An Analytical Approximation for Pricing VWAP Options

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This Talk is Based on

1. Funahashi, H. and Kijima, M. (2015c), "An analytical approximation for pricing VWAP options," Working Paper.
2. Funahashi, H. and Kijima, M. (2015b), "A unified approach for the pricing of options related to averages," Working Paper.
3. Funahashi, H. and Kijima, M. (2014), "An extension of the chaos expansion approximation for the pricing of exotic basket options," *Applied Mathematical Finance*, **21** (**2**), 109–139.
4. Funahashi, H. and Kijima, M. (2015a), "A chaos expansion approach for the pricing of contingent claims," *Journal of Computational Finance*, **18** (**3**), 27–58.

Formulation

- Let S_t and v_t be the time-t price and trading volume, respectively, of the underlying asset.
- VWAP (volume weighted average price) is determined by

$$M_T = \frac{\int_0^T v_t S_t \mathrm{d}t}{\int_0^T v_t \mathrm{d}t},$$

  where M_T is called the VWAP of the time interval $[0, T]$.
- The standard definition of a continuous VWAP call option is given by

$$\mathrm{VC}(S, v, K, T) = \mathrm{e}^{-rT}\mathbb{E}[(M_T - K)^+]$$

  where $S = S_0$ is the initial price, $v = v_0$ is the initial trading volume, K is a strike, T is a maturity, and r is the short rate, $\mathbb{E}$ is the risk-neutral expectation operator.
- Existing papers try to approximate the distribution of M_T directly.

Motivation

- VWAP Options are becoming increasingly popular in options markets, since they can help corporate firms hedge risks arising from market disruption when entering large buy or sell orders.
- Their prices assign more weight to periods of high trading than to periods of low trading in its calculation.
- Hence, VWAP options differ conceptually from Asian options because the resulting payoff is not a linear combination of underlying prices.
- As a result, the pricing of VWAP options is significantly more difficult than Asians and few pricing models have been proposed in the literature, despite their popularity in practice.
- See, e.g., Buryak and Guo (2014), Novikov et al. (2013) for details.

Literature Review

- It is common to model the underlying S_t by using the geometric Brownian motion (GBM) for simplicity.
- For the trading volume v_t, Stace (2007) proposes a mean-reverting process; Novikov et al. (2013) use a squared Ornstein–Uhlenbeck (OU) process; and Buryak and Guo (2014) suggest a simple gamma process, respectively, for the trading volume process.
- Under the GBM assumption, these papers produce approximated pricing formulas by utilizing the moment-matching technique for M_T.
- On the other hand, Novikov and Kordzakhia (2013) derive very tight upper and lower bounds for the price of VWAP options.

In This Talk

- Other than the GBM case, these approaches seem difficult to apply for deriving an approximation formula of VWAP options.
- Funahashi and Kijima (2015b) apply the chaos expansion technique to derive a unified approximation method for pricing any type of Asian options when the underling process follows a diffusion.
- In this talk, not of the VWAP itself, but we try to approximate the distribution of $\widetilde{M}_T$,

$$\widetilde{M}_T(x) = \int_0^T v_t S_t \mathrm{d}t - x \int_0^T v_t \mathrm{d}t,$$

  when the underling asset price and trading volume processes follow a local volatility model and a mean-reverting model, respectively.

The Setup

- We assume that the price S_t and the trading volume v_t of the underlying asset are modeled by the following SDE:

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} & = & r(t)\mathrm{d}t + \sigma(S_t, t)\mathrm{d}W_t \\ \mathrm{d}v_t & = & (\theta(t) - \kappa(t)v_t)\mathrm{d}t + \gamma(v_t)\mathrm{d}W_t^v \end{cases}$$

  under the risk-neutral measure $\mathbb{Q}$, where $\{W_t\}$ and $\{W_t^v\}$ are the standard Brownian motions with correlation $\mathrm{d}W_t\mathrm{d}W_t^v = \rho\mathrm{d}t$.
- The volatility functions $\sigma(S, t)$ and $\gamma(v)$ are sufficiently smooth with respect to (S, t) and v, respectively.
- $r(t)$, $\theta(t)$ and $\kappa(t)$ are some deterministic functions of time t.

Key Observation

- Denote the cumulative distribution function (CDF) of M_T by $F_M(x) = \mathbb{Q}\{M_T \le x\}$, $x > 0$.
- The VWAP call price can be written as

$$\mathrm{VC}(S, v, K, T) = \mathrm{e}^{-rT} \int_K^\infty (1 - F_M(x)) \mathrm{d}x$$

- For each $x > 0$, let $F_{\widetilde{M},x}(y)$ be the CDF of the random variable

$$\widetilde{M}_T(x) = \int_0^T v_t S_t \mathrm{d}t - x \int_0^T v_t \mathrm{d}t$$

- It follows from the definition of VWAP that

$$F_M(x) = F_{\widetilde{M},x}(0), \qquad x > 0$$

- Therefore, it suffices to know the CDF $F_{\widetilde{M},x}(y)$ at $y = 0$.
- To this end, we apply the chaos expansion approach to approximate the distribution of the random variable $\widetilde{M}_T(x)$.

Approximation 1

Employing the same idea as in Theorem 3.1 of Funahashi and Kijima (2015a), S_t is approximated by the following formula:

Lemma

Let $F(0,t) = S\mathrm{e}^{\int_0^t r(u)\mathrm{d}u}$ *be the forward price of the underlying asset with delivery date* t*. Then,*

$$\begin{aligned} S_t \approx\; & F(0,t)\Big[1 + \int_0^t p_1(s)\mathrm{d}W_s + \int_0^t p_2(s)\left(\int_0^s \sigma_0(u)\mathrm{d}W_u\right)\mathrm{d}W_s \\ & + \int_0^t p_3(s)\left(\int_0^s \sigma_0(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s \\ & + \int_0^t p_4(s)\left(\int_0^s p_5(u)\left(\int_0^u \sigma_0(r)\mathrm{d}W_r\right)\mathrm{d}W_u\right)\mathrm{d}W_s\Big] \end{aligned}$$

Approximation 1, Continued

where

$$
\begin{aligned}
p_1(s) &:= \sigma_0(s) + F(0,s)\sigma_0'(s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right) \\
&\qquad + \frac{1}{2}F^2(0,s)\sigma_0''(s)\left(\int_0^s \sigma_0^2(u)\mathrm{d}u\right) \\
p_2(s) &:= \sigma_0(s) + F(0,s)\sigma_0'(s) \\
p_3(s) &:= \sigma_0(s) + 3F(0,s)\sigma_0'(s) + F^2(0,s)\sigma_0''(s) \\
p_4(s) &:= \sigma_0(s) + F(0,s)\sigma_0'(s) \\
p_5(s) &:= F(0,s)\sigma_0'(s)
\end{aligned}
$$

with $\sigma_0'(t) := \partial_x \sigma(x,t)|_{x=F(0,t)}$ and $\sigma_0''(t) := \partial_{xx} \sigma(x,t)|_{x=F(0,t)}$

Approximation 2

Employing the successive substitution used in Funahashi (2014), we obtain the following result. Let $E(0,t) = \mathrm{e}^{\int_0^t \kappa(u)\mathrm{d}u}$, $\bar{E}(t) = 1/E(t)$, and

$$V_t = \bar{E}(t)\left(v_0 + \int_0^t E(s)\theta(s)\mathrm{d}s\right)$$

Lemma

The trading volume v_t is approximated as $v_t \approx$

$$V(0,t) + \bar{E}(t)\int_0^t p_6(s)\mathrm{d}W_s^v + \bar{E}(t)\int_0^t \gamma_0'(s)\left(\int_0^s E(u)\gamma_0(u)\mathrm{d}W_u^v\right)\mathrm{d}W_s^v$$
$$+\bar{E}(t)\int_0^t \bar{E}(s)\gamma_0''(s)\left(\int_0^s E(u)\gamma_0(u)\left(\int_0^u E(r)\gamma_0(r)\mathrm{d}W_r^v\right)\mathrm{d}W_u^v\right)\mathrm{d}W_s^v$$
$$+\bar{E}(t)\int_0^t \gamma_0'(s)\left(\int_0^s \gamma_0'(u)\left(\int_0^u E(r)\gamma_0(r)\mathrm{d}W_r^v\right)\mathrm{d}W_u^v\right)\mathrm{d}W_s^v,$$

where $\gamma_0(t) := \gamma(V_t)$, $\gamma_0'(t) := \partial_x\gamma(x)|_{x=V_t}$, $\gamma_0''(t) := \partial_{xx}\gamma(x)|_{x=V_t}$, *and*

$$p_6(t) := E(t)\gamma_0(t) + \frac{1}{2}\bar{E}(t)\gamma_0''(t)\left(\int_0^t E^2(s)\gamma_0^2(s)\mathrm{d}s\right)$$

Approximation 3

Using the approximation results for S_t and v_t, we obtain the following.

Lemma

The traded value $v_t S_t$ is approximated as

$$v_t S_t \approx I_0(t) + I_1(t) + I_2(t) + I_3(t),$$

where $I_0(t) = V_t F(0,t) + \rho F(0,t)\bar{E}(t)\int_0^t p_6(s)p_1(s)\mathrm{d}s$,

$$\begin{aligned} I_1(t) &= V(0,t)F(0,t)\int_0^t p_1(s)\mathrm{d}W_s + F(0,t)\bar{E}(t)\int_0^t p_{10}(t,s)\mathrm{d}W_s \\ &\quad + F(0,t)\bar{E}(t)\int_0^t \left(p_6(s) + p_9(t,s)\right)\mathrm{d}W_s^v, \end{aligned}$$

and others (omitted).

Approximation 4

By changing the order of integration, $\widetilde{M}_T$ can be approximated by a truncated sum of iterated Itô stochastic integrals as follows.

Lemma

For each $x > 0$*, the random variable* $\widetilde{M}_T$ *can be approximated as*

$$\begin{aligned}\widetilde{M}_T &= \int_0^T v_t S_t \mathrm{d}t - x \int_0^T v_t \mathrm{d}t \\ &\approx J_0(x,T) + J_1(x,T) + J_2(x,T) + J_3(x,T),\end{aligned}$$

where

$$\begin{aligned}J_0(x,T) &= \int_0^T V_t \left(F(0,t) - x\right) \mathrm{d}t \\ &+ \rho \int_0^T F(0,t)\bar{E}(t) \left(\int_0^t p_6(s) p_1(s) \mathrm{d}s\right) \mathrm{d}t,\end{aligned}$$

Approximation 4, Continued

$$
\begin{aligned}
J_1(x,T) &= \int_0^T p_1(t)\left(\int_t^T V(0,s)F(0,s)\mathrm{d}s\right)\mathrm{d}W_t \\
&+ \int_0^T \left(\int_t^T p_{10}(s,t)F(0,s)\bar{E}(s)\mathrm{d}s\right)\mathrm{d}W_t^v \\
&+ \int_0^T \left(\int_t^T \left(p_6(t)+p_9(s,t)F(0,s)\bar{E}(s)\right)\mathrm{d}s\right)\mathrm{d}W_t^v \\
&- x\int_0^T p_6(t)\left(\int_t^T \bar{E}(s)\mathrm{d}s\right)\mathrm{d}W_t^v,
\end{aligned}
$$

$$
\begin{aligned}
J_2(x,T) &= \int_0^T r_1(t)\left(\int_0^t \sigma_0(s)\mathrm{d}W_s\right)\mathrm{d}W_t + \int_0^T r_2(t)\left(\int_0^t p_6(s)\mathrm{d}W_s^v\right)\mathrm{d}W_t \\
&+ \int_0^T r_3(t)\left(\int_0^t p_1(s)\mathrm{d}W_s\right)\mathrm{d}W_t^v + \int_0^T r_4(x,t)\left(\int_0^t E(s)\gamma_0(s)\mathrm{d}W_s^v\right)\mathrm{d}W_t^v,
\end{aligned}
$$

and others (omitted).

Option Pricing Formula

- Let us define $Y_t = \widetilde{M}_t - J_0(x,t)$, and denote its probability density function (PDF) by $f_{Y_T,x}(y)$.
- We can obtain the PDF by applying the following lemma.

Lemma

The PDF of Y_T is approximated as

$$
\begin{aligned}
f_{Y_T,x}(y) \quad \approx \quad & n\left(y;0,V_x(T)\right) - \frac{\partial}{\partial y}\left\{\mathbb{E}[J_2(x,T)|J_1(x,t)=y]n\left(y;0,V_x(T)\right)\right\} \\
& - \frac{\partial}{\partial y}\left\{\mathbb{E}[J_3(x,t)|J_1(x,t)=y]n\left(y;0,V_x(T)\right)\right\} \\
& + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left\{\mathbb{E}[J_2(x,t)^2|J_1(x,t)=y]n\left(y;0,V_x(T)\right)\right\},
\end{aligned}
$$

where $n(y;a,b)$ denotes the normal density with mean a and variance b.

Option Pricing Formula, Continued

- The conditional expectations can be evaluated explicitly.
- Using the approximated density function of $f_{Y_T,x}(y)$, we can approximate the CDF $F_{Y_T,x}(y)$ of Y_T.
- But, from the relation $F_{\widetilde{M},x}(y) = F_{Y_T,x}(y - J_0(x,T))$, we have

$$F_M(x) = F_{\widetilde{M},x}(0) = F_{Y_T,x}(-J_0(x,T))$$

- It follows that the VWAP call option price can be approximated as

$$\mathrm{VC}(S,v,K,T) \;\approx\; \mathrm{e}^{-rT}\int_K^\infty (1 - F_{Y_T,x}(-J_0(x,T)))\mathrm{d}x$$

Numerical Examples: CEV Case

- We suppose that the volatilities are specified as

$$\boldsymbol{\sigma}(S,t) = \boldsymbol{\sigma} S^{\boldsymbol{\beta}-1}, \quad \gamma(v) = \boldsymbol{\nu}\, v^{\boldsymbol{\lambda}-1},$$

  where $\boldsymbol{\sigma}$, $\boldsymbol{\beta}$, $\boldsymbol{\nu}$, and $\boldsymbol{\lambda}$ are some constants.

- The base-case parameters are set to be $S = 100$, $K = 100$, $T = 1$, $r(t) = 3.0\%$, $v = 100$, and $\rho = 0.3$.
- Also, we set $\theta(t) = 10$ and $\kappa(t) = 0.1$, i.e., the long-run average of the trading volume is 100.
- As to the volatilities, we consider (H) high and (L) low volatility cases in which we set $\boldsymbol{\sigma} S^{\boldsymbol{\beta}-1} = 30\%$ and $\boldsymbol{\nu}\, v^{\boldsymbol{\lambda}-1} = 30\%$ for case (H) and $\boldsymbol{\sigma} S^{\boldsymbol{\beta}-1} = 15\%$ and $\boldsymbol{\nu}\, v^{\boldsymbol{\lambda}-1} = 15\%$ for case (L), respectively.

Numerical Examples; Accuracy Check

- We consider two cases; (1) log-normal case ($\boldsymbol{\beta = 1}$ and $\boldsymbol{\lambda = 1}$), and (2) square-root case ($\boldsymbol{\beta = 0.5}$ and $\boldsymbol{\lambda = 0.5}$).
- Figure 1 shows option prices for (L) with short maturity ($\boldsymbol{T = 0.5}$) when (1), whereas Figure 2 depicts for (2).
- Through the numerical experiments, it is observed that the effect of volatility and maturity appears only around ATM ($\boldsymbol{K = 100}$), and the volatility effect is stronger than the maturity effect.
- As to the accuracy of our approximation, we find that the difference between our approximation and the Monte Carlo result are very small.
- The error becomes slightly larger for long maturity and high volatility cases; however, for practical uses, the errors are sufficiently small.

Figure 1 (GBM, low vol, $T = 0.5$)

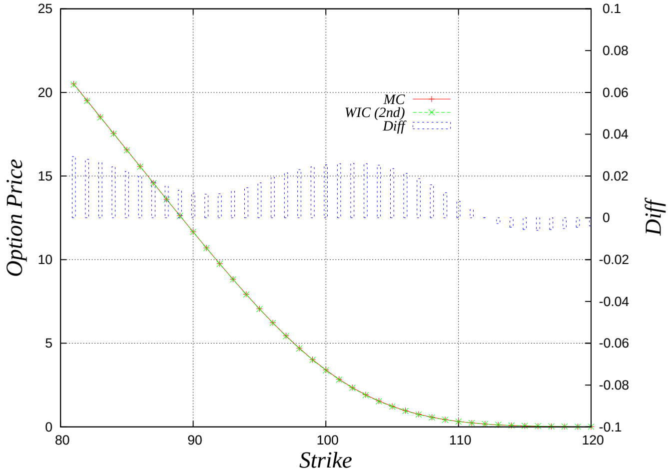
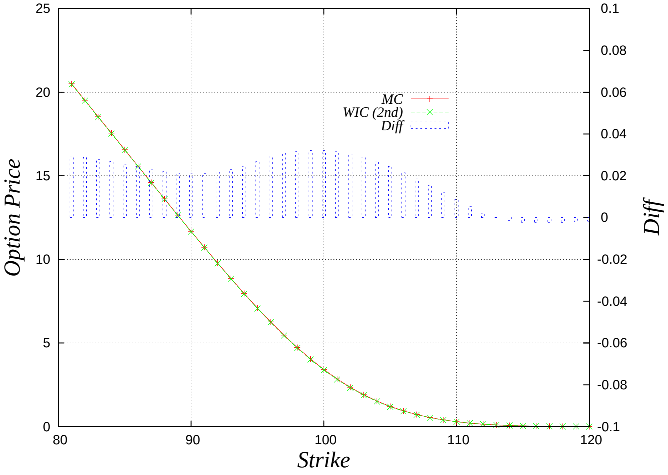


Figure 2 (Square-Root, low vol, $T = 0.5$)



Other Findings; Effect of Correlation

- The effect of correlation gets bigger as κ, the speed of mean reversion, becomes smaller.
- When κ is large, the trading volume v_t sticks around the long-run average so as to behave as if it were uncorrelated to the stock price.
- Stace (2007) sets $\kappa = 100$ under the assumption $\rho = 0$.
- Our result suggests that, when $\kappa = 100$, the impact of correlation on the VWAP call option prices is negligible.
- This result may be an important message for practitioners, because it is in general very difficult to estimate the correlation accurately.
- The effect of correlation gets bigger as the maturity T becomes longer and the volatility σ of the asset price becomes larger.
- These results can be understood by the fact that the effect of correlation is bigger as more uncertainty is involved.

Other Findings; Effect of Model Choice

Recall that

$$\boldsymbol{\sigma}(S,t) = \boldsymbol{\sigma} S^{\boldsymbol{\beta}-1}, \quad \gamma(v) = \boldsymbol{\nu}\, v^{\boldsymbol{\lambda}-1}$$

- The effect of $\boldsymbol{\beta}$ gets bigger as $\boldsymbol{\kappa}$ becomes smaller, the maturity becomes longer and the asset volatility becomes larger.
- These results can be explained by the exactly same reason as above.
- The effect of $\boldsymbol{\lambda}$ gets bigger as $\boldsymbol{\kappa}$ becomes smaller, and has less impact on the others.
- Compared with the impact of the underlying asset price, the maturity as well as the volatility of trading volume has less impact on the VWAP option prices.

Some Extensions

- Squared OU Model for Trading Volume as in Novikov et al. (2014).

$$v_t = X_t^2 + \gamma, \qquad \mathrm{d}X_t = (\theta - \kappa X_t)\mathrm{d}t + \beta \mathrm{d}W_t^v,$$

  where θ, κ, γ and β are some constants. In this case, we have

$$\begin{aligned} X_t^2 &= V_t^2 + 2V_t\bar{E}(t)\int_0^t \beta E(s)\mathrm{d}W_s^v + \bar{E}^2(t)\left(\int_0^t \beta^2 E^2(s)\mathrm{d}s\right) \\ &\quad + 2\bar{E}^2(t)\left(\int_0^t \beta E(s)\left(\int_0^s \beta E(u)\mathrm{d}W_u^v\right)\mathrm{d}W_s^v\right) \end{aligned}$$

- Generalized VWAP $M_T = \frac{\int_0^T w_t^1 v_t S_t \mathrm{d}t}{\int_0^T w_t^2 v_t \mathrm{d}t}$, where w_t^i is a deterministic function of time t. Consider a floating-strike VWAP option defined by

$$\mathrm{VC}(S, v, K, T) = \mathrm{e}^{-rT}\mathbb{E}[(M_T - S_T)^+]$$

Conclusion

- In this talk, we develop a unified approximation method for options whose payoff depends on a volume weighted average price (VWAP).
- Compared to the previous works, our method is applicable to the local volatility model, not just for the geometric Brownian motion case.
- Moreover, our method can be used for any special type of VWAP option, including ordinary Asian and Australian options, with fixed-strike, floating-strike, continuously sampled, discretely sampled, forward starting, and in-progress transactions.
- Through numerical examples, we show that the accuracy of the second-order approximation is high enough for practical use.
- Our approximation get slightly worse for long maturity and high volatility case; in such a case, 3rd-order may be required.

Thank You for Your Attention