Stochastic Radner equilibria and a system of quadratic BSDEs

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joint work with
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Advanced Methods in Mathematical Finance, Angers, September 4, 2015

Problem

Agent: for $i = 1, \ldots, d$,

1. utility: $U_i(x) = -e^{-x/\delta_i}$, $\delta_i > 0$,
2. random endowment: $E^i \in \mathbb{L}^0(\mathcal{F}_T)$.

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Equilibrium: λ, $(\pi_i)_{1 \le i \le d}$,

1. Utility maximization: $\mathbb{E}\left[U_i(\pi_i \cdot B_T^{(\lambda)} + E^i)\right] \to \text{Max}$;
2. Market clearing: $\sum_{i=1}^d \pi_i = 0$.

Completeness

All future risk can be exchanged for upfront cash.

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- Representative agent method

$$U_{rep}(c;\gamma) := \sup_{\sum c^i = c} \sum_{i=1}^{d} \gamma_i U^i(c^i).$$

The problem reduces to find the weight $(\gamma_i)_i$.

- Equilibrium is Pareto optimal.
- All agents share the same pricing measure:

$$M_T^{com} \propto U'_{rep}(c;\gamma).$$

[Breeden 79]

Incompleteness

Discrete time:

[Radner 82] extended the classical Arrow-Debreu model.

[Hart 75] gave a counter-example that equilibrium may not exist.

[Duffie-Shafer 85, 86] showed equilibrium exists for generic endowments.

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Continuous time: long standing open problem

[Cuoco-He 94]

[Žitković 12]

[Zhao 12], [Choi-Larsen 14]

[Christensen-Larsen-Munk 12], [Christensen-Larsen 14]

Our results

Our goal: Global existence

1. Non-Markovian case: (http://arxiv.org/abs/1505.07224)
   - unbounded endowment
   - equilibrium exists, when endowments are close to Pareto optimality
   - equilibrium exists when
     i) many similar agents, or
     ii) small time horizon

2. Markovian case: [Benoussan-Frehse 02]
   working progress with G. Žitković
   - bounded terminal condition
   - global existence
   - add probabilistic flavor to the proof of [Benoussan-Frehse 02]

Risk-aware reparametrization

Define

$$G^i = \frac{1}{\delta^i} E^i \quad \text{and} \quad \rho^i = \frac{1}{\delta^i} \pi^i.$$

Then the market clearing condition is

$$A[\rho] = \sum_i \alpha^i \rho^i = 0,$$

where $\alpha^i = \delta^i / (\sum_j \delta^j)$ with $\sum_i \alpha^i = 1$.

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We look for equilibrium λ in $\mathtt{bmo}$ (or $H_{\mathtt{BMO}}$).

$$\mathtt{bmo} = \left\{ \mu \ : \ \sup_\tau \left\| \mathbb{E}_\tau \left[\int_\tau^T |\mu_u|^2 du \right] \right\|_{\mathbb{L}^\infty} < \infty \right\}.$$

Assumptions on endowments

We assume, following [Delbaen et al. 02],

G is bounded from above with $\mathbb{E}[e^{-(1+\epsilon)G}] < \infty$ for some $\epsilon > 0$.

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Define

$$X_t^G = -\log \mathbb{E}_t[\exp(-G)], \quad t \in [0, T],$$

and (m, n) via the following BSDE

$$dX_t^G = m_t dB_t + n_t dW_t + \frac{1}{2}(m_t^2 + n_t^2)dt, \quad X_T^G = G.$$

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We assume

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In particular, when G is bounded, these assumptions are satisfied.

BSDE characterization of equilibria

Certainty-equivalent process

$$\exp(-Y_t^{\lambda,G}) = \text{ess sup}_\rho \mathbb{E}_t[\exp(-\rho \cdot B_T^\lambda + \rho \cdot B_t^\lambda - G)], \quad t \in [0, T].$$

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Theorem

For $\lambda \in$ bmo, the following are equivalent:

1. *λ is an equilibrium;*
2. *$\lambda = A[\mu] = \sum_i \alpha^i \mu^i$ for some solution $(Y^i, \mu^i, \nu^i)_i$ of the BSDE system*

$$dY_t^i = \mu_t^i dB_t + \nu_t^i dW_t + \left(\frac{1}{2}(\nu_t^i)^2 - \frac{1}{2}\lambda_t^2 + \lambda_t \mu_t^i\right) dt,$$
$$Y_T^i = G^i, \quad i \in \{1, 2, \dots, I\},$$

and $(\mu^i, \nu^i) \in$ bmo for all i.

System of quadratic BSDEs

Open problem: [Peng 99]

- [Darling 95], [Blache 05, 06]: Harmonic map
- [Tang 03]: Riccati system
- [Tevzadze 08]: existence when terminal condition is small
- [Frei-dos Reis 11]: counter example
- [Cheridito-Nam 14]: generator $f + z\,g$, f and g are Lipschitz
- [Hu-Tang 14]: diagonally quadratic
- [Jamneshan-Kupper-Luo 15]: cases not covered by [Tevzadze 08]

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Applications:

- Stochastic differential game: [Bensoussan-Frehse 02], [El Karoui-Hamadène 03]
- Relative performance: [Espinosa-Touzi 13], [Frei-dos Reis 11], [Frei 14]:
- Equilibrium pricing: [Cheridito-Horst-Kupper-Pirvu 12]:
- Market making: [Kramkov-Pulido 14]

Pareto optimality

$(\xi^i)_i$ is Pareto optimal if there is no $\sum_i \alpha^i \xi^i$-feasible allocation which is strictly better off.

Lemma

$(G^i)_i$ is Pareto optimal if and only if there exists ξ^c and constants $(c^i)_i$ such that

$$G^i = \xi^c + c^i, \quad \text{for all } i.$$

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Distance to Pareto optimality:

$$H(G) = \inf_{\xi^c} \max_i \left\| (m^i - m^c, n^i - n^c) \right\|_{\mathtt{bmo}(\mathbb{P}^c)},$$

where $d\mathbb{P}^c / d\mathbb{P} = \mathcal{E}(-m^c \cdot B - n^c \cdot W)_T = \exp(-\xi^c)/\mathbb{E}[\exp(-\xi^c)]$.

First main result (non-Markovian)

Theorem

Suppose that

$$H(G) < \frac{3}{2} - \sqrt{2} \approx 0.0858.$$

Then, there exists a unique equilibrium $\lambda \in \mathit{bmo}$.

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- Global uniqueness, similar to [Kramkov-Pulido 14].
- Uniqueness for the quadratic system as well.

Two corollaries

Smallness in size:

If

$$\inf_{\xi^c} \max_i \left\| G^i - \xi^c \right\|_{\mathbb{L}^\infty} < \left(\frac{3 - 2\sqrt{2}}{4} \right)^2 .$$

Then $\exists!$ equilibrium.

For a given total endowment $E_\Sigma \in \mathbb{L}^\infty$, equilibrium exists among sufficient more sufficient homogeneous agent.

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Smallness in time:

If $D^b(G^i - \xi^c), D^w(G^i - \xi^c) \in \mathcal{S}^\infty$, for some ξ^c and all i. Then a unique equilibrium exists when

$$T < T^* = \frac{\left(\frac{3}{2} - \sqrt{2}\right)^2}{\max_i \left(\|D^b(G^i - \xi^c)\|^2_{\mathcal{S}^\infty} + \|D^w(G^i - \xi^c)\|^2_{\mathcal{S}^\infty} \right)}.$$

Outline of proof

$$dY_t^i = \mu_t^i dB_t + \nu_t^i dW_t + \left(\frac{1}{2}(\nu_t^i)^2 - \frac{1}{2}\lambda_t^2 + \lambda_t \mu_t^i \right) dt, \quad Y_T^i = G^i.$$

where $\lambda = A[\mu]$.
Consider the excess-demand map

$$F : \lambda \mapsto A[\mu].$$

A fixed point in $\mathtt{bmo}$ gives a solution.

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where $\lambda = A[\mu]$.
Consider the excess-demand map

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A fixed point in bmo gives a solution.

1. A priori estimate: if λ is an equilibrium, then

$$\|\lambda\|_{\mathtt{bmo}} \leq \max_i \left\|(m^i, n^i)\right\|_{\mathtt{bmo}}.$$

2. Suppose $\max_i \left\|(m^i, n^i)\right\|_{\mathtt{bmo}} \leq \epsilon$,

$$F \text{ is a contraction on } B(a\epsilon) \text{ for some } a > 1.$$

Second main results (non-Markovian)

An allocation G is pre-Pareto if there exists an equilibrium λ such that

$$\tilde{G} = G + \rho^{\lambda, G} \cdot B_T^{\lambda}$$

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Fix a pre-Pareto G^p, consider the relative system.

Theorem

If G is "close" to a pre-Pareto G^p, then an equilibrium exists.

Markovian case

$$
\begin{aligned}
dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t, \\
dY_t &= -f(t, X_t, Z_t)dt + Z_t dW_t, \quad Y_T = G(X_T),
\end{aligned}
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where X is d-dim and Y is n-dim.

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where X is d-dim and Y is n-dim.

Assumption:

1. $b, \sigma\sigma'$ bounded and uniformly elliptic
2. G locally Hölder
3. $f = (f^1, \dots, f^n)$ satisfies

$$f^i(t, x, z) = g^i(t, x, z) \cdot z^i + h^i(t, x, z) + \ell^i(t, x, z) + k^i(t, x),$$

$$\|g^i\| \leq C_i \|z\|,$$
$$|\ell^i| \leq C_i \|z\|^{\beta_i}, \quad \text{for some } \beta \in [0, 2),$$
$$k^i \in \mathbb{L}^\infty,$$
$$|h^i| \leq \sum_{j=1}^{i} C_{ij} \|z^j\|^2,$$

where z^i is the i-th column of z.

Main result (Markovian)

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Example (Equilibrium)

Two agents (n=2)

- Y^1, Y^2 are bounded from below, $Y^1 + Y^2$ is bounded from above.
- Let $\tilde{Y}^1 = Y^1 - Y^2$ and $\tilde{Y}^2 = Y^1 + Y^2$. The previous structural condition is satisfied.

Therefore, equilibrium exists for all time.

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- Truncation into Lipschitz system.
  $Y^n = v^n(\cdot, X_\cdot)$. Uniform bounds on $\|v^n\|_\infty$.
- $\exists$ local uniform convergence subsequence $(v^n)_n$. (Key compactness)
- Convergence of semi-martingale [Barlow-Potter 90].

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Campanato space:

$$\sup_{(t_0,x_0)} \sup_R R^{-d-2-\alpha} \int_{Q_{\delta,R}(t_0,x_0)} \|v - \overline{v}\|^2 < \infty,$$

where $Q_{\delta,R}(t_0, x_0)$ is a parabolic domain and $\overline{v}$ is the average of v on Q.

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Campanato $\sim$ Hölder.

Proof cont.

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Multi-dim: [Bensoussan-Freshe 02]: Consider $\alpha(u) = e^u + e^{-u} - 2$.

Define the map $H : \mathbb{R}^n \to \mathbb{R}^n$ via

$$H^n(y) = \exp(\alpha(\gamma^n y^n)),$$
$$H^i(y) = \exp\left(\alpha(\gamma^i y^i) + H^{i+1}(y)\right), \quad i = 1, \ldots, n-1.$$

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Define $H_t^1 = H^1(t, Y_t)$ and apply Itô's formula to H^1 to obtain

$$
dH_t^1 \geq \|Z_t\|^2 dt - k(t, X_t)dt + \text{local martingale}.
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[Bensoussan-Frehse 02] used integration by part. [Barles-Lesigne 97].

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Define $H^1_t = H^1(t, Y_t)$ and apply Itô's formula to H^1 to obtain

$$dH^1_t \geq \|Z_t\|^2 dt - k(t, X_t)dt + \text{local martingale}.$$

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Step 2: "Hole-filling" technique by [Struwe 81].

Conclusion

1. We study a continuous time equilibrium in an incomplete market.
2. Translate the problem to a system of quadratic BSDE.
3. Non-Markovian: local existence + global uniqueness
4. Markovian: global existence.

Thanks for your attention!