Model-free Arbitrage and Superhedging in Discrete Time

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Three approaches

We have two extreme cases.

1. We are completely sure about the reference probability measure P.
2. We face complete uncertainty about any probabilistic model and therefore we describe our model independently by any probability: **Model-free approach**
   - Hobson 1998, Brown Hobson Rogers 2001, Davis Hobson 2007, Cox Obloj 2011, Riedel 2011, Acciaio, Beiglböck, Penkner, Schachermayer 2013.

Between cases 1. and 2., there is the possibility to accept that the model could be described in a probabilistic setting, but we cannot assume the knowledge of a specific reference probability measure but at most of a set of priors, which leads to the theory of

3. Quasi-sure Stochastic Analysis: Peng, Touzi, Zhang, Dolinski, Soner, Kardaras, Bouchard, Nutz, Biagini S., Denis, Martini, Bion-Nadal, Cohen...

Model-free Superhedging Duality

Theorem

Let Ω be Polish, $t \in \{0, 1, ..., T < \infty\}$, $g : \Omega \mapsto \mathbb{R}$ be $\mathcal{F}_T$-measurable:

$$\begin{aligned}
& \inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T \geq g \ \mathcal{M}\text{-q.s.}\} \\
= \ & \inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega) \ \forall \omega \in \Omega_*\} \\
= \ & \sup_{Q \in \mathcal{M}_f} E_Q[g] = \sup_{Q \in \mathcal{M}} E_Q[g],
\end{aligned}$$

- $(H \cdot S)_t := \sum_{u=1}^{t} H_u(S_u - S_{u-1}) = \sum_{j=1}^{d} \sum_{u=1}^{t} H_u^j(S_u^j - S_{u-1}^j)$;
- $\mathcal{F}_t := \bigcap_{P \in \mathcal{P}} (\mathcal{F}_t^S \vee \mathcal{N}_t^P)$, with: $\mathcal{N}_t^P := \{N \subseteq A \in \mathcal{F}_t^S \mid P(A) = 0\}$;
- $\mathcal{H} := \{\text{all } \mathbb{F}\text{-predictable proc.}\}$, $\mathbb{F} := \{\mathcal{F}_t\}_t$ Universal Filtration; $\mathcal{P} := \{\text{all probabilities on } (\Omega, \mathcal{B}(\Omega))\}$;

$$\begin{aligned}
\mathcal{M} \ &:= \left\{ Q \in \mathcal{P} \mid S \text{ is an } \mathbb{F}^S\text{-martingale under } Q \right\}; \\
\mathcal{M}_f \ &:= \{Q \in \mathcal{M} \mid Q \text{ has finite support}\};
\end{aligned}$$

- $\Omega_* := \{\omega \in \Omega \mid \exists Q \in \mathcal{M} \text{ s.t. } Q(\omega) > 0\}$.

Remarks: model free setup

- No reference to any a priori assigned probability measure and the notions of $\mathcal{M}$, $\mathcal{H}$ and Ω_* only depend on the measurable space $(\Omega, \mathcal{F})$ and the price process S. In general, the class $\mathcal{M}$ is not dominated.

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- In an example, we show that the initial cost of the cheapest portfolio that dominates a contingent claim g on *every possible path*

$$\inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ such that } x + (H \cdot S)_T(\omega) \geq g(\omega)\ \forall \omega \in \Omega\}$$

can be strictly greater than $\sup_{Q \in \mathcal{M}} E_Q[g]$, unless some artificial assumptions are imposed on g or on the market.
To avoid such restrictions it is crucial to select the correct set of paths (i.e. the set Ω_* of those $\omega \in \Omega$ which are weighted by at least one martingale measure $Q \in \mathcal{M}$ or $Q \in \mathcal{M}_f$).

Remarks: about Omega*

- The family of $\mathcal{M}$-polar sets is:

$$\mathcal{N} := \{N \subseteq A \in \mathcal{F} \mid Q(A) = 0 \ \forall Q \in \mathcal{M}\}$$

- Recall that a property is said to hold *quasi surely* (q.s.) if it holds outside a polar set.
- We show the existence of the **maximal** $\mathcal{M}$-polar set N_*, namely a set $N_* \in \mathcal{N}$ containing any other set $N \in \mathcal{N}$. Moreover

$$\Omega_* = (N_*)^C.$$

- The inequality

$$x + (H \cdot S)_T \geq g \quad \mathcal{M}\text{-q.s.}$$

is therefore equivalent to the inequality

$$x + (H \cdot S)_T(\omega) \geq g(\omega) \ \forall \omega \in \Omega_*,$$

which justifies the first equality in the Theorem.

Remarks: Omega* is an analytic set

- The set Ω_* can be equivalently determined via the set $\mathcal{M}_f$ of **martingale measures with finite support**:

$$\Omega_* := \{\omega \in \Omega \mid \exists Q \in \mathcal{M}_f \text{ s.t. } Q(\omega) > 0\},$$

  a property that turns out to be crucial in several proofs.
- One of the main technical results of the paper is the proof that the set Ω_* is an **analytic set** (it can be written as the nucleous of a Souslin scheme), and so our findings show that the natural setup for studying this problem is $(\Omega, S, \mathbb{F}, \mathcal{H})$, with
  - $\mathbb{F} = \{\mathcal{F}_t\}_t$ the Universal filtration $\mathcal{F}_t := \bigcap_{P \in \mathcal{P}} (\mathcal{F}_t^S \vee \mathcal{N}_t^P)$,
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  - $\mathcal{H} := \{\mathbb{F}\text{-predictable processes}\}$.
- Financial interpretation: Ω_* is the set of points where is not possible to build 1p Arbitrage opportunities.

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- No need to assume $\mathcal{M} \neq \varnothing$ (we shall discuss this when dealing with No Arbitrage)
- No restriction on S, so that it may describe stocks and/or options. However, in the above Theorem the class $\mathcal{H}$ of admissible trading strategies requires dynamic trading in all assets.
- In the theorem below we easily extend this setup to the case of semi-static trading on a finite number of options.

Semi-static superhedging with options

Theorem

Let $g : \Omega \mapsto \mathbb{R}$ *be the* $\mathcal{F}_T$ *measurable claim and* $\phi^j : \Omega \mapsto \mathbb{R}$, $j = 1, ..., k$, *be* $\mathcal{F}_T$ *measurable random variables representing the payoff of* k *given options traded at zero price. Then*

$$\pi_\Phi(g) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g].$$

where

$$\begin{aligned} \Omega_\Phi \quad &: \quad = \{\omega \in \Omega \mid \exists Q \in \mathcal{M}_\Phi \text{ s.t. } Q(\omega) > 0\} \subseteq \Omega_* \\ \mathcal{M}_\Phi \quad &: \quad = \{Q \in \mathcal{M}_f \mid E_Q(\phi^j) = 0 \ \forall j = 1, ..., k\} \subseteq \mathcal{M}_f \end{aligned}$$

$$\pi_\Phi(g) := \inf \left\{ \begin{array}{c} x \in \mathbb{R} \mid \exists (H, h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that} \\ x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g(\omega) \ \forall \omega \in \Omega_\Phi \end{array} \right\}.$$

Literature

- **On classical Superhedging** (a probability P is fixed): El Karoui and Quenez (95); Karatzas (97); ....
- **Model free set up and robust hedging**: Hobson (98), Brown Hobson Rogers (01), Davis Hobson (07), Hobson (09), Hobson (11), Cox Obloj (11), Riedel (11), ...
- **Optimal mass transport**: Beiglböck, Dolinsky, Galichon, Henry-Labordère, Hobson, Hou, Nutz, Obloj, Penker, Rogers, Soner, Spoida, Tan, Touzi...
- **Superhedging with respect to a non dominated class of probability measures** $\mathcal{P}' \subseteq \mathcal{P}$: Bouchard Nutz (13), Biagini S. Bouchard Kardaras Nutz (14)
- **Superhedging via model-free Arbitrage**: Acciaio Beiglböck Penker Schachermayer (13).

Bouchard and Nutz 2013

Superhedging Duality Theorem w.r.to a family $\mathcal{P}' \subseteq \mathcal{P}$. If $g : \Omega \to \mathbb{R}$ is upper semianalytic (Borel measurable) then

$$\inf \left\{ x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s.t. } x + (H \cdot S)_T \geq g \ \mathcal{P}'\text{-q.s.} \right\} = \sup_{Q \in \mathcal{M}(\mathcal{P}')} E_Q[g].$$

where $\mathcal{M}(\mathcal{P}') := \{ Q \in \mathcal{M} \mid \exists P \in \mathcal{P}' \text{ s.t. } Q \ll P \}$.
The theorem is obtained under two technical hypothesis:

- $\Omega = \Omega_1^T$, where Ω_1 is Polish and Ω_1^t is the t-fold product space;
- The set of priors $\mathcal{P}'$ have the form $P := P_0 \otimes \ldots \otimes P_T$ where every P_t is a measurable selector of a certain random class $\mathcal{P}'_t \subseteq \mathcal{P}(\Omega_1)$. $\mathcal{P}'_t(\omega)$ is the set of possible models, given state ω at time t and the graph$(\mathcal{P}'_t)$ **must be an analytic subset of** $\Omega_1^t \times \mathcal{P}(\Omega_1)$.

In our setting we do not impose restrictions on the state space Ω and even in the case of $\Omega = \Omega_1^T$, the class of martingale probability measures $\mathcal{M}$ is **endogenously determined by the market** and we do not require that it satisfies any additional restrictions.

Acciao et al. 2013 Superhedging on the whole Omega

Same discrete time market as ours, but S is a one dimensional canonical process on the path space $\Omega = [0, \infty)^T$.

Theorem

Assume No Model Independent Arbitrage. Let $\phi^j = f^j(S_T)$*, with* $f^j : \mathbb{R}_+ \mapsto \mathbb{R}$*,* $j = 0, 1, \dots$ *be the payoff of options traded at zero price and let* f^0 *be convex and* **super linear***. If* $g = f(S_T)$ *with* $f : \mathbb{R}_+ \mapsto \mathbb{R}$ **upper semicountinuous** *then:*

$$\widehat{\pi}(g) = \sup_{Q \in \mathcal{M}_\Phi} E_Q[g] \qquad \mathcal{M}_\Phi := \{Q \in \mathcal{M} \mid E_Q(\phi^j) = 0 \ \forall j\}$$

$$\widehat{\pi}(g) \triangleq \inf \left\{ \begin{array}{c} x \in \mathbb{R} \mid \exists (H, h) \in \mathcal{H} \times \mathbb{R}^k \text{ such that} \\ x + (H \cdot S)_T(\omega) + h\Phi(\omega) \geq g(\omega) \ \forall \omega \in \Omega \end{array} \right\}$$

- Superhedging on Ω, but restrictions on g and on the market.
- **Example** where this duality doesn't hold if f **is not usc**, a property with no financial meaning.

On aggregation

To prove our theorem we need the following aggregation results:

Proposition

Let $g : \Omega \mapsto \mathbb{R}$ *be* $\mathcal{F}_T$ *measurable and define*

$$\begin{aligned} \pi(g) &\triangleq \inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s.t. } x + (H \cdot S)_T(\omega) \geq g(\omega) \ \forall \omega \in \Omega_* \\ \pi_Q(g) &\triangleq \inf \{x \in \mathbb{R} \mid \exists H \in \mathcal{H} \text{ s.t. } x + (H \cdot S)_T \geq g \quad Q\text{- a.s. } \} . \end{aligned}$$

Then

$$\pi(g) = \sup_{Q \in \mathcal{M}_f} \pi_Q(g)$$

In particular, if $\pi(g) < +\infty$ *the* inf *is a* min.

On aggregation

Corollary (Super replication)

If for every $Q \in \mathcal{M}_f$ *there exists* $H^Q \in \mathcal{H}$ *such that*

$$x + (H^Q \cdot S)_T \geq g \quad Q\text{-a.s},$$

then there exists $H \in \mathcal{H}$ *such that*

$$x + (H \cdot S)_T(\omega) \geq g(\omega) \text{ for every } \omega \in \Omega_*.$$

Corollary (Replication)

If for every $Q \in \mathcal{M}_f$ *there exists* $H^Q \in \mathcal{H}, x^Q \in \mathbb{R}$ *such that*

$$x^Q + (H^Q \cdot S)_T = g \quad Q - a.s.$$

then there exists $H \in \mathcal{H}, x \in \mathbb{R}$ *such that*

$$x + (H \cdot S)_T(\omega) = g(\omega) \text{ for every } \omega \in \Omega_*,$$

Results based on BFM2014

The results presented so far can be found in:

Burzoni M., Frittelli M., Maggis M., Model-free Superhedging Duality, 2015.

and are based on the theory about Model-free arbitrage in discrete time developed in:

Burzoni M., Frittelli M., Maggis M., Universal Arbitrage Aggregator in Discrete Time Markets under Uncertainty, *Fin. Stoch.*, forthcoming.

which we now illustrate.

On Model-free Arbitrage in Discrete Time

- In a model independent discrete time financial market, we discuss the richness of the family of martingale measures in relation to different notions of Arbitrage, generated by a class of **significant** sets $\mathcal{S}$, which we call **Arbitrage de la classe** $\mathcal{S}$.

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- The choice of $\mathcal{S}$ reflects into the intrinsic properties of the class of polar sets of martingale measures.
- In particular for
  - $\mathcal{S} = \{\Omega\}$, absence of Model Independent Arbitrage $\Longleftrightarrow \mathcal{M} \neq \varnothing$
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- We provide the dual representation of "Open Arbitrage" in terms of weakly open sets of probability measures, which highlights the robust nature of such concept.
- These results are obtained by adopting a technical filtration enlargement and by constructing a universal arbitrage aggregator of all arbitrage opportunities.
- We introduce the notion of **market feasibility** and characterize it.

Our setting

We consider the market model as initially described:

$$(\Omega, I, S, \mathbb{F}, \mathcal{H})$$

and we assume that Ω is a Polish space, $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel sigma algebra, $I = \{0, ..., T\}$, S is a d-dimensional stochastic process, $\mathbb{F} = \mathbb{F}^S$, the trading strategies $H \in \mathcal{H}$ are $\mathbb{R}^d$-valued $\mathbb{F}$-predictable stoc. proc.

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Set:

$$\begin{aligned} \mathcal{P}_+ &: = \{Q \in \mathcal{P} \mid Q \text{ has full support}\} \\ \mathcal{M} &: = \{Q \in \mathcal{P} \mid S \text{ is a martingale under } Q\} \\ \mathcal{M}_+ &: = \mathcal{M} \cap \mathcal{P}_+, \end{aligned}$$

$\mathcal{M}_+$ is the set of *martingale probability measures with full support.*
No reference probability measure is required.

Arbitrage de la Classe S

Let:

$$\mathcal{V}_H^+ := \{\omega \in \Omega \mid \ V_T(H)(\omega) > 0\}.$$

Definition

Let $\mathcal{S}$ be a class of measurable subsets of Ω such that $\varnothing \notin \mathcal{S}$. A trading strategy $H \in \mathcal{H}$ is an Arbitrage de la classe $\mathcal{S}$ if $V_0(H) = 0$ and

- $V_T(H)(\omega) \geq 0 \ \forall \omega \in \Omega$ and $\mathcal{V}_H^+$ contains a set de la classe $\mathcal{S}$.

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- The class $\mathcal{S}$ has the role to translate mathematically the meaning of a "true gain": there is a "true gain" if the set $\mathcal{V}_H^+$ contains a set considered **significant**.
- When a "reference Probability" P is given, then a true gain is: $P(V_T(H) > 0) > 0$
- When a subset $\mathcal{P}'$ of probability measures is given, one replace the P-a.s. conditions with $\mathcal{P}'$-q.s conditions, as in [BN13].

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- H is a $\mathcal{P}'$-q.s. Arbitrage when $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0 \text{ for some } P \in \mathcal{P}'\}$, for a fixed family $\mathcal{P}' \subseteq \mathcal{P}$.
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- H is a P-a.s. Arbitrage when $\mathcal{S} = \{C \in \mathcal{F} \mid P(C) > 0\}$ for fixed $P \in \mathcal{P}$.
- H is a **model independent Arbitrage** when when $\mathcal{S} = \{\Omega\}$

Obviously,

$$\text{No 1p A} \Rightarrow \text{No A de la Classe } \mathcal{S} \ \Rightarrow \text{No Model Ind. A}$$

and A de la Classe $\mathcal{S}$ are not necessarily related to a probabilistic model.

Technical Filtration

In order to show in which context the **FTAP** holds true we need to build up a filtration enlargement $\widetilde{\mathbb{F}} := (\widetilde{\mathcal{F}}_t)_{t\in I}$ which follows directly from the market structure and

- It preserves the sets of martingale measures $\mathcal{M}(\widetilde{\mathbb{F}}) \rightleftarrows \mathcal{M}(\mathbb{F})$
  - The restriction of any $\widetilde{Q} \in \mathcal{M}(\widetilde{\mathbb{F}})$ to $\mathcal{F}_T$ belongs to $\mathcal{M}(\mathbb{F})$.
  - Any $Q \in \mathcal{M}(\mathbb{F})$ can be uniquely extended to an element of $\mathcal{M}(\widetilde{\mathbb{F}})$.
- It ensure the existence of an $\widetilde{\mathbb{F}}$-predictable multifunction $\mathbb{H}_t$ which is an aggregator for all P-a.s. arbitrage opportunities.
- We provide many examples where there is no equivalence between "No Arbitrage" and existence of martingale measures with reasonable properties, without the enlargement of the filtration.
- $\widetilde{\mathbb{F}}$ is built from $\mathbb{F}$ by "adding" one step ahead $\mathcal{M}$-polar sets.

Some Quotes

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Consider two call options C_1, C_2 on the same underlying asset with same initial price $p_1 = p_2$ but strikes $K_2 > K_1$.

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Strategy A: $C_1 - C_2$ $\quad V_T(H) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ > 0 & \text{if } S_T > K_1 \end{cases}$

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Strategy B: $-C_1 \qquad V_T(H) = p_1 > 0 \qquad \text{if } S_T \leq K_1$

which happens with probability one.

Main Result

Theorem (Model Free FTAP)

Let $(\Omega, \widetilde{\mathcal{F}}_T, (\widetilde{\mathcal{F}}_t)_{t\in I})$ *be the enlarged filtered space. Let* $\mathcal{N}$ *the family of polar sets of* $\mathcal{M}$. *Then*

No Arbitrage de la classe $\mathcal{S}$ *in* $\widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{N}$ *does not contain sets of* $\mathcal{S}$

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$$\text{No Arbitrage de la classe } \mathcal{S} \text{ in } \widetilde{\mathcal{H}} \iff \mathcal{N} \text{ does not contain sets of } \mathcal{S}$$

In particular for $\mathcal{S} = \{\text{open non-empty sets}\}$ we obtain:

Corollary (FTAP for Open Arbitrage)

Let $(\Omega, \widetilde{\mathcal{F}}_T, (\widetilde{\mathcal{F}}_t)_{t\in I})$ *as before.*

$$\begin{aligned} \text{No Open Arbitrage in } \widetilde{\mathcal{H}} &\iff \text{there are no open } \mathcal{M}\text{-polar sets} \\ &\iff \mathcal{M}_+ \neq \varnothing \end{aligned}$$

where $\mathcal{M}_+$ *is the class of martingale measure with full support.*

On full support martingale measures

- Riedel 2011, pointed out the relevance of the concept of full support martingale measures in the model-free setting:
- He proved, in **one-period model** and under the assumption that the **price process is continuous** with respect to the state variable ω, that:

$$\text{No 1p arbitrage} \iff \mathcal{M}_+ \neq \varnothing$$

- In such setting, from the continuity of S one also deduces that

$$\text{No open arbitrage in } \mathcal{H} \iff \text{ No 1p arbitrage} \iff \mathcal{M}_+ \neq \varnothing$$

- However, both equivalences are no longer true in the multi-period setting (even with S continuous in ω).
- To recover the equivalence between No open arbitrage and $\mathcal{M}_+ \neq \varnothing$ one needs the filtration enlargement.

For $\mathcal{S} = \{\Omega\}$ we obtain:

Corollary (Model Independent FTAP)

Let $(\Omega, \widetilde{\mathcal{F}}_T, (\widetilde{\mathcal{F}}_t)_{t\in I})$ *as before:*

$$\text{No Model Independent Arbitrage in } \widetilde{\mathcal{H}} \Longleftrightarrow \mathcal{M} \neq \varnothing$$

Technical aspects

For any trajectory $z \in Mat(d, T+1)$, study the level set of the price process S i.e.

$$\Sigma_{t-1}^z := \{\omega \in \Omega \mid S_{0:t-1}(\omega) = z_{0:t-1}\} \in \mathcal{F}_{t-1}^S$$

If $0 \notin ri(co(conv(\Delta S_t(\Sigma_{t-1}^z))) \cup \{0\})$ we may decompose the level set as $\Sigma_{t-1}^z = B^* \cup \bigcup_{i=1}^{\beta} B^i$.

The difference between the sets B^i and B^*:

- Restricted to the time interval $[t-1, t]$, a probability measure whose mass is concentrated on B^* admits an equivalent martingale measure
- For those probabilities that assign positive mass to at least one B^i an arbitrage opportunity can be constructed.
  - $H^i \cdot \Delta S_t(\omega) > 0 \quad \forall \omega \in B^i$
  - $H^i \cdot \Delta S_t(\omega) \geq 0 \quad \forall \omega \in \cup_{j=i}^{\beta} B^j$

Remark: β is finite and $\beta \leq d$

Two technical results on Omega* and polar sets

1. $$\Omega_* = \bigcap_{t=1}^{T} \left(\bigcup_{z \in \mathbf{Z}} B^*_{t,z} \right).$$

   and

   $$\mathcal{M} \neq \varnothing \Longleftrightarrow \Omega_* \neq \varnothing \Longleftrightarrow \mathcal{M} \cap \mathcal{P}_f \neq \varnothing,$$

2. Fix $t \in I$ and $Q \in \mathcal{M}$. The set

   $$\mathfrak{B}_t := \bigcup_{z \in \mathbf{Z}} \left\{ \bigcup_{i=1}^{\beta_{t,z}} B^i_{t,z} \right\}$$

   is a subset of a $\mathcal{F}_t$-measurable Q null set.

Example of separation of atoms

Let $(\Omega, \mathcal{F}) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Consider the market $S_0 = [1, 1, 1]$ and $S_t = [S_t^1, S_t^2, S_t^3]$ with

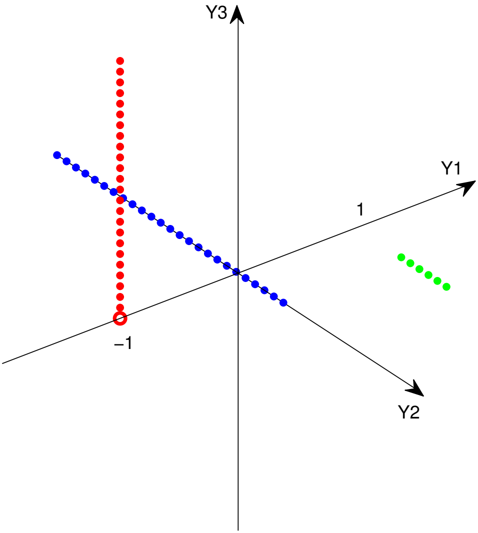
$$S_1^1(\omega) = \begin{cases} 0 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q} \cap (1/2, +\infty) \\ 2 & \omega \in \mathbb{Q} \cap (0, 1/2) \end{cases} \quad S_1^2(\omega) = \begin{cases} 1 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ \omega^2 & \omega \in \mathbb{Q} \cap (1/2, +\infty) \\ \omega^2 & \omega \in \mathbb{Q} \cap (0, 1/2) \end{cases}$$

$$S_1^3(\omega) = \begin{cases} 1 + \omega^2 & \omega \in \mathbb{R}^+ \setminus \mathbb{Q} \\ 1 & \omega \in \mathbb{Q} \cap (1/2, +\infty) \\ 1 & \omega \in \mathbb{Q} \cap (0, 1/2) \end{cases}$$

Example of separation of atoms

Set $Y_i := S_1^i - S_0^i$

In this example:
$B^1 = \mathbb{R}^+ \setminus \mathbb{Q}$
$B^2 = \mathbb{Q} \cap (0, 1/2)$
$B^* = \mathbb{Q} \cap (1/2, +\infty)$



Universal Arbitrage Aggregation

Let $B^*, B^i,\ i = 1, \ldots, \beta$ the disintegration of the level set Σ^z_{t-1}.

$$\mathbb{H}_t(\omega) := \left\{ H \in \mathbb{R}^d \mid H \cdot \Delta S_t(\widehat{\omega}) \geq 0 \text{ for any } \widehat{\omega} \in \cup_{j=i(\omega)}^{\beta} B^j \right\}$$

Theorem

If P is not absolutely continuous w.r.to some $Q \in \mathcal{M}$ then there exists an $\mathbb{F}^P$-predictable trading strategy H^P which is a P-Classical Arbitrage and

$$H^P(\omega) \in \mathbb{H}(\omega) \qquad P\text{-a.s.}$$

where $\mathcal{F}^P_t$ denote the P-completion of $\mathcal{F}_t$ and $\mathbb{F}^P := \{\mathcal{F}^P_t\}_{t \in I}$.

Robustness of Open Arbitrage

Proposition

H is an Open Arbitrage iff there exists a non empty $\sigma(\mathcal{P}, C_b)$-open set $\mathcal{U} \subseteq \mathcal{P}$ such that

$$V_T(H) \geq 0 \ P\text{-a.s.} \ \forall P \in \mathcal{U} \quad \textit{and} \quad P(V_T(H) > 0) > 0 \ \forall P \in \mathcal{U}. \qquad (1)$$

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- No a priori fixed class of probability measures
- If $(H, \mathcal{U})$ satisfy (1) and we disregard any finite subset of probabilities then H still satisfy (1).
- If $(H, \mathcal{U})$ satisfy (1), $\mathcal{U}$ will contain a full support probability P under which H is a P-Arbitrage.
- If the $\sigma(\mathcal{P}, C_b)$ topology is replaced by the one induced by $\| \cdot \|$ we obtain the equivalence for 1p-Arbitrage.

Market Feasibility

- How large is the set of probability measures such that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfy No Classical Arbitrage (w.r.to P) ?

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$$\mathcal{P}_0 = \{P \in \mathcal{P} \mid NA(P) \text{ holds}\} = \{P \in \mathcal{P} \mid \mathcal{M}^e(P) \neq \varnothing\}$$

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We introduce the following:

Definition

The market is **feasible** if

$$\overline{\mathcal{P}_0}^{\sigma(\mathcal{P}, C_b)} = \mathcal{P}.$$

Characterization of feasibility

We can relate the notion of feasibility and No arbitrage as follows:

Theorem

Let $(\Omega, \widetilde{\mathcal{F}}_T, (\widetilde{\mathcal{F}}_t)_{t\in I})$ *be the enlarged filtered space. TFAE:*

1. $\mathcal{M}_+ \neq \varnothing$;
2. *No Open Arbitrage holds w.r.to the strategies in* $\widetilde{\mathcal{H}}$.
3. *The market is feasible:* $\overline{\mathcal{P}_0}^{\sigma(\mathcal{P}, C_b)} = \mathcal{P}$

Thank you for the attention!