# Robust Detection of Unobservable Disorder Time in Poisson Rate

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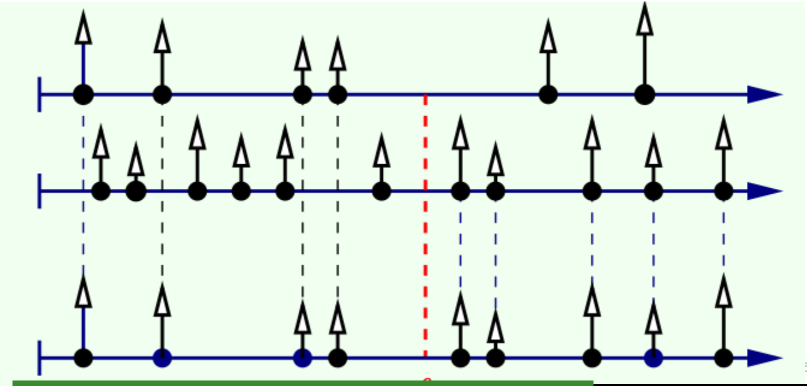
# Plan

# Motivation

The Poisson disorder problem is less formally stated as follows

- Observe a trajectory of the Poisson process $(N_t)$ whose intensity changes from $\lambda$ to $\rho\lambda$ at some (unknown) time $\theta$.
- The problem is to find a rule to detect $\theta$ as quickly as possible with a limited number of false alarms



# Where do the disorder problems arise ?

## Industrial Problems

- Quality control and maintenance, fraud and computer intrusion detection... etc.
- See also other application discussed during this workshop

## Our Main Motivation

- Non-life Insurance: Recalculate the premiums for the future sales of insurance policies when the risk structure changes ($\rho > 1$)
- Pension funds: Large exposure to change in mortality risk : ($\rho < 1$)
- Life insurance: Monitoring and surveillance of mortality dynamics
  - Sequential information on death occurrences
  - Updating mortality assumptions

## The observation process

- Let $N = (N_t)_{t \geq 0}$ be a counting process (of claims, or deaths) with intensity $\lambda = (\lambda_t)_{t \geq 0}$.
- A change in the intensity occurs at an unobservable date $\theta$, from $\lambda$ to $\rho\lambda, \rho > 0$.

## Change of Point $\theta$

- Random with known prior (Bayesian)
- Deterministic but unknown (Non-Bayesian but Robust)

## Known statistics $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ $(\lambda, \rho\lambda)$

- Under $\mathbb{P}$, no change, $(\lambda_t)$ holds
- Under $\widetilde{\mathbb{P}}$, immediate change, $(\rho\lambda_t)$ holds
- Under $\mathbb{P}^\theta$, $\lambda_t^\theta = \mathbf{1}_{t<\theta}\, \lambda_t + \mathbf{1}_{t \geq \theta}\, \rho\lambda_t$

- Based on the conditional distribution of the time of change,
- Formulated as an optimal stopping problem for partially observable process

Brownian framework with abrupt change in the drift

- Page (1954), Shiryaev (1963), Roberts (1966), Beibel (1988), Moustakides (2004), and Dayanik (2006),...

Poisson framework with abrupt change in intensity

- More recent studies : Gal (1971), Gapeev (2005), Bayraktar (2005, 2006), Dayanik (2006) for compound Poisson, Peskir, Shyriaev (2009) and others

New methods using particle filters

Andrieu, Legland (2004), Zhang (2005)

# Robust Detection Problem

## Robust Detection

- Non-Bayesian framework, mainly motivated by the lack of any prior on the statistical behavior of the change-point.
- Concerned with general counting process, in particular inhomogeneous Poisson process

## Robust Criterium, Lorden (1971)

- Let $T$ be a stopping time, candidate for the estimation of $\theta$
- The robust Lorden criterium with worst case detection delay

$$C^{\text{Lorden}}(T) = \sup_{\theta \in [0,\infty]} \operatorname{ess\,sup}_\omega \mathbb{E}_\theta\left[(T-\theta)^+ \middle| \mathcal{F}_\theta\right]$$

## Min-Max Robust Optimisation Problem

- Find $T^*$ such that $C^{\text{Lorden}}(T^*) = \min_T C^{\text{Lorden}}(T)$, s.t. the false alarm constraint $\mathbb{E}[T] \geq \pi$

# A min-max problem under constraint

## Our Lorden-modified criterium

- Well-adapted to our general framework

$$C(T) = \sup_{\theta \in [0,\infty]} \operatorname{ess\,sup}_{\omega} \mathbb{E}_{\theta}\left[ (N_T - N_{\theta})^+ \middle| \mathcal{F}_{\theta} \right]$$

with the false alarm constraint $\mathbb{E}[N_T] \geq \pi$.

## Stable by time rescaling:

- Let $\tau(t)$ be a time rescaling process, $\hat{N}_t = N_{\tau(t)}$ the rescaled counting process, with intensity $\hat{\Lambda}_t = \Lambda_{\tau(t)} = \int_0^{\tau(t)} \lambda_s ds$,
- If $T^*$ is an optimal stopping rule for the min-max problem, then $\tau^{-1}(T^*)$ is optimal for the modified criterium associated with $\hat{N}$
- It follows that we can only consider the case of constant intensity (Poisson case, i.e. $\tau(t) = \Lambda_t^{-1}$)

# Likelihood formulation

Conditional probability ratio between $\mathbb{P}$ =no change, $\widetilde{\mathbb{P}}$= immediate change.

- The conditional probability ratio process of $\widetilde{\mathbb{P}}$ w.r to $\mathbb{P}$ is

$$d\widetilde{\mathbb{P}}/d\mathbb{P} = \mathcal{E}_t = \exp\big(\log(\rho)(N_t - (\rho - 1)\Lambda_t)$$

- Put $U_t^\rho = N_t - \beta(\rho)\Lambda_t$, so $\rho^{U_t^\rho}$ is a $\mathbb{P}$-martingale.
- where $\beta(\rho) = \frac{\rho-1}{\log(\rho)}$, with $\beta(\rho) = \int_0^1 \rho^u du.$= Laplace transform of $U[0,1]$.
- Put $\tilde{\beta}(\rho) = \beta(1/\rho) = \beta(\rho)/\rho, \quad$ and $\tilde{\rho} = 1/\rho$.

  Then the CPR of $\mathbb{P}$ w.r to $\widetilde{\mathbb{P}}$ is : $\rho^{-U_t^\rho} = (1/\rho)^{U_t^\rho}$

## Sequential conditional probability ratio

- between $\mathbb{P}^\theta$ w.r to $\mathbb{P}$ is $d\mathbb{P}_\theta/d\mathbb{P} = \mathcal{E}_t^\theta$ with

$$\begin{aligned} \mathcal{E}_t^\theta &= \exp\Big(\log(\rho)(N_{t\vee\theta} - N_\theta) - (\rho - 1)\int_\theta^{t\vee\theta} \lambda_s\, ds\Big) \\ &= \rho^{U^\rho_{t\vee\theta} - U^\rho_\theta} \end{aligned}$$

- Useful in test hypothesis on intensity $\lambda$

## The Cumulative Sum rule (cusum)= Max in time of Likelihood

- Based on $\max_{s\leq t}(\rho^{U_t - U_s})$, ( sign of $\ln(\rho)$)
- Cusum stopping time, $\tau_m = \inf\{t\,;\mathrm{cusum}_t \geq m\}$ ($\mathrm{cusum}_t$ defined later)
- Main question: The optimality of $\tau_m$, if the false alarm constraint is achieved, $\mathbb{E}(N_{\tau_m}) = \mathbb{E}(N_T) = \pi$.

# Connection to insurance theory

Surplus process (Abundant literature)

- premium rate $(\rho - 1)\lambda_t$ and constant size of claims $\log \rho$
- Surplus by claim is $X_t(z) = z - N_t + \beta(\rho)\Lambda_t = z - U_t$

Ruin problem and Viability condition

- $\beta > 1$ $(\rho > 1)$ is called the *security loading condition*, and $\mathbb{E}(X_t) = z + (\beta(\rho) - 1)\mathbb{E}(\Lambda_t)$ drifts to $\infty$
- $\mathbb{P}(\sup_t U_t \leq m) = \bar{u}(m)$ is finite and of main interest. Well known for a long time (Feller 1971) as scale function

Equity process, $U_t(z)$ or dual risk model

- premia viewed as costs, and claims as profits coming suddently as in RD
- Viability condition $\mathbb{E}(U_1) > 0$ or $\beta < 1$

# Plan

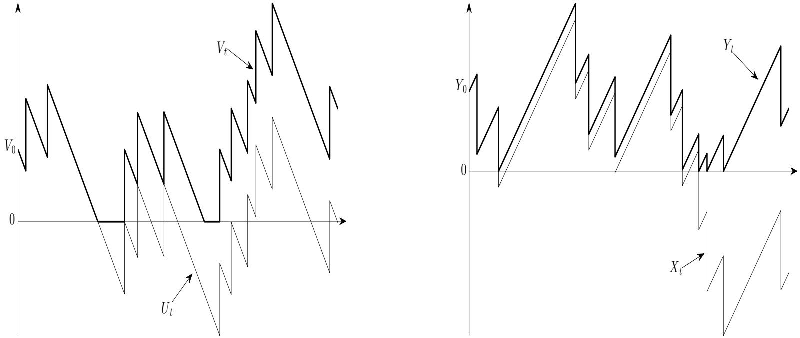
## Running supremum

- Running supremum $\bar{Z}_t = \sup_{s \leq t} Z_s$
- Put $X_t = -U_t = N_t - \beta\Lambda_t$. So, $\bar{X}_t$ is continuous.
- $\bar{U}_t$ is not continuous and increases only at jumps of $N$, such that $\bar{U}_t = U_t$

## Reflected processes and Cusum processes

- Cusum rule based on $\max_{s \leq t} \rho^{U_t - U_s}$
- $\rho > 1$: Reflected process $X$ at its maximum,
$$V_t = \sup_{\theta \leq t}(U_t - U_\theta) = \bar{X}_t - X_t$$
- $\rho < 1$: Reflected process $U$ at its maximum:
$$Y_t = \sup_{\theta \leq t}(X_t - X_\theta) = \bar{U}_t - U_t,$$

# Typical paths for $\rho > 1$ and $\rho < 1$




(a) $\rho > 1$ (b) $\rho < 1$

Figure: Sample paths of the processes $V$, $U$, $X$ and $Y$ when $\lambda$ is time-homogeneous.

Definitions with initial conditions

$$\begin{aligned} V_t(Z_0) &= U_t + \sup\{Z_0, \bar{X}_t\} = U_t(Z_0) + (\bar{X}_t - Z_0)^+, \\ Y_t(Z_0) &= X_t + \sup\{Z_0, \bar{U}_t\} = X_t(Z_0) + (\bar{U}_t - Z_0)^+, \\ \bar{X}_t^{ad} &= (\bar{X}_t - Z_0)^+, \quad \bar{U}_t^{ad} = (\bar{U}_t - Z_0)^+. \end{aligned}$$

Differential point of view of reflected processes $(j(y) = y \wedge 1)$

- $V$ and $Y$ are solutions of the ODE's driven by $N$,

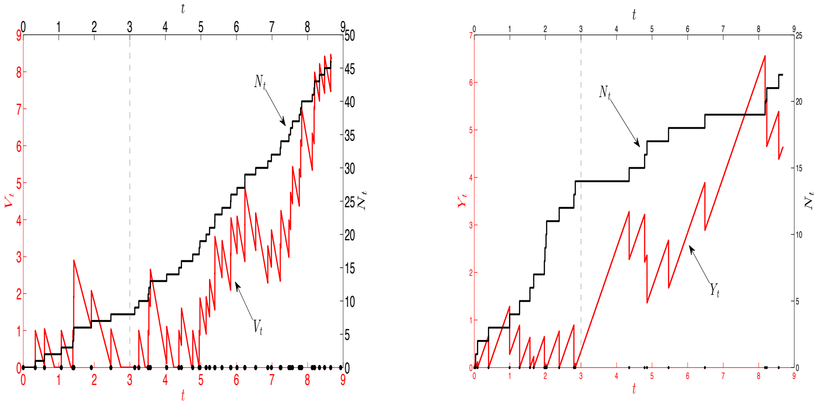
$$dV_t = dN_t - \beta\, \mathbf{1}_{(0,\infty)}(V_t) d\Lambda_t, \quad d\bar{X}_t^{ad} = \beta \mathbf{1}_{\{V_t=0\}} d\Lambda_t$$

$$dY_t = -j(Y_{t-})dN_t + \beta d\Lambda_t, \quad d\bar{U}_t^{ad} = \mathbf{1}_{\{Y_t=0\}}(1 - Y_{t-})dN_t$$

Exit times: For any cadlag process $Z$

- $\tau_m^Z = \inf\{t : Z_t \geq m\}$ and $\sigma_b^Z = \inf\{t : Z_t \leq b\}$
- $\rho > 1$: $\bar{u}(m) = \mathbb{P}(\tau_m^U = +\infty)$ and $\bar{u}(m - U_t)$ is a martingale.

# Typical paths with change of regime at date 3




(a) Processes $N$ and $V_t$ (b) Processes $N$ and $Y_t$

Figure: Sample paths, for $\rho = 1.5$, of the cusum processes $N$, $V^{\rho}$ (left) and $N$, $Y_t^{\rho}$ for $\rho = 0.5$ (right ).

# Performance analysis for $\rho > 1$, and $\tau_m^V$

## Performance functions of the $V$-cusum rule

- The performance of the cusum stopping is based on

$$\Gamma_t^m(x) = \widetilde{\mathbb{E}}_x\left[\mathbf{1}_{\tau_m^V \geq t}(N_{\tau_m^V} - N_t)\,|\mathcal{F}_\theta\right] = \tilde{h}_m(V_t(x)) \quad \widetilde{\mathbb{P}}\ a.s.$$

- $\widetilde{H}_t(x,m) = \tilde{h}_m(V_t) + N_t$ is a $\widetilde{\mathbb{P}}_x$-local martingale on $[0, \tau_m^V)$,
- Similar definition under $\mathbb{P}_x$, with $h_m(x) = \mathbb{E}_x(N_{\tau_m^V})$ and $H_t(x,m) = h_m(V_t) + N_t$ is a $\mathbb{P}_x$-local martingale on $[0, \tau_m)$

## Performance functions of the $Y$-cusum rule

- The performance of the cusum stopping is based on

$$\Gamma_t^m(x) = \widetilde{\mathbb{E}}_x\left[\mathbf{1}_{\tau_m^Y \geq t}(N_{\tau_m^Y} - N_t)\,|\mathcal{F}_\theta\right] = \tilde{g}_m(Y_t(x)) \quad \widetilde{\mathbb{P}}\ a.s.$$

- $\widetilde{G}_t(x,m) = \tilde{g}_m(Y_t) + N_t$ is a $\widetilde{\mathbb{P}}_x$-local martingale on $[0, \tau_m)$
- Similar definition under $\mathbb{P}_x$, with $g_m(x) = \mathbb{E}_x(N_{\tau_m^Y})$ and $G_t(x,m) = g_m(Y_t) + N_t$ is a $\mathbb{P}_x$-local martingale on $[0, \tau_m)$

Two problems:

- Extension of the martingale property for any $T$
- Computation of the functions $h_n, \tilde{h}_m, g_m, \tilde{g}_m$

# Plan

# Stochastic differential calculus

Typical stochastic processes and Itô's formula for continuous functions

- $dZ_t = \sigma(Z_{t-})dN_t + b(Z_t)d\Lambda_t$
- $d\phi(Z_t) = (\phi(Z_{t-} + \sigma(Z_{t-})) - \phi(Z_{t-}))\, dN_t + \phi'(Z_t)\, b(Z_t)\, d\Lambda_t.$

Example with the $\bar{u}$ function, $\rho > 1$

- $\bar{u}(x) = \mathbb{P}(\bar{U}_\infty \leq x)$ and $\bar{u}(m - U_t)$ is the martingale on $[0, \tau_m^U]$

  $d\bar{u}(m - U_t) = (\bar{u}(m - U_{t-} - 1) - \bar{u}(m - U_{t-})dN_s + \beta\bar{u}'(m - U_{t-})d\Lambda_t$
- if $\beta\bar{u}'(x) = \bar{u}(x) - \bar{u}(x-1)$= delayed equation

Extension to monotonic functions with one jumps, $\delta\phi(m)$

- $J_t^{d,Z}$= number of down-crossings of $m$ continuously
- $d\phi(Z_t) = standartpart + (\phi(m) - \phi(m-))dJ_t^{d,Z}$

Example with $V$-performance functions

- $dH^m_t = dh_m(V_t) + dN^m_t - h_m(m-)dJ^{d,V}_t$ is the martingale
  $= (h_m(V_{t-}+1) - h_m(V_{t-}))dN_t - h'_m(V_{t-})\beta \mathbf{1}_{(0,\infty)}(V_{t-})d\Lambda_t$
- if $\beta h'_m(x) = h_m(x+1) - h_m(x) + 1, \quad x \in (0,m), \quad h'(0) = 0,$

Example with $Y$-performance functions

- $dG^m_t = dg_m(Y_t) + dN^{Y,m}_t$ is the martingale
  $= (g_m((Y_{t-}-1)^+) - g_m(Y_{t-}))dN_t - g'_m(y_{t-})\beta d\Lambda_t$
- if $\beta g'_m(x) = g_m(x) - g_m((x-1)^+) - 1, \quad x \in (0,m)$

Same result for the tilded functions with $\tilde{\beta}$

### Delayed equation

- Delayed equation with continuous solution on $(0, \infty)$ ($0$ for $x < 0$) with only one jump at 0.

$$\beta u'(x) = u(x) - u(x-1), \quad \beta > 0.$$

### Delayed equation properties

- If $\beta = \beta(\rho)$, then $\rho^x u(x)$ is solution of DDE with $\tilde{\beta}(\rho) = \beta(1/\rho)$
- $\hat{u}(x) = \int_0^x u(z)dz$ is solution of DDE

$$\beta \hat{u}'(x) = \hat{u}(x) - \hat{u}(x-1) + \beta u(0) \quad \beta > 0, \quad u(0) = \hat{u}'(0).$$

- $\bar{u}(x) = \mathbb{P}(\bar{U}_\infty \leq x)$ is solution of the DDE by martingale property.

## Old result, Feller (1971), $\beta > 1$

- The derivative is a solution of the convolution equation,

$$u'(x) = (1/\beta)\,\mathbf{1}_{[0,1)}(x)u(0) + (1/\beta)\int_0^1 u'(x-z)dz$$

- When $\beta > 1$, let $S_n$ be sum of i.i.d. unif on $[0,1]$, $S_n$, and $\nu$ an indep. geometric r.v. with , $\mathbb{P}(\nu = j) = (1-1/\beta)\beta^{-j}$.
- $u'(x) = u(0)/(\beta-1)\mathbb{P}(S_\nu \in [x-1,x))$,
- $\bar{u}(x) = \mathbb{P}(\bar{U}_\infty \le x)$ is equal to $\mathbb{P}(S_\nu \le x)$
- $\frac{1}{(\beta-1)}\bar{u}(x)$ is a scale function $W(x)$ (Bertoin (1996)).

## Scale functions

- For $\rho > 1$, $W(x) = \frac{1}{(\beta-1)}\mathbb{P}(S_\nu \leq x)$ and $\widetilde{W}(x) = \rho^x\, W(x)$
- For $\rho < 1$, $W(x) = \rho^{-x}\widetilde{W}(x)$ and $\widetilde{W}(x) = \frac{1}{\rho(\tilde{\beta}-1)}\,\widetilde{\mathbb{P}}(\bar{U}_\infty^{\tilde{\rho}} \leq x)$.

### $Y$-and $V$ performance functions

**$Y$-performance, $\rho < 1$**

- $g_m(y) = \int_y^m W(z)dz, \quad \tilde{g}_m(y) = \int_y^m \rho\widetilde{W}(z)dz, \quad y \in [0, m].$

**$V$-performance, $\rho > 1$**

- $h_m(m-) = W(0)\frac{W(m)}{W'(m)}, \quad \tilde{h}_m(m-) = \rho W(0)\frac{\widetilde{W}(m)}{\widetilde{W}'(m)}.$
- $h_m(x) = W(m-x)\frac{W(m)}{W'(m)} - \int_0^{m-x} W(y)dy$
- $\tilde{h}_m(x) = \rho\left(\widetilde{W}(m-x)\frac{\widetilde{W}(m)}{\widetilde{W}'(m)} - \int_0^{m-x}\widetilde{W}(y)dy\right)$

# Plan

Using similar arguments of Shiryaev (1996) and Moustakides (2004).

Integration by parts

- $\Gamma_t^T := \widetilde{\mathbb{E}}_x(\int_t^T dN_s \big| \mathcal{F}_t)$, and $(\bar{Z}_t)$ an increasing process
- By integration of $\Gamma_t^T$ with respect to $\rho^{\bar{Z}_t}$
  $\widetilde{\mathbb{E}}_x\big[\int_t^T \Gamma_\alpha^T d\rho^{\bar{Z}_\alpha} \big| \mathcal{F}_t\big] = \mathbb{E}_x\big[\int_t^T (\rho^{\bar{Z}_{s-}} - \rho^{\bar{Z}_t}) dN_s \big| \mathcal{F}_t\big]$.

Applications to lower bounds

- for $\rho > 1$, $\rho\,\mathbb{E}\big[\int_t^T \rho^{V_{s-}} dN_s \big| \mathcal{F}_t\big] \leq C(T)\,\mathbb{E}\big(\rho^{V_T} \big| \mathcal{F}_t\big)$
- for $\rho < 1$, $\rho\,\mathbb{E}\big[\int_t^T \rho^{-Y_{s-}} dN_s \big| \mathcal{F}_t\big] \leq C(T)\,\mathbb{E}\big(\rho^{-Y_T} \big| \mathcal{F}_t\big)$

Applications to performance functions

- $\rho^x(\tilde{h}_m(x) - \tilde{h}_m(0)) = \rho\,\mathbb{E}_x\big(\int_0^{\tau_m} \rho^{V_{s-}} dN_s\big) - \tilde{h}_m(0)\,\mathbb{E}_x(\rho^{V_{\tau_m}})$,
- $\rho^{-y}(\tilde{g}_m(y) - \tilde{g}_m(0)) = \rho\,\mathbb{E}_y\big(\int_0^{\tau_m} \rho^{-Y_{s-}} dN_s\big) - \tilde{g}_m(0)\,\mathbb{E}_y(\rho^{-Y_{\tau_m}})$.

Example based on the process $V$

- False alarm (with the notation $N_t^{m,V} = \int_0^t \mathbf{1}_{[0,m)}(V_{s-})dN_s$):
$\mathbb{E}(N_T) = \mathbb{E}(N_{\tau_m^V}) = h_m(0) = \mathbb{E}(N_T^{m,V} - h_m(m-)J_T^{d,V} + h_m(V_T))$,
so that $\mathbb{E}(h_m(V_T)) = \mathbb{E}(\int_0^T \mathbf{1}_{[m,\infty)}(V_{s-})dN_s + h_m(m-)J_T^{d,V})(*)$,
- By the martingale property
$\mathbb{E}(\int_0^T \rho^{V_s} dN_s^{m,V} - \tilde{h}_m(0)\rho^{V_T}) = \mathbb{E}(\rho^m \tilde{h}_m(m-)J_T^{d,V} - \tilde{h}_m(V_T))\rho^{V_T})$
- So the problem is reduced to show that, given $(*)$
$\mathbb{E}(\int_0^T \rho^{V_s}\mathbf{1}_{[m,\infty)}(V_{s-})dN_s + \rho^m \tilde{h}_m(m-)J_T^{d,V} - \tilde{h}_m(V_T))\rho^{V_T}] \geq 0$
- We eliminate $J_T^{d,V}$ in this inequality by multiplying the false alarm $(*)$ by $\rho^m \tilde{h}_m(m-)/h_m(m-)$

# Argument for the proofs

## Reduction to functions comparison

- $\mathbb{E}(\int_0^T \rho^{V_s}\mathbf{1}_{[m,\infty)}(V_{s-})dN_s) \geq \rho^m \frac{\tilde{h}_m(m-)}{h_m(m-)}\mathbb{E}(\int_0^T \mathbf{1}_{[m,\infty)}(V_{s-})dN_s)$.
- $\mathbb{E}(\rho^m \frac{\tilde{h}_m(m-)}{h_m(m-)} h_m(V_T) - \rho^{V_T}\tilde{h}(V_T)) \geq 0$

## Example based on the process $Y$

- Similar arguments apply but without the discontinuities, and the comparison reduces to
- $\mathbb{E}(\int_0^T \rho^{-Y_{s-}}\mathbf{1}_{[m,\infty)}(Y_{s-})dN_s) \geq \rho^{-m}\mathbb{E}(\int_0^T \mathbf{1}_{[m,\infty)}(Y_{s-})dN_s)$.
- True if $\mathbb{E}(\rho\rho^{-m}g_m(Y_T) - \rho^{-Y_T}\tilde{g}_m(Y_T)) \geq 0$

## Comparison based on the scale function representation

- $\psi(y) = \rho^{-(m-y)}g_m(y) - \tilde{g}_m(y)/\rho$ is positive if $\rho < 1$.
- $\phi_m(m-z) = \frac{\tilde{h}_m(m-)}{h_m(m-)}\rho^{m-z}\, h_m(z) - \tilde{h}_m(z)$ is positive for $\rho > 1$ and $\tilde{h}'_m(x) = \rho^{m-x}\frac{\tilde{h}_m(m-)}{h_m(m-)}h'_m(x)$

## Optimality of the cusum rule

### Bounds of the cusum rule

- $\tilde{h}_m(0)$ and $\tilde{g}_m(0)$ are respectively the cusum bounds of the stopping times $\tau_m^V$ $(\rho > 1)$ and $\tau_m^Y$ $(\rho < 1)$.

### Optimality for a decrease in intensity

- Let $T$ be a stopping times with finite cusum bound, such that $\mathbb{E}(N_T) = \mathbb{E}(N_{\tau_m^Y}) = g_m(0)$., then
- $\mathbb{E}\big( \int_0^T \rho^{1-Y_{s-}} dN_s \big) \geq \tilde{h}_m(0)\, \mathbb{E}\big(\rho^{-Y_T}\big),$

### Optimality for an increase in intensity

- Let $T$ be a stopping times with finite cusum bound,such that $\mathbb{E}(N_T) = \mathbb{E}(N_{\tau_m^V}) = h_m(0)$., then
- $\mathbb{E}\big( \int_0^T \rho^{V_s} dN_s \big) \geq \tilde{h}_m(0)\, \mathbb{E}\big(\rho^{V_T}\big),$

# Numerical instability of $\tilde{h}_m$

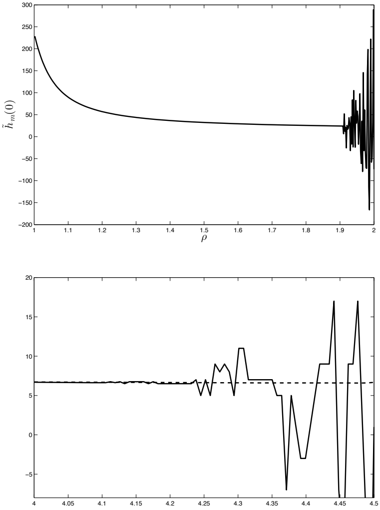



Figure: Function $\tilde{h}_m(0)$ for different values of $\rho$.

# Conclusion

- The CUSUM detection rule is optimal in the case of non-homogeneous Poisson process with a modified Lorden criterion
- Very easy to implement
- Non asymptotic criterium
- Based on fine properties of scale functions, easy to extend to Lévy processes
- The proof provide lower bound for some conditional ratio (Basseville)
- Applications in non-life insurance and life insurance
- Further research on possible extensions to Lévy process

# Detection Procedure – Algorithm

Step 1: Fix the input parameters: The post-change intensity through the specification of $\rho$ and the false alarm constraint $\pi$.

Step 2: Determine the threshold $m$ as the solution of the equation $\mathbb{E}_\infty[N_{\tau_m}] = \pi$.

Step 3: For each new observation at time $t$ compute the value of the CUSUM process $V$ given by the iterative relation $V_{t+1} = (V_{t-1} + U_t)^+$.

Step 4: Compare the current value of $V$ to the threshold $m$ and stop the procedure once $V_t \geq m$ and sound an alarm. Hence $\tau_m(0) = t$.

### Evolution of the global population

$$\lambda_t = a\big(1+\exp(-(t-b)/c)\big)^{-1},\ t \geq 0,$$

where $a, b$ and $c$ are some constant parameters given in

$$a = 13,80\ b = 11,65\ c = 26,40,\ \ d = 0.0907$$

- 10000 simulations for $\rho =\ 1.1, 1.5, 2$

| | $\rho$ | 1.1 | 1.5 | 2 |
|---|---|---|---|---|
| $m = 5$ | $\tilde{h}_m(0)$ | 25.01 | 17.77 () | 14.15 |
| | $h_m(0)$ | 32.92 () | 58.52 () | 116.82 |
| $m = 10$ | $\tilde{h}_m(0)$ | 81.48 () | 44.49 () | 31.97 |
| | $h_m(0)$ | 147.73 () | 643.29 () | 4174.49 |

# Thank You!

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