An Analytical Approximation for Pricing VWAP Options

Hideharu Funahashi and Masaaki Kijima

Graduate School of Social Sciences, Tokyo Metropolitan University

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Formulation

- Let $S_t$ and $v_t$ be the time-$t$ price and trading volume, respectively, of the underlying asset.
- VWAP (volume weighted average price) is determined by

$$M_T = \frac{\int_0^T v_t S_t dt}{\int_0^T v_t dt},$$

where $M_T$ is called the VWAP of the time interval $[0, T]$.
- The standard definition of a continuous VWAP call option is given by

$$VC(S, v, K, T) = e^{-rT} \mathbb{E}[(M_T - K)^+]$$

where $S = S_0$ is the initial price, $v = v_0$ is the initial trading volume, $K$ is a strike, $T$ is a maturity, and $r$ is the short rate, $\mathbb{E}$ is the risk-neutral expectation operator.
- Existing papers try to approximate the distribution of $M_T$ directly.
Motivation

- VWAP Options are becoming increasingly popular in options markets, since they can help corporate firms hedge risks arising from market disruption when entering large buy or sell orders.
- Their prices assign more weight to periods of high trading than to periods of low trading in its calculation.
- Hence, VWAP options differ conceptually from Asian options because the resulting payoff is not a linear combination of underlying prices.
- As a result, the pricing of VWAP options is significantly more difficult than Asians and few pricing models have been proposed in the literature, despite their popularity in practice.
- See, e.g., Buryak and Guo (2014), Novikov et al. (2013) for details.
It is common to model the underlying $S_t$ by using the geometric Brownian motion (GBM) for simplicity.

For the trading volume $v_t$, Stace (2007) proposes a mean-reverting process; Novikov et al. (2013) use a squared Ornstein–Uhlenbeck (OU) process; and Buryak and Guo (2014) suggest a simple gamma process, respectively, for the trading volume process.

Under the GBM assumption, these papers produce approximated pricing formulas by utilizing the moment-matching technique for $M_T$.

On the other hand, Novikov and Kordzakhia (2013) derive very tight upper and lower bounds for the price of VWAP options.
In This Talk

- Other than the GBM case, these approaches seem difficult to apply for deriving an approximation formula of VWAP options.

- Funahashi and Kijima (2015b) apply the chaos expansion technique to derive a unified approximation method for pricing any type of Asian options when the underlying process follows a diffusion.

- In this talk, not of the VWAP itself, but we try to approximate the distribution of $\tilde{M}_T$,

$$\tilde{M}_T(x) = \int_0^T v_t S_t dt - x \int_0^T v_t dt,$$

when the underlying asset price and trading volume processes follow a local volatility model and a mean-reverting model, respectively.
The Setup

- We assume that the price $S_t$ and the trading volume $v_t$ of the underlying asset are modeled by the following SDE:

\[
\begin{align*}
\frac{dS_t}{S_t} &= r(t)dt + \sigma(S_t, t)dW_t \\
\phantom{\frac{dS_t}{S_t}} &= r(t)dt + \sigma(S_t, t)dW_t \\
dv_t &= (\theta(t) - \kappa(t)v_t)dt + \gamma(v_t)dW_t^v
\end{align*}
\]

under the risk-neutral measure $\mathbb{Q}$, where $\{W_t\}$ and $\{W_t^v\}$ are the standard Brownian motions with correlation $dW_t dW_t^v = \rho dt$.

- The volatility functions $\sigma(S, t)$ and $\gamma(v)$ are sufficiently smooth with respect to $(S, t)$ and $v$, respectively.

- $r(t)$, $\theta(t)$ and $\kappa(t)$ are some deterministic functions of time $t$. 

Key Observation

- Denote the cumulative distribution function (CDF) of $M_T$ by $F_M(x) = \mathbb{Q}\{M_T \leq x\}$, $x > 0$.
- The VWAP call price can be written as
  \[
  VC(S, v, K, T) = e^{-rT} \int_{K}^{\infty} (1 - F_M(x)) \, dx
  \]
- For each $x > 0$, let $F_{\widetilde{M},x}(y)$ be the CDF of the random variable
  \[
  \widetilde{M}_T(x) = \int_{0}^{T} v_t S_t \, dt - x \int_{0}^{T} v_t \, dt
  \]
- It follows from the definition of VWAP that
  \[
  F_M(x) = F_{\widetilde{M},x}(0), \quad x > 0
  \]
- Therefore, it suffices to know the CDF $F_{\widetilde{M},x}(y)$ at $y = 0$.
- To this end, we apply the chaos expansion approach to approximate the distribution of the random variable $\widetilde{M}_T(x)$. 
Employing the same idea as in Theorem 3.1 of Funahashi and Kijima (2015a), $S_t$ is approximated by the following formula:

**Lemma**

Let $F(0, t) = S e^{\int_0^t r(u) du}$ be the forward price of the underlying asset with delivery date $t$. Then,

$$S_t \approx F(0, t) \left[ 1 + \int_0^t p_1(s) dW_s + \int_0^t p_2(s) \left( \int_0^s \sigma_0(u) dW_u \right) dW_s ight.$$

$$+ \int_0^t p_3(s) \left( \int_0^s \sigma_0(u) \left( \int_0^u \sigma_0(r) dW_r \right) dW_u \right) dW_s$$

$$+ \int_0^t p_4(s) \left( \int_0^s p_5(u) \left( \int_0^u \sigma_0(r) dW_r \right) dW_u \right) dW_s \right]$$
where

\[ p_1(s) := \sigma_0(s) + F(0, s)\sigma_0'(s) \left( \int_0^s \sigma_0^2(u)du \right) \]

\[ + \frac{1}{2} F^2(0, s)\sigma_0''(s) \left( \int_0^s \sigma_0^2(u)du \right) \]

\[ p_2(s) := \sigma_0(s) + F(0, s)\sigma_0'(s) \]

\[ p_3(s) := \sigma_0(s) + 3F(0, s)\sigma_0'(s) + F^2(0, s)\sigma_0''(s) \]

\[ p_4(s) := \sigma_0(s) + F(0, s)\sigma_0'(s) \]

\[ p_5(s) := F(0, s)\sigma_0'(s) \]

with \( \sigma_0'(t) := \partial_x \sigma(x, t)\big|_{x=F(0,t)} \) and \( \sigma_0''(t) := \partial_{xx} \sigma(x, t)\big|_{x=F(0,t)} \)
Employing the successive substitution used in Funahashi (2014), we obtain the following result. Let \( E(0, t) = e^{\int_0^t \kappa(u)du} \), \( \bar{E}(t) = \frac{1}{E(t)} \), and 
\[ V_t = \bar{E}(t) \left( v_0 + \int_0^t E(s) \theta(s) ds \right) \]

Lemma

The trading volume \( \nu_t \) is approximated as \( \nu_t \approx \)
\[ V(0, t) + \bar{E}(t) \int_0^t p_6(s) dW_s^\nu + \bar{E}(t) \int_0^t \gamma'_0(s) \left( \int_0^s E(u) \gamma_0(u) dW_u^\nu \right) dW_s^\nu \]
\[ + \bar{E}(t) \int_0^t \bar{E}(s) \gamma''_0(s) \left( \int_0^s E(u) \gamma_0(u) \left( \int_0^u E(r) \gamma_0(r) dW_r^\nu \right) dW_u^\nu \right) dW_s^\nu \]
\[ + \bar{E}(t) \int_0^t \gamma'_0(s) \left( \int_0^s \gamma'_0(u) \left( \int_0^u E(r) \gamma_0(r) dW_r^\nu \right) dW_u^\nu \right) dW_s^\nu, \]
where \( \gamma_0(t) := \gamma(V_t) \), \( \gamma'_0(t) := \partial_x \gamma(x)|_{x=V_t} \), \( \gamma''_0(t) := \partial_{xx} \gamma(x)|_{x=V_t} \), and 
\[ p_6(t) := E(t) \gamma_0(t) + \frac{1}{2} \bar{E}(t) \gamma''_0(t) \left( \int_0^t E^2(s) \gamma_0^2(s) ds \right) \]
Using the approximation results for $S_t$ and $v_t$, we obtain the following.

**Lemma**

The traded value $v_t S_t$ is approximated as

$$v_t S_t \approx I_0(t) + I_1(t) + I_2(t) + I_3(t),$$

where $I_0(t) = V_t F(0, t) + \rho F(0, t) \bar{E}(t) \int_0^t p_6(s) p_1(s) \, ds$,

$$I_1(t) = V(0, t) F(0, t) \int_0^t p_1(s) \, dW_s + F(0, t) \bar{E}(t) \int_0^t p_{10}(t, s) \, dW_s$$

$$+ F(0, t) \bar{E}(t) \int_0^t (p_6(s) + p_9(t, s)) \, dW_s^\nu,$$

and others (omitted).
By changing the order of integration, $\tilde{M}_T$ can be approximated by a truncated sum of iterated Itô stochastic integrals as follows.

**Lemma**

For each $x > 0$, the random variable $\tilde{M}_T$ can be approximated as

$$\tilde{M}_T = \int_0^T v_t S_t dt - x \int_0^T v_t dt \approx J_0(x, T) + J_1(x, T) + J_2(x, T) + J_3(x, T),$$

where

$$J_0(x, T) = \int_0^T V_t (F(0, t) - x) dt$$

$$+ \rho \int_0^T F(0, t) \bar{E}(t) \left( \int_0^t p_6(s)p_1(s)ds \right) dt,$$
\[ J_1(x, T) = \int_0^T p_1(t) \left( \int_t^T V(0, s)F(0, s)ds \right) dW_t + \int_0^T \left( \int_t^T p_{10}(s, t)F(0, s)\bar{E}(s)ds \right) dW_t^\nu + \int_0^T \left( \int_t^T \left( p_6(t) + p_9(s, t)F(0, s)\bar{E}(s) \right) ds \right) dW_t^\nu - x \int_0^T p_6(t) \left( \int_t^T \bar{E}(s)ds \right) dW_t^\nu, \]

\[ J_2(x, T) = \int_0^T r_1(t) \left( \int_0^t \sigma_0(s)dW_s \right) dW_t + \int_0^T r_2(t) \left( \int_0^t p_6(s)dW_s^\nu \right) dW_t + \int_0^T r_3(t) \left( \int_0^t p_1(s)dW_s \right) dW_t^\nu + \int_0^T r_4(x, t) \left( \int_0^t E(s)\gamma_0(s)dW_s^\nu \right) dW_t^\nu, \]

and others (omitted).
Option Pricing Formula

- Let us define $Y_t = \tilde{M}_t - J_0(x, t)$, and denote its probability density function (PDF) by $f_{Y_T, x}(y)$.
- We can obtain the PDF by applying the following lemma.

**Lemma**

The PDF of $Y_T$ is approximated as

$$f_{Y_T, x}(y) \approx n(y; 0, V_x(T)) - \frac{\partial}{\partial y} \left\{ \mathbb{E}[J_2(x, T) | J_1(x, t) = y]n(y; 0, V_x(T)) \right\}$$

$$- \frac{\partial}{\partial y} \left\{ \mathbb{E}[J_3(x, t) | J_1(x, t) = y]n(y; 0, V_x(T)) \right\}$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial y^2} \left\{ \mathbb{E}[J_2(x, t)^2 | J_1(x, t) = y]n(y; 0, V_x(T)) \right\},$$

where $n(y; a, b)$ denotes the normal density with mean $a$ and variance $b$. 
The conditional expectations can be evaluated explicitly.

Using the approximated density function of \( f_{Y_T,x}(y) \), we can approximate the CDF \( F_{Y_T,x}(y) \) of \( Y_T \).

But, from the relation \( F_{\tilde{M},x}(y) = F_{Y_T,x}(y - J_0(x, T)) \), we have

\[
F_M(x) = F_{\tilde{M},x}(0) = F_{Y_T,x}(-J_0(x, T))
\]

It follows that the VWAP call option price can be approximated as

\[
VC(S, v, K, T) \approx e^{-rT} \int_{K}^{\infty} (1 - F_{Y_T,x}(-J_0(x, T)))dx
\]
Numerical Examples: CEV Case

- We suppose that the volatilities are specified as

\[ \sigma(S, t) = \sigma S^{\beta-1}, \quad \gamma(v) = \nu v^{\lambda-1}, \]

where \( \sigma, \beta, \nu, \) and \( \lambda \) are some constants.

- The base-case parameters are set to be \( S = 100, \ K = 100, \ T = 1, \ r(t) = 3.0\%, \ v = 100, \) and \( \rho = 0.3. \)

- Also, we set \( \theta(t) = 10 \) and \( \kappa(t) = 0.1, \) i.e., the long-run average of the trading volume is 100.

- As to the volatilities, we consider (H) high and (L) low volatility cases in which we set \( \sigma S^{\beta-1} = 30\% \) and \( \nu v^{\lambda-1} = 30\% \) for case (H) and \( \sigma S^{\beta-1} = 15\% \) and \( \nu v^{\lambda-1} = 15\% \) for case (L), respectively.
Numerical Examples; Accuracy Check

• We consider two cases; (1) log-normal case ($\beta = 1$ and $\lambda = 1$), and (2) square-root case ($\beta = 0.5$ and $\lambda = 0.5$).

• Figure 1 shows option prices for (L) with short maturity ($T = 0.5$) when (1), whereas Figure 2 depicts for (2).

• Through the numerical experiments, it is observed that the effect of volatility and maturity appears only around ATM ($K = 100$), and the volatility effect is stronger than the maturity effect.

• As to the accuracy of our approximation, we find that the difference between our approximation and the Monte Carlo result are very small.

• The error becomes slightly larger for long maturity and high volatility cases; however, for practical uses, the errors are sufficiently small.
Figure 2 (Square-Root, low vol, $T = 0.5$)
Other Findings; Effect of Correlation

- The effect of correlation gets bigger as $\kappa$, the speed of mean reversion, becomes smaller.
- When $\kappa$ is large, the trading volume $v_t$ sticks around the long-run average so as to behave as if it were uncorrelated to the stock price.
- Stace (2007) sets $\kappa = 100$ under the assumption $\rho = 0$.
- Our result suggests that, when $\kappa = 100$, the impact of correlation on the VWAP call option prices is negligible.
- This result may be an important message for practitioners, because it is in general very difficult to estimate the correlation accurately.
- The effect of correlation gets bigger as the maturity $T$ becomes longer and the volatility $\sigma$ of the asset price becomes larger.
- These results can be understood by the fact that the effect of correlation is bigger as more uncertainty is involved.
Recall that

\[ \sigma(S, t) = \sigma S^{\beta - 1}, \quad \gamma(v) = \nu v^{\lambda - 1} \]

- The effect of \( \beta \) gets bigger as \( \kappa \) becomes smaller, the maturity becomes longer and the asset volatility becomes larger.
- These results can be explained by the exactly same reason as above.
- The effect of \( \lambda \) gets bigger as \( \kappa \) becomes smaller, and has less impact on the others.
- Compared with the impact of the underlying asset price, the maturity as well as the volatility of trading volume has less impact on the VWAP option prices.
Some Extensions

- **Squared OU Model** for Trading Volume as in Novikov et al. (2014).

\[
v_t = X_t^2 + \gamma, \quad dX_t = (\theta - \kappa X_t)dt + \beta dW_t^v,
\]

where \(\theta, \kappa, \gamma\) and \(\beta\) are some constants. In this case, we have

\[
X_t^2 = V_t^2 + 2V_t \tilde{E}(t) \int_0^t \beta E(s)dW_s^v + \tilde{E}^2(t) \left( \int_0^t \beta^2 E^2(s)ds \right) \\
+ 2\tilde{E}^2(t) \left( \int_0^t \beta E(s) \left( \int_s^t \beta E(u)dW_u^v \right) dW_s^v \right)
\]

- **Generalized VWAP** \(M_T = \frac{\int_0^T w_t^1 v_t S_t dt}{\int_0^T w_t^2 v_t dt}\), where \(w_t^i\) is a deterministic function of time \(t\). Consider a **floating-strike VWAP option** defined by

\[
VC(S, v, K, T) = e^{-rT} \mathbb{E}[(M_T - S_T)^+] \]
Conclusion

- In this talk, we develop a unified approximation method for options whose payoff depends on a volume weighted average price (VWAP).
- Compared to the previous works, our method is applicable to the local volatility model, not just for the geometric Brownian motion case.
- Moreover, our method can be used for any special type of VWAP option, including ordinary Asian and Australian options, with fixed-strike, floating-strike, continuously sampled, discretely sampled, forward starting, and in-progress transactions.
- Through numerical examples, we show that the accuracy of the second-order approximation is high enough for practical use.
- Our approximation get slightly worse for long maturity and high volatility case; in such a case, 3rd-order may be required.
Thank You for Your Attention