On the maximum of a fractional Brownian motion

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arXiv:1508.00099

Angers, September 2, 2015

Introduction

Fractional Brownian motion B_t^H with Hurst parameter $H \in (0,1)$ is a centered continuous Gaussian process with covariance function

$$\mathrm{E}\big(B_s^H B_t^H\big) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \qquad B_0 = 0.$$

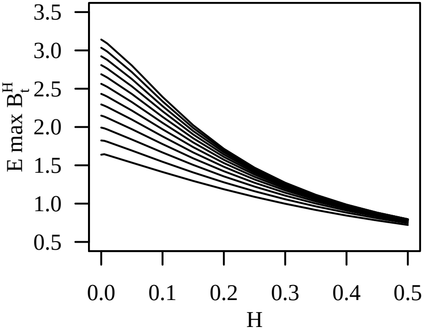
This paper was motivated by studying the expected maximum of B_t^H

$$\mathrm{E} \max_{t \leqslant 1} B_t^H$$

and trying to approximate it numerically by

$$\mathrm{E} \max_{t \leqslant 1} B_t^H \approx \mathrm{E} \max_{i \leqslant n} B_{i/n}^H, \qquad n \to \infty$$

On the picture: $\mathrm{E}\max_{i\leqslant n} B^H_{i/n}$ for $n = 2^4, 2^5, \dots, 2^{16}$



- The convergence is very slow for small H, making the numerical computation of $\mathrm{E}\max_{t\leqslant 1} B^H_t$ difficult
- Not clear whether the limit $\lim_{H\to 0}\mathrm{E}\max_{t\leqslant 1} B^H_t$ is finite or infinite

The results of the paper

We work with continuous Gaussian processes X_t such that

$$C_1|t-s|^{H_1} \leqslant \sqrt{\mathrm{Var}(X_t - X_s)} \leqslant C_2|t-s|^{H_2} \text{ for all } t, s \geqslant 0 \quad (*)$$

The main results

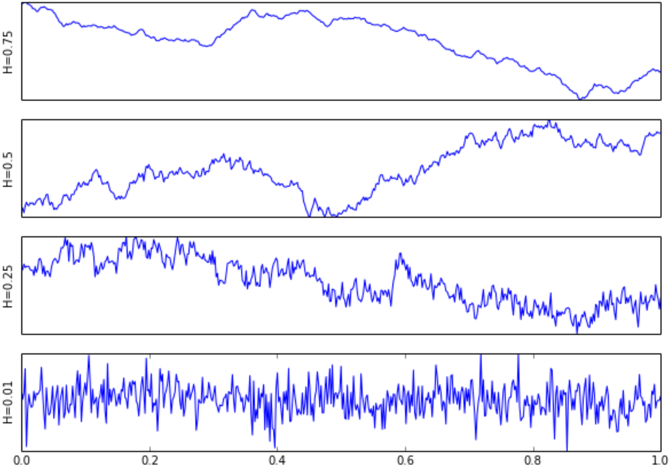
1. Bounds for the expected maximum

$$K_1 \cdot \frac{C_1}{\sqrt{H_1}} \leqslant \mathrm{E} \max_{t \leqslant 1} X_t \leqslant K_2 \cdot \frac{C_2}{\sqrt{H_2}}$$

2. Bound for the discrete approximation

$$\mathrm{E} \max_{t \leqslant 1} X_t - \mathrm{E} \max_{i \leqslant n} X_{i/n} \leqslant K_3 \cdot \frac{C_2 \sqrt{\ln n}}{n^{H_2}}$$

Remark: what happens with B_t^H when $H \to 0$



Remark (cont.)

Let ξ_t be a standard Gaussian white noise, i.e. ξ_t are i.i.d $N(0,1)$. Define

$$Z_t = \frac{1}{\sqrt{2}}(\xi_t - \xi_0), \qquad t \geqslant 0$$

Then B_t^H converges to Z_t in the sense of finite-dimensional distributions

$$(B_{t_1}, \ldots, B_{t_m}^H) \xrightarrow{d} (Z_{t_1}, \ldots Z_{t_m}), \qquad H \to 0.$$

In particular,

$$\mathrm{E} \max_{i \leqslant n} B_{i/n}^H \to \frac{1}{\sqrt{2}} \mathrm{E}\Big(\max_{1 \leqslant i \leqslant n} \xi_i\Big)^+, \qquad H \to 0.$$

Upper and lower bounds for the expected maximum

Theorem 1

Let X be a Gaussian process satisfying $(*)$. Then

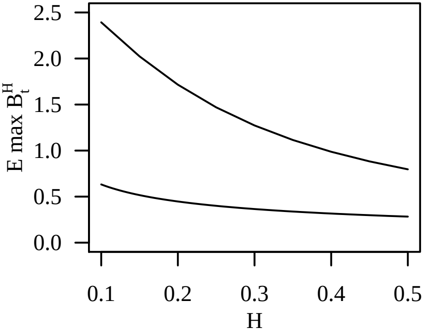
$$C_1 \cdot \frac{1}{5\sqrt{H_1}} < \mathrm{E} \max_{t \leqslant 1} X_t < C_2 \cdot \frac{17}{\sqrt{H_2}}.$$

Remark: slightly more accurate constants are

$$C_1 \cdot \frac{1}{\sqrt{4 H_1 \pi e \ln 2}} \leqslant \mathrm{E} \max_{t \leqslant 1} X_t \leqslant C_2 \cdot L \sqrt{\frac{2\pi}{H_2 \ln^3 2}},$$

where $L < 3.75$ is the constant from the generic chaining bound (see Talagrand, 2005).

The bounds are both of order of magnitude $1/\sqrt{H}$ for small H, but the lower bound is much closer to the true values.



On the picture: the lower bound and the true values for B_t^H.
The upper bound is far above.

Remark: literature

A powerful method to obtain tight bounds for the expected maximum of a Gaussian process is the generic chaining (see Talagrand, 2005).

$$\mathrm{E} \max_{t \leqslant 1} X_t \asymp \sup_{t \leqslant 1} \sum_{n \geqslant 0} 2^{n/2} d(t, T_n)$$

with some $T_1 \subseteq T_2 \subseteq \ldots \subseteq [0,1]$, where $|T_n| \leqslant 2^{2^n}$, and the metric $d(t,s)$ given by

$$d(t,s) = \sqrt{\mathrm{Var}(X_t - X_s)}$$

Proof of Theorem 1

- To prove the upper bound we use the generic chaining bound in the following simpler form (Dudley's entropy bound):

$$\mathrm{E}\max_{t\leqslant 1} X_t \leqslant L\sum_{n\geqslant 0} 2^{n/2}\max_{t\leqslant 1} d(t,T_n)$$

  for $T_n = \{i/2^{2^n},\ i = 1,\ldots,2/2^{2^n}\}$

- For the lower bound, the generic chaining is unnecessary complicated. Instead we use Sudakov's inequality

$$\mathrm{E}\max_{t\leqslant 1} X_t \geqslant \mathrm{E}\max_{i\leqslant n} X_{i/n} \geqslant \sqrt{\frac{\log_2(n+1)}{n^{2H}2\pi}}$$

  and maximize over n.

An upper bound for the discrete approximation

Theorem 2

Let X_t be a Gaussian process such that $\sqrt{\mathrm{Var}(X_t - X_s)} \leqslant |t-s|^H$ and $n \geqslant 2^{1/H}$. Then

$$\mathrm{E}\max_{t\leqslant 1} X_t - \mathrm{E}\max_{i\leqslant n} X_{i/n} \leqslant \frac{2\sqrt{\ln n}}{n^H}\left(1 + \frac{4}{n^H} + \frac{0.008}{(\ln n)^{3/2}}\right) \leqslant \frac{7\sqrt{\ln n}}{n^H}$$

Example: according to this estimate, to compute the expected maximum of fBm for $H = 0.1$ with $\pm 10\%$ accuracy we need at least $n = 2^{20}$.

Remark: literature

- There are many results on pathwise approximation of continuous-time processes by discrete-time processes, e.g. $\mathrm{E}\max(X_t - X_t^{(n)})$.
- Pathwise approximations give bounds for the tail probabilities, which can be used to obtain something like

$$\mathrm{E}\max_{t\leqslant 1} X_t - \mathrm{E}\max_{i\leqslant n} X_{i/n} \leqslant K(H)\frac{\sqrt{\ln n}}{n^H}$$

  with $K(H)$ depending on H.
- Piterbarg & Ivanov (preprint 2004) used a direct method for expected maxima of exponential functionals of fBm and obtained bounds of order $K(H)\sqrt{\ln n}/n^H$.

Remark: a bound with $K = K(H)$

For fBm, it is much easier to obtain a bound with a constant K that depends on H:

$$\mathrm{E}\max_{t\leqslant 1} B_t^H - \mathrm{E}\max_{i\leqslant n} B_{i/n}^H \leqslant \frac{\sqrt{K(H)+\ln n}}{n^H}.$$

where $K(H)\to\infty$ as $H\to 0$.

Idea: pathwise approximation + self-similarity + Jensen's inequality.

$$\begin{aligned}\mathrm{E}\max_{t\leqslant 1} B_t^H - \mathrm{E}\max_{i\leqslant n} B_{i/n}^H &\leqslant \mathrm{E}\max_{0\leqslant t\leqslant 1}(B_t^H - B_t^{H,n}) =: \mathrm{E}\max_{1\leqslant i\leqslant n} Z_i^{H,n} \\ &\leqslant \frac{1}{\lambda}\ln\big(n\mathrm{E}e^{\lambda (Z_1^{H,n_1})^2}\big),\end{aligned}$$

for any λ, and $\lambda = n^{2H}$ gives the bound with $K(H) = \ln \mathrm{E}\max_{t\leqslant 1} e^{(\widehat{B}_t^H)^2}$.

Proof of Theorem 2

The proof is in the spirit of generic chaining. First,

$$\mathrm{E}\max_{t\leqslant 1} X_t - \mathrm{E}\max_{i\leqslant n} X_{i/n} = \sum_{k=1}^{\infty}\left[\mathrm{E}\max_{i\leqslant n^{k+1}} X_{\frac{i}{n^{k+1}}} - \mathrm{E}\max_{i\leqslant n^k} X_{\frac{i}{n^k}}\right].$$

Then each term is estimated by using the inequality

$$\mathrm{P}(X_t - X_s > x) \leqslant \frac{|t-s|^H}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2|t-s|^{2H}}\right)$$

A lower bound for the discrete approximation

We don't know any good lower bound... but still we have the following

Theorem 3.

Let X_t be a Gaussian process such that $\sqrt{\mathrm{Var}(X_t - X_s)} \geqslant |t-s|^H$. Suppose $n(H) \geqslant 2$ is such that

$$\sup_{H\to 0}\Big[\mathrm{E}\max_{t\leqslant 1} X_t - \mathrm{E}\max_{i\leqslant n(H)} X_{i/n(H)}\Big] = \varepsilon < \infty.$$

Then

$$\liminf_{H\to 0}(n(H))^H > 1,$$

i.e. $n(H)$ is of the order $c^{1/H}$ for some $c = c(\varepsilon) > 1$.

Remark: standard Brownian motion

For standard Brownian motion it is known that that

$$\mathrm{E}\max_{t\leqslant 1} B_t - \mathrm{E}\max_{i\leqslant n} B_{i/n} \asymp \frac{1}{\sqrt{n}}$$

(A. Borovkov, T. Arak, *Theory Probab Appl*, 1973–1975)

Open problem: the optimal partition

What is the partition

$$\{t_0, t_1, \ldots, t_n\} \subseteq [0,1]$$

which maximizes

$$\mathrm{E} \max_{i \leqslant n} B^H_{t_i} \ ?$$

Preliminary results

- We can prove that $t_0 = 0$, $t_n = 1$:
  - For t_n – scaling argument
  - For t_0 – use that $\mathrm{E}\max_{t\leqslant 1} B_t^H = \mathrm{E}\max_{t\leqslant 1}(B_t^H - B_1^H)$ and scaling
- For the case $n = 3$ the optimal partition is $(0,\ 1/2,\ 1)$ – this follows from the formula (suggested by G. Shevchenko)

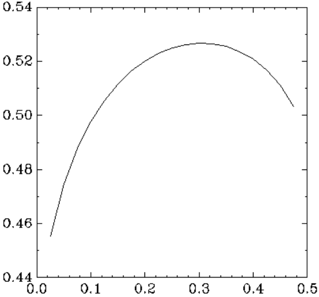
$$\mathrm{E}\big(\max\{X, Y, Z\}\big) = \frac{\sqrt{\mathrm{E}(X-Y)^2} + \sqrt{\mathrm{E}(Y-Z)^2} + \sqrt{\mathrm{E}(Z-X)^2}}{2\sqrt{2\pi}}$$

for any jointly Gaussian random variables X, Y, Z with zero mean.

Preliminary results (cont.)

Numerically we found out that the optimal partition is not uniform for standard Brownian motion and $n = 4, 5, 6$.

On the picture: the expected maximum for the partition $(0,\ t,\ 1-t,\ 1)$.



Thank you