Exponential integral

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Let X be a smooth algebraic variety over field $\mathbf{k}$, and $f \in \Gamma(\mathcal{O}_X)$ be a function on X. One can treat the pair (X, f) as a generalization of a variety (plain varieties correspond to the case $f = 0$). For the example, if $k = \mathbb{F}_q$ is a finite field of characteristic p, then the generalization of the number of points is the exponential sum

$$\sum_{x \in X(\mathbb{F}_q)} e^{Trace_{\mathbb{F}_q \to \mathbb{F}_p} f(x)\, 2\pi i/p}$$

For $k = \mathbb{C}$ there are Betti and de Rham cohomology theories together with a comparison isomorphism given essentially by *exponential integrals*

$$\int_\gamma e^f \omega$$

Here ω is an algebraic differential form on X killed by the twisted differential $d + df \wedge \cdot$ (e.g. any top-degreee form), and γ is a non-compact closed chain in $X(\mathbb{C})$ which goes to infinity in the directions where $Re(f) \to -\infty$.

I will introduce several other cohomology theories for (X, f) based on theory of vanishing cycles, as well as Hodge filtration on de Rham cohomology whose terms are labeled by rational numbers. Then I'll go to a generalization in which function f is replaced by a closed algebraic 1-form α $(= df)$, and a wall-crossing formalism. This formalism leads to a nice class of resurgent functions.

The wall-crossing picture make sence in some infinite-dimensional situation and allows to define under favorable circumstances a semi-infinite cohomology for (X, α) where X an infinite-dimensional complex manifold and α is a closed holomorphic 1-form on X. An important example is X being the space of C^∞ paths connecting two algebraic Lagrangian submanifolds L_0, L_1 in an algebraic symplectic manifold M, and 1-form α obtained by integration of the symplectic form. The cohomology of (X, α) in this case can be called *holomorphic Floer cohomology* $HF^\bullet_{hol}(L_0, L_1)$ and give rise to a non-perturbative quantization of M. When M is cotangent bundle $M = T^*Y$ the constrtuction can be reduced to Gaiotto-Moore-Neitzke spectral networks (when Y is a curve) and gives a new interpretation of it.

If time permits, I'll discuss examples of complexified and holomorphic Chern-Simons theories.