 \notin [a, b];\ c \in \mathbb{R}\} \cup \{0\}$

For any system $\mathscr{T} \subseteq L^2(\mu)$, we put $\mathscr{X}_{\mathscr{T}} := \{X^{(f)}, \quad f \in \mathscr{T}\}$.

The next theorem is an intermediate step.

Theorem I

The system $\mathscr{X}_{\mathscr{C}}$ satisfies the conditions (A), (B), (C) of the main theorem and hence possesses the CRP w.r.t. $\mathbb{F}^{\mathscr{X}_{\mathscr{C}}} = \mathbb{F}^L$.

Applications: Lévy Processes

Based on Theorem I, the following criterion can be derived for $\mathscr{X}_{\mathscr{T}}$ to satisfy the CRP.

Theorem II

Suppose $\mathscr{T} \subseteq L^2(\mu)$. Then the family $\mathscr{X}_{\mathscr{T}}$ possesses the CRP w.r.t. $\mathbb{F}^L$ if and only if $\mathscr{T}$ is total in $L^2(\mu)$.

Idea of Proof (Sufficiency):

- Taking $\mathscr{C}$ as above, we get $\mathscr{H}^2_T = \mathscr{J}(\mathscr{X}_{\mathscr{C}}) \subseteq \mathscr{J}(\mathscr{X}_{L^2(\mu)})$.
- From $\mathrm{cl}(\mathrm{span}(\mathscr{T}))_{L^2(\mu)} = L^2(\mu)$ it can be shown that

$$\mathrm{cl}(\mathrm{span}(\mathscr{X}_{\mathscr{T}}))_{\mathscr{H}^2_T} = \mathscr{X}_{L^2(\mu)}$$

which implies $\mathscr{J}(\mathscr{X}_{L^2(\mu)}) = \mathscr{J}(\mathscr{X}_{\mathscr{T}})$.

- Summarizing, it follows $\mathscr{H}^2_T = \mathscr{J}(\mathscr{X}_{\mathscr{T}})$, hence $\mathscr{X}_{\mathscr{T}}$ possesses the CRP. □

Applications: Lévy Processes

For complete orthonormal systems $\mathscr{T} = \{f_n,\ n \geq 1\}$ in $L^2(\mu)$, from Theorem II we get the following orthogonal decomposition:

Theorem III (CRP for Lévy Processes)

Let $\mathscr{T} := \{f_n,\ n \geq 1\}$ be a complete orthonormal system in $L^2(\mu)$. Then the associated family $\mathscr{X}_{\mathscr{T}}$ has the CRP on $L^2(\Omega, \mathscr{F}^L_T, \mathbb{P})$ and the following orthogonal decompositions hold:

$$L^2(\Omega, \mathscr{F}^L, \mathbb{P}) = \mathbb{R} \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\dots,j_n)\in\mathbb{N}^n} \mathscr{J}_{n,T}^{(f_{j_1},\dots,f_{j_n})} \right),$$

$$\mathscr{H}^2_T(\mathbb{F}^L) = \mathbb{R} \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\dots,j_n)\in\mathbb{N}^n} \mathscr{J}_{n}^{(f_{j_1},\dots,f_{j_n})} \right),$$

where $\mathscr{J}_n^{(f_{j_1},\dots,f_{j_n})}$ denotes the closed linear space of n-fold iterated integrals with respect to $(X^{(f_{j_1})}, \dots, X^{(f_{j_n})})$; $n \geq 1$.

Example: Teugels Martingales

Nualart & Schoutens (2000): Let $(L, \mathbb{F}^L)$ be a Lévy process with characteristic triplet (β, σ^2, ν).

Additional Condition: $\exists \varepsilon, \lambda > 0$ with $x \mapsto \mathrm{e}^{\frac{\lambda}{2}|x|} 1_{\{|x|>\varepsilon\}} \in L^2(\nu)$

- $x \mapsto x^n$ belongs to $L^2(\nu)$, $n \geq 1$.
- Set $h_n(x) = \delta_{1,n} 1_{\{0\}}(x) + x^n$, $x \in \mathbb{R}$, $n \geq 1$.
- $\mathscr{T} := \{h_n,\ n \geq 1\}$ is total in $L^2(\mu)$ with $\mu := \sigma^2\delta_0 + \nu$.
- $X^{(h_n)}$ are just the so-called Teugels martingales.
- The family $\mathscr{X}_{\mathscr{T}} := \{X^{(h_n)},\ n \geq 1\}$ possesses the CRP.
- $\mathscr{X}_{\mathscr{T}} := \{X^{(h_n)},\ n \geq 1\}$ is compensated-covariation stable.
- Let $\mathscr{P}$ be the system of polynomials obtained by the Gram–Schmidt orthogonalization of $\mathscr{T}$ in $L^2(\mu)$.
- The associated family $\mathscr{X}_{\mathscr{P}}$ of martingales is the one of *orthogonalized* Teugels martingales: It possesses the CRP on $L^2(\Omega, \mathscr{F}^L_T, \mathbb{P})$, with orthogonal decomposition as above.

Further Examples

Further examples of families $\mathscr{X}_{\mathscr{T}}$ of martingales possessing the CRP can be provided using any complete orthonormal system $\mathscr{T}$ of $L^2(\mu)$.

Useful complete orthonormal systems $\mathscr{T}$ can be constructed:

- On the basis of Hermite polynomials (H_n) if the Lévy measure ν is equivalent to the Lebesgue measure, with density h, setting $\mathscr{T} = \{P_n,\ n \geq 0\}$ with

  $$P_n = g(0)H_n(0)\, 1_{\{0\}} + 1_{\mathbb{R}\setminus\{0\}}\, gH_n$$

  where $g(x) := h(x)^{-1/2} \exp(-x^2/2),\ x \in \mathbb{R}$.
- Transforming Haar functions appropriately for arbitrary Lévy measure ν.

6. Brief Outlook to Future Research

- Extension of the approach to families $\mathscr{X}$ of square integrable martingales X such that X_0 is an arbitrary square integrable r.v. (done).
- Extension of the approach to families $\mathscr{X}$ such that $\langle X, Y \rangle$ are no longer deterministic. Important case: $\langle X, Y \rangle$ is $\mathscr{F}_0^{\mathscr{X}}$-measurable.
- Example: To derive results for Cox Processes (instead of Lévy processes).
- Applications to the completion of Lévy markets.
- Applications to BSDEs driven by Lévy processes (or Cox processes).
- Investigation of different versions of the Malliavin Calculus in dependence of the choice of a "martingale basis", i.e., in case of an underlying Lévy process, in dependence of the choice of a complete orthonormal system $\{f_n : \ n \geq 1\}$ of $L^2(\mu)$.

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THANK YOU FOR YOUR ATTENTION!

The Product Formula

For all $m \geq 2$ and $\alpha_1, \ldots, \alpha_n \in \Lambda$, recursively we define

$$X^{(\alpha_1,\ldots,\alpha_n)} = [X^{(\alpha_1,\ldots,\alpha_{n-1})}, X^{(\alpha_n)}] - \langle X^{(\alpha_1,\ldots,\alpha_{n-1})}, X^{(\alpha_n)} \rangle .$$

Theorem (Product Formula) $\mathscr{X}$ compensated-covariation stable, $m \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_m \in \Lambda$. Then

$$\prod_{i=1}^m X^{(\alpha_i)} = \sum_{i=1}^m \sum_{1 \leq j_1 < \ldots < j_i \leq m} \Big(\prod_{\substack{k=1 \\ k \neq j_1,\ldots,j_i}}^m X_-^{(\alpha_k)} \Big) \cdot X^{(\alpha_{j_1},\ldots,\alpha_{j_i})}$$

$$+ \sum_{p=0}^{m-2} \sum_{i=p+2}^m \sum_{1 \leq j_1 < \ldots < j_i \leq m} \Big\{ \Big(\prod_{\substack{k=1 \\ k \neq j_1,\ldots,j_i}}^m X_-^{(\alpha_k)} \Big) \Big(\prod_{\ell=i-p+1}^{i} \Delta X^{(\alpha_{j_\ell})} \Big) \Big\} \cdot \langle X^{(\alpha_{j_1},\ldots,\alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})} \rangle .$$

Integrals of Products

We introduce the following systems:

$$\mathscr{R} := \{ (\prod_{i=1}^{m} X_{-}^{(\alpha_i)}) \cdot X^{(\alpha)},\ \alpha, \alpha_1, \ldots, \alpha_m \in \Lambda,\ m \geq 0 \} \cup \{1\}\,,$$
$$\mathscr{R}_T := \{ (\prod_{i=1}^{m} X_{-}^{(\alpha_i)}) \cdot X_T^{(\alpha)},\ \alpha, \alpha_1, \ldots, \alpha_m \in \Lambda,\ m \geq 0 \} \cup \{1\}\,.$$

Important Fact:
If $\mathscr{X}$ is compensated-covariation stable then $\mathscr{R} \subseteq \mathscr{J}$.

Idea of Proof of the Main Theorem

- Since $\mathscr{R} \subseteq \mathscr{J}$, to prove the theorem it is sufficient to verify that $\mathscr{R}_T$ is total in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$.
- Without loss of generality we can assume that $\mathscr{X}$ is stable under stopping w.r.t. deterministic stopping times.
- This additional property yields that every monomial (i.e., every element of $\mathscr{K}$) appears on the left hand side of the product formula evaluated at time T.
- Let $\xi \in L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ be orthogonal to $\mathscr{R}_T$. Using the product formula it is shown inductively on the order of the monomials from $\mathscr{K}$ that ξ is orthogonal to $\mathscr{K}$. Indeed, the first part on the right hand side of the product formula is an element from $\mathscr{R}$ and the second part only contains products of lower order (integrated with respect to deterministic processes of finite variation).
- ξ being orthogonal to $\mathscr{K}$, the totality of $\mathscr{K}$ in $L^2(\mathbf{P})$ yields $\xi = 0$ $\mathbf{P}$-a.s., proving the claim. □

Continuous Gaussian Families

The following theorem is a version of P. Lévy's theorem on the characterization of Brownian motion.

Theorem

The following conditions are equivalent:

(i) *The family* $\mathscr{X} = \{X^{(\alpha)} : \alpha \in \Lambda\}$ *is pairwise Gaussian.*

(ii) *For every* $\alpha, \beta \in \Lambda$, $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ *is deterministic.*

If one (and therefore both) of these conditions is satisfied, then $\mathscr{X}$ *is a Gaussian family with expectation zero and covariance structure*

$$\mathrm{Cov}\,(X_t^{(\alpha)}, X_s^{(\beta)}) = \langle X^{(\alpha)}, X^{(\beta)} \rangle_{t \wedge s}, \quad \forall s, t, \alpha, \beta\,.$$

Corollary

Every pairwise Gaussian family of continuous local martingales is a Gaussian family of true martingales.