Free energy and BPS Wilson loop in $\mathcal{N} = 2$ superconformal theories

Arkady Tseytlin

M. Beccaria, AT

• $1/N$ expansion of circular Wilson loop in $\mathcal{N} = 2$ superconformal $SU(N) \times SU(N)$ quiver, 2102.07696

M. Beccaria, G. Dunne, AT

• BPS Wilson loop in $\mathcal{N} = 2$ superconformal $SU(N)$ "orientifold" gauge theory and weak-strong coupling interpolation, 2104.12625

• Strong coupling expansion of free energy and BPS Wilson loop in $\mathcal{N} = 2$ superconformal models with fundamental hypermultiplets, 2105.14729

• $\mathcal{N} = 4$ SYM $\leftrightarrow$ $\mathrm{AdS}_5 \times S^5$ superstring
quantitative understanding of duality
in planar limit based on integrability

N = 4 SYM	**String theory in $\mathbf{AdS^5 \times S^5}$**
Yang-Mills coupling: g_{YM} Number of colors: N	String coupling: g_s String tension: T
Level 1: Exact equivalence	
$g_s = g_{YM}^2/4\pi, \quad T = \sqrt{g_{YM}^2 N}/2\pi$	
Level 2: Equivalence in the 't Hooft limit	
$N \to \infty, \quad \lambda = g_{YM}^2 N$-fixed (planar limit)	$g_s \to 0, \quad T$-fixed (non-interacting strings)
Level 3: Equivalence at strong coupling	
$N \to \infty, \quad \lambda \gg 1$	$g_s \to 0, \quad T \gg 1$

• beyond planar limit?
limited progress: BPS observable $\langle \mathcal{W} \rangle$ from localization matrix model
$\rightarrow$ exact in N, $\lambda = g_{\rm YM}^2 N$
but string loops in $\text{AdS}_5 \times S^5$ are hard

• beyond $\mathcal{N} = 4$ susy:
$\mathcal{N} = 2$ superconformal models planar-equivalent to $\mathcal{N} = 4$ SYM
$\rightarrow$ models to study duality including $1/N$ corrections

• $\mathcal{N} = 2$ localization matrix model not exactly solvable (non-gaussian)
but can extract leading $1/N$ corrections, expand at $\lambda \gg 1$ and
compare to dual string theory (orbifolds/orientifolds of $\text{AdS}_5 \times S^5$)
$\rightarrow$ match some universal string predictions

Plan

- $\mathcal{N}=4$ SYM: free energy F and circular Wilson loop $\langle\mathcal{W}\rangle$ vs string theory

- $\mathcal{N}=2$ superconformal models planar-equivalent to $\mathcal{N}=4$ SYM
    - $SU(N)\times SU(N)$ quiver
    - $SU(N)$ SA-model (symm+antisymm hypers)
    - $SU(N)$ FA-model (fund+antisymm hypers)
    - $Sp(2N)$ FA-model

- $\mathcal{N}=2$ localization matrix model representation for F and $\langle\mathcal{W}\rangle$

- $1/N$ corrections from matrix model: relations between F and $\langle\mathcal{W}\rangle$

- comments on dual string theory interpretation

$SU(N)$ $\mathcal{N}=4$ SYM: free energy on S^4 is scheme dependent

$$F = -\log Z(S^4) = 4\text{a}\log(\Lambda_{\text{UV}}\,\text{r}) + F_0\,, \qquad \text{a} = \tfrac{1}{4}(N^2-1)$$

• localization $\to$ Gaussian matrix model with λ-independent measure

$\to Z(S^4)$ in special scheme (r $=1$)

$$F = -2\text{a}\log\lambda + C(N) = -\tfrac{1}{2}(N^2-1)\log\lambda + C(N)\,, \qquad \lambda = N g^2_{\text{YM}}$$

• AdS/CFT: F on S^4 should match Z_{str} in AdS$_5 \times S^5$

planar (2-sphere) order: $Z_{\text{str}} \sim$ IIB sugra action (+ α'-corrections)

leading term $\sim$ vol(AdS$_5$) $\to$ IR divergent $\sim$ a $\log\Lambda_{\text{IR}}$

[reproduces N^2 part of SYM conformal anomaly [Liu, AT 98; Henningson, Skenderis 98]]

• matching N^2 term in F for special "AdS/CFT motivated" Λ_{IR} [Russo, Zarembo 2012]

[Λ_{IR} in units of L; Λ_{UV} in SYM as limit $\sim \sqrt{\alpha'}$; ratio: $\frac{L}{\sqrt{\alpha'}} = \lambda^{1/4}$]

• $N^2 \to N^2-1$: one-loop (torus) term in $Z_{\text{str}} \sim$ regularized vol(AdS$_5$)

only from short multiplets $\to$ same as 1-loop sugra [Beccaria, AT 2014]

$\frac{1}{2}$ BPS Wilson-Maldacena loop in $SU(N)$ SYM

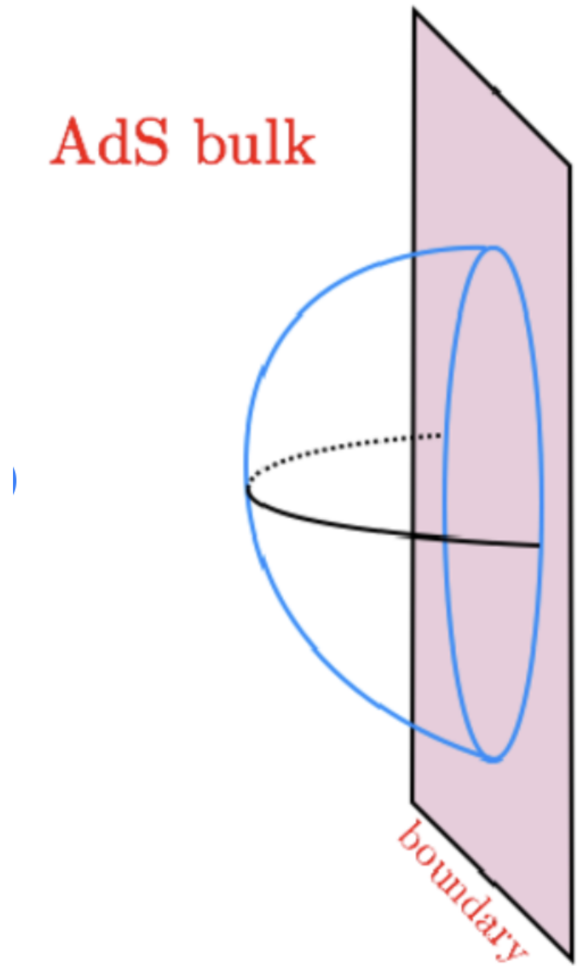
$$W(C) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left[\oint_C d\tau \, \left(iA_\mu(x)\dot{x}^\mu + \Phi_i(x)\theta^i |\dot{x}| \right) \right].$$

• Localization $\rightarrow$ Gaussian matrix model:
exact expression for circular WL [Erickson, Semenoff, Zarembo 00; Drukker, Gross 00; Pestun 07]

$$\mathcal{W} = \mathrm{Tr}\, P e^{\int (iA+\Phi)}, \qquad \langle \mathcal{W} \rangle = e^{\frac{\lambda}{8N}(1-\frac{1}{N})} L^1_{N-1}(-\tfrac{\lambda}{4N})$$

$$\langle \mathcal{W} \rangle_{N\to\infty} = N \tfrac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \overset{\lambda \gg 1}{=} W_0 + ..., \quad W_0 = N\sqrt{\tfrac{2}{\pi}}\, \lambda^{-3/4} e^{\sqrt{\lambda}}$$

• $N \gg 1, \lambda \gg 1$: compare to $AdS_5 \times S^5$ string theory



expansion near AdS_2 minimal surface

disk partition function:

$$\langle \mathcal{W} \rangle = Z_{\rm str} = \tfrac{1}{g_{\rm s}} Z_1 + \mathcal{O}(g_{\rm s})\,, \qquad\qquad Z_1 = \int [dx]...\, e^{-T\int d^2\sigma\, L}$$

$$g_{\rm s} = \frac{g^2_{\rm YM}}{4\pi} = \frac{\lambda}{4\pi N}\,, \qquad T = \frac{L^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}\,, \qquad \lambda = g^2_{\rm YM} N$$

$$\langle \mathcal{W} \rangle = W_0 \Big[1 + \mathcal{O}(T^{-1})\Big] + \mathcal{O}(g_{\rm s})$$

$$W_0 = c_1\, \frac{\sqrt{T}}{g_s}\; e^{2\pi T}\,, \qquad\qquad c_1 = \frac{1}{2\pi}$$

- $e^{2\pi T} = e^{-T{\rm vol}({\rm AdS}_2)}$ – area of minimal surface [Berenstein, Corrado, Fischler, Maldacena 98]
- $c_1 = \frac{1}{2\pi}$ – fluct. det's in Z_1 and measure factor

[Drukker, Gross, AT 00; Kruczenski, Tirziu 08; Buchbinder, AT 14; Medina-Rincon, Zarembo, AT 18]

- $\sqrt{T}$ prefactor – from universal dependence of Z_1 on AdS radius

$$\log Z_{1-{\rm loop}} = -\tfrac{1}{2} \log \frac{[\det(-\nabla^2 + 2)]^3\; [\det(-\nabla^2)]^5}{[\det(-\nabla^2 + \frac{1}{2})]^8}$$

$$\log Z_{1-\mathrm{loop}} = B_2 \log(L\,\Lambda) + \log c_1\ , \qquad\qquad B_2 = \tfrac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\,R^{(2)} = \chi$$

- $B_2 = \zeta_{\rm tot}(0) = \chi = 1-2p$ is universal : for any genus p [Giombi, AT 20]
- 2d UV ∞ canceled by universal string measure contribution $\log(\sqrt{2\pi\alpha'}\Lambda)$

finite part of $\log Z_p$: $\quad -\chi \log\frac{L}{\sqrt{2\pi\alpha'}} = -\chi\log\sqrt{T}$

$$Z_p \sim (\sqrt{T})^{\chi}\ , \qquad\qquad T = \tfrac{L^2}{2\pi\alpha'}$$

- disk with p handles: $g_s^{-1} \to g_s^{\chi}, \quad \sqrt{T} \to (\sqrt{T})^{\chi}, \quad \chi = 1-2p$
- thus expect to find for g_s expansion at large T [Giombi, AT 20]

$$\langle \mathcal{W}\rangle = e^{2\pi T}\sum_{p=0}^{\infty} c_{p+1}\Big(\frac{g_s}{\sqrt{T}}\Big)^{2p-1}\ \Big[1+\mathcal{O}(T^{-1})\Big]$$

- indeed consistent with $1/N,\ \lambda \gg 1$ expansion of exact SYM result

$$\langle \mathcal{W} \rangle = e^{\frac{\lambda}{8N}(1-\frac{1}{N})} L^1_{N-1}(-\tfrac{\lambda}{4N}) = e^{\sqrt{\lambda}} \sum_{p=0}^{\infty} \tfrac{\sqrt{2}}{96^p \sqrt{\pi}\, p!} \frac{\lambda^{\frac{6p-3}{4}}}{N^{2p-1}} \Big[1 + \mathcal{O}(\tfrac{1}{\sqrt{\lambda}})\Big]$$

$$= W_1 \Big(1 + \tfrac{1}{96} \frac{\lambda^{3/2}}{N^2} \big[1 + \mathcal{O}(\tfrac{1}{\sqrt{\lambda}})\big] + \mathcal{O}(\tfrac{1}{N^4})\Big), \qquad W_1 = \langle \mathcal{W} \rangle_{N\to\infty} = W_0 + ...$$

- leading large λ corrections at each order in $1/N^2$ exponentiate:

$$\langle \mathcal{W} \rangle = W_0\, e^H \Big[1 + \mathcal{O}(T^{-1})\Big], \qquad W_0 = \frac{1}{2\pi} \frac{\sqrt{T}}{g_s} e^{2\pi T}, \qquad H \equiv \frac{\pi}{12} \frac{g_s^2}{T}$$

- handle insertion operator $\to \exp H$: "dilute handle gas" approx.

• structure should be universal:
should apply to circular WL in string theories in $\mathrm{AdS}_5 \times M^5$
as AdS_2 minimal surface lies in AdS_5
(also applies to $\mathrm{AdS}_3 \times S^3 \times T^4$ string dual to ABJM [Giombi, AT 20])

• find that indeed in $\mathcal{N} = 2$ models planar-equivalent to $\mathcal{N} = 4$ SYM
which are dual to orbifolds of $\mathrm{AdS}_5 \times S^5$ superstring

$$\langle \mathcal{W} \rangle = \langle \mathcal{W} \rangle_{\rm SYM} \Big(1 + c \frac{\lambda^{3/2}}{N^2} \big[1 + \mathcal{O}(\tfrac{1}{\sqrt{\lambda}}) \big] + \mathcal{O}(\tfrac{1}{N^4}) \Big),$$
$$\frac{\lambda^{3/2}}{N^2} \sim \frac{g_{\rm s}^2}{T}$$

$\mathcal{N}=2$ superconformal models planar-equivalent to $\mathcal{N}=4$ SYM

- conformal invariance of $SU(N)$ $\mathcal{N}=2$ model with hypers in adjoint, fundamental, rank-2 symm, and rank-2 antisymm reps [Koh, Rajpoot 83; Howe, Stelle, West 83]

$$SU(N): \qquad \beta_1 = 2N - 2N\, n_{\mathrm{Adj}} - n_{\mathrm{F}} - (N+2)\, n_{\mathrm{S}} - (N-2)\, n_{\mathrm{A}} = 0$$

  - $n_{\mathrm{Adj}} = 1$: $\quad n_{\mathrm{F}} = n_{\mathrm{A}} = n_{\mathrm{S}} = 0 \;\rightarrow \mathcal{N}=4$ SYM
  - $n_{\mathrm{Adj}} = 0$: $\quad n_{\mathrm{F}} = 2N - (N+2)\, n_{\mathrm{S}} - (N-2)\, n_{\mathrm{A}}$

- planar equivalence with $\mathcal{N}=4$ SYM ($\rightarrow$ simple AdS dual):
  - $n_{\mathrm{F}} = 2N, \;\; n_{\mathrm{S}} = n_{\mathrm{A}} = 0 \;\rightarrow\;\; SU(N) \times SU(N)$ quiver ($\beta_1 = 0$ for $\lambda_{1,2}$)
  - n_{F} not depending on N $\rightarrow n_{\mathrm{S}} + n_{\mathrm{A}} = 2$:

  only two solutions: SA= symm+antisymm and FA= fund+antisymm

$$\mathrm{SA}: \quad (n_{\mathrm{F}}, n_{\mathrm{S}}, n_{\mathrm{A}}) = (0,1,1)\,, \qquad \mathrm{FA}: \quad (n_{\mathrm{F}}, n_{\mathrm{S}}, n_{\mathrm{A}}) = (4,0,2)$$

- SA and FA $\mathcal{N}=2$ theories dual to certain orbifold/orientifold projections of $\mathrm{AdS}_5 \times S^5$ string [Ennes, Lozano, Naculich, Schnitzer 2000]

• conformal anomaly a and c coeffs of $\mathcal{N} = 2$ models
= free-theory values fixed by numbers of hypers

$$\mathrm{a} = \tfrac{5}{24}\,\mathrm{n_v} + \tfrac{1}{24}\,\mathrm{n_h}\,, \qquad \mathrm{c} = \tfrac{1}{6}\,\mathrm{n_v} + \tfrac{1}{12}\,\mathrm{n_h}$$

$SU(N) \times SU(N)$ quiver: (same as $2\times$ SA)

$$\mathrm{a} = \tfrac{1}{2}N^2 - \tfrac{5}{12}\,, \qquad \mathrm{c} = \tfrac{1}{2}N^2 - \tfrac{1}{3}$$

$SU(N)$	a	c
$\mathcal{N} = 4$ SYM	$\frac{1}{4}N^2 - \frac{1}{4}$	$\frac{1}{4}N^2 - \frac{1}{4}$
$\mathcal{N} = 2$ SA	$\frac{1}{4}N^2 - \frac{5}{24}$	$\frac{1}{4}N^2 - \frac{1}{6}$
$\mathcal{N} = 2$ FA	$\frac{1}{4}N^2 + \frac{1}{8}N - \frac{5}{24}$	$\frac{1}{4}N^2 + \frac{1}{4}N - \frac{1}{6}$

- $\mathcal{N} = 2\ Sp(2N)$ models

$Sp(2N)$ =compact symplectic group $\equiv USp(2N) = U(2N) \cap Sp(2N, C)$
$\dim \mathrm{Adj} = N(2N+1),\ \dim \mathrm{F} = 2N,\ \dim \mathrm{A} = N(2N-1)-1.$
conformal invariance condition:

$$Sp(2N): \qquad \beta_1 = 2N + 2 - (2N+2)\, n_{\mathrm{Adj}} - n_{\mathrm{F}} - (2N-2)\, n_{\mathrm{A}} = 0$$

- $n_{\mathrm{Adj}} = 1$: $\quad n_{\mathrm{F}} = n_{\mathrm{A}} = 0 \ \to \mathcal{N} = 4$ SYM
- $n_{\mathrm{Adj}} = 0$: $\quad$ planar equivalence to $\mathcal{N} = 4$ SYM $\ \to n_{\mathrm{F}} = 2N + 2$
  or n_{F} independent of N: $\quad n_{\mathrm{F}} = 4,\ n_{\mathrm{A}} = 1 -$ FA model

$$Sp(2N) \quad \mathrm{FA}: \qquad (n_{\mathrm{F}}, n_{\mathrm{A}}) = (4, 1)$$

$Sp(2N)$	a	c
$\mathcal{N} = 4$ SYM	$\frac{1}{2}N^2 + \frac{1}{4}N$	$\frac{1}{2}N^2 + \frac{1}{4}N$
$\mathcal{N} = 2$ FA	$\frac{1}{2}N^2 + \frac{1}{2}N - \frac{1}{24}$	$\frac{1}{2}N^2 + \frac{3}{4}N - \frac{1}{12}$

$\mathcal{N}=2$ localization matrix model

- free energy of superconformal model on S^4 of radius r

$$\hat{F} = -\log \hat{Z} = 4\mathrm{a} \log(\Lambda\, \mathrm{r}) + F(\lambda, N)$$

renormalized F depends on subtraction scheme

- localization matrix model for $\mathcal{N}=2$ model on S^4 [Pestun 2007]

$$Z = e^{-F} = \int Da\, e^{-S_0(a)}\, \mathbb{Z}_{\text{1-loop}}(a)\,, \qquad \langle \mathcal{W} \rangle = \langle \mathrm{Tr}\, e^{2\pi \mathrm{r}\, a} \rangle$$

$$S_0 = \frac{8\pi^2 N\, \mathrm{r}^2}{\lambda} \mathrm{Tr}\, a^2\,, \qquad \mathbb{Z}_{\text{1-loop}}(a) = e^{-S_{\text{int}}(a)}$$

a= const scalar matrix: $\int_{S^4} \frac{1}{6} R\, \mathrm{Tr}\, \phi^2 \to \mathrm{Tr}\, a^2$; $\quad \lambda = g^2_{\text{YM}} N$ only in S_0

- $\mathcal{N}=4$ SYM: $\quad \mathbb{Z}_{\text{1-loop}} = 1$: Gaussian, λ-independent measure $\to$

$$Z_{\text{SYM}} = C(N) \left(\frac{N\mathrm{r}^2}{\lambda}\right)^{-\frac{1}{2}\dim G}, \quad F_{\text{SYM}} = -2\mathrm{a} \log(\lambda \mathrm{r}^{-2}) + f(N)\,, \quad \mathrm{a} = \tfrac{1}{4} \dim G$$

for $G = SU(N)$ in this special scheme ($\mathrm{r}=1$)

$$F_{\text{SYM}} = -2\mathrm{a} \log \lambda + f(N) = -\tfrac{1}{2}(N^2-1) \log \lambda + f(N)$$

• $\mathcal{N}=2$ model with hypers in $R=\oplus R_i$:   (ignore instantons in $1/N$)
$\hat{\mathbb{Z}}_{\text{1-loop}}$: 1-loop dets on S^4 in $a=$const backgr. – depend on r (no λ)

$$\hat{\mathbb{Z}}_{\text{1-loop}}(a,\mathrm{r})=\prod_{n=1}^{\infty}\Big(\frac{\prod_{\alpha\in\mathrm{roots}(\mathfrak{g})}\big[\mathrm{r}^{-2}n^2+(\alpha\cdot a)^2\big]}{\prod_{w\in\mathrm{weights}(R)}\big[\mathrm{r}^{-2}n^2+(w\cdot a)^2\big]}\Big)^n$$

• $\prod_{\text{roots}}$ includes "massless" contributions: Cartan directions $\alpha\cdot a=0$ for which a-dependence and r dependence not correlated

• regularized value of $\hat{\mathbb{Z}}$ [Pestun 2007]

$$\mathbb{Z}_{\text{1-loop}}(a\,\mathrm{r})=e^{-S_{\text{int}}(a\,\mathrm{r})}=\frac{\prod_{\alpha\in\mathrm{roots}(\mathfrak{g})}\mathrm{H}(i\,\alpha\cdot a\,\mathrm{r})}{\prod_{w\in\mathrm{weights}(R)}\mathrm{H}(i\,w\cdot a\,\mathrm{r})}$$

$\mathrm{H}(x)\equiv\mathrm{G}(1+x)\,\mathrm{G}(1-x)$ – product of Barnes G-functions
• no "massless" contributions as $\mathrm{H}(0)=1$
$\to Z$ depends on r as in $\mathcal{N}=4$ SYM
• $\mathcal{N}=2$ free energy: coeff. of $\log\lambda$ in regularized matrix integral
not full a-anomaly coefficient beyond planar limit

- to get full conformal anomaly go back to unregularized expression

$$\prod_{n=1}^{\infty}\left(\mathrm{r}^{-2}n^2+\mu^2\right)^n=\prod_{n=1}^{\infty}\mathrm{r}^{-2n}\ \prod_{n=1}^{\infty}\left(n^2+\mathrm{r}^2\mu^2\right)^n\,,\quad \mu=\alpha\cdot a\ \text{ or }\ w\cdot a$$

$$\prod_{n=1}^{\infty}\mathrm{r}^{-2n}=e^{-2\zeta(-1)\log\mathrm{r}}=e^{\frac{1}{6}\log\mathrm{r}}$$

$$\hat{\mathbb{Z}}_{\text{1-loop}}(a,\mathrm{r})\ \to\ e^{\frac{1}{6}(\dim G-\dim R)\log\mathrm{r}}\,\mathbb{Z}_{\text{1-loop}}(a\,\mathrm{r})$$

- after $\mathrm{r}\,a\to a$ find total dependence on r

$$F=\left[\dim G-\tfrac{1}{6}(\dim G-\dim R)\right]\log\mathrm{r}+\ldots=4\mathrm{a}\log\mathrm{r}+\ldots$$

$$\mathrm{a}=\tfrac{5}{24}\dim G+\tfrac{1}{24}\dim R$$

- "bare" matrix model integral reproduces full a-anomaly term in F
- correlation between r and λ only in Gaussian part of matrix integral

Matrix model for $\mathcal{N}=2$ $SU(N)$ SA and FA models

$N \times N$ hermitian traceless matrix a (eigenvalues $\{a\}_{r=1}^{N}$)

$$Z \equiv e^{-F} = \int \mathcal{D}a \, e^{-S_0(a) - S_{\text{int}}(a)}\,, \qquad S_0(a) = \frac{8\pi^2 N}{\lambda} \operatorname{Tr} a^2, \qquad \lambda = g_{\text{YM}}^2 N$$

$$\mathcal{D}a \equiv \prod_{r=1}^{N} da_r \, \delta\Big(\sum_{s=1}^{N} a_s\Big) \, \big[\Delta(a)\big]^2\,, \qquad \Delta(a) = \prod_{1 \le r < s \le N} (a_r - a_s)$$

- $\mathcal{N}=2$ model with $n_{\text{F}}, n_{\text{S}}, n_{\text{A}}$ hypers [Beccaria, Billo, Galvagno, Hasan, Lerda 20]

$$S_{\text{int}}(a) = \sum_{r=1}^{N} \Big[n_{\text{F}} \log H(a_r) + n_{\text{S}} \log H(2a_r) \Big]$$
$$+ \sum_{r<s=1}^{N} \Big[(n_{\text{S}} + n_{\text{A}}) \log H(a_r + a_s) - 2 \log H(a_r - a_s) \Big]$$

$$H(x) = \prod_{n=1}^{\infty} \Big(1 + \frac{x^2}{n^2}\Big)^n e^{-\frac{x^2}{n}} = e^{-(1+\gamma_{\text{E}})\, x^2}\, \mathrm{G}(1+ix)\, \mathrm{G}(1-ix)$$

$$\log H(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \zeta_{2n+1}\, x^{2n+2}$$

$SU(N)$ SA model: $n_{\rm S} = n_{\rm A} = 1,\; n_{\rm F} = 0 \qquad$ (after $a \to \sqrt{\frac{\lambda}{N}}\,\hat{a}$)

$$S_{\rm int}(a) = \sum_{i,j=1}^{N} \Big[\log H(a_i + a_j) - \log H(a_i - a_j)\Big]$$

$$= \sum_{i,j=1}^{\infty} C_{ij}(\lambda)\,\mathrm{Tr}\,\Big(\frac{\hat{a}}{\sqrt{N}}\Big)^{2i+1}\mathrm{Tr}\,\Big(\frac{\hat{a}}{\sqrt{N}}\Big)^{2j+1}$$

$$C_{ij}(\lambda) = \Big(\frac{\lambda}{8\pi^2}\Big)^{i+j+1}\zeta_{2i+2j+1}\,\frac{4(-1)^{i+j}\,\Gamma(2i+2j+2)}{\Gamma(2i+2)\,\Gamma(2j+2)}$$

- WL defined by $\mathcal{N}=2$ vector multiplet fields: $\langle \mathcal{W}\rangle = \langle \mathrm{Tr}\, e^{2\pi a}\rangle$
- free energy and WL: leading $1/N^2$ correction

$$\langle e^{-S_{\rm int}}\rangle_0 = \frac{Z_{\rm SA}}{Z_{\rm SYM}} = e^{-\Delta F}\,, \qquad \Delta F(\lambda; N) = F_{\rm SA} - F_{\rm SYM}$$

$$\frac{\langle \mathcal{W}\rangle_{\rm SA}}{\langle \mathcal{W}\rangle_{\rm SYM}} = \frac{\langle e^{-S_{\rm int}}\,\mathrm{Tr}\,e^{2\pi a}\rangle_0}{\langle e^{-S_{\rm int}}\rangle_0\,\langle \mathrm{Tr}\,e^{2\pi a}\rangle_0} = 1 + \frac{1}{N^2}\Delta q(\lambda) + \mathcal{O}\Big(\frac{1}{N^4}\Big)$$

$$\langle \mathcal{W}\rangle_{\rm SYM} = \langle \mathrm{Tr}\,e^{2\pi a}\rangle_0\,, \qquad \langle \ldots\rangle_0 \equiv \int Da\, e^{-S_0(a)}\ldots$$

- non-planar correction from "connected" part of $\langle e^{-S_{\rm int}}\,{\rm Tr}\, e^{2\pi a}\rangle_0$

$$\langle e^{-S_{\rm int}}\,{\rm Tr}\, e^{2\pi a}\rangle_0 = N\langle e^{-S_{\rm int}}\rangle_0 + 2\pi^2\langle e^{-S_{\rm int}}\,{\rm Tr}\, a^2\rangle_0 + ...$$

- insertion of ${\rm Tr}\, a^2 = \frac{\lambda}{8\pi^2 N} S_0$ same as $\frac{d}{d\lambda} Z$ [Beccaria, Dunne, AT 2021]

key relation between $\langle \mathcal{W}\rangle$ and F:

$$\Delta q = -\tfrac{1}{4}\lambda^2 \frac{d}{d\lambda}\Delta F(\lambda)$$

$$\Delta F(\lambda) = \lim_{N\to\infty} \Delta F(\lambda; N)$$

- main task is to compute $1/N^2$ correction $\Delta F(\lambda)$

$SU(N)$ FA model $(n_{\mathrm{F}} = 4,\, n_{\mathrm{S}} = 0,\, n_{\mathrm{A}} = 2)$:

$$S_{\mathrm{int}}(a) = 4\sum_{i=1}^{N} \log H(a_i) + \sum_{i,j=1}^{N} \Big[\, \log H(a_i + a_j) - \log H(a_i - a_j)\Big]$$

$$= \sum_{i=1}^{\infty} B_i(\lambda)\ \mathrm{Tr}\,\Big(\frac{\hat{a}}{\sqrt{N}}\Big)^{2i+2} + \sum_{i,j=1}^{\infty} C_{ij}(\lambda)\ \mathrm{Tr}\,\Big(\frac{\hat{a}}{\sqrt{N}}\Big)^{2i+1}\mathrm{Tr}\,\Big(\frac{\hat{a}}{\sqrt{N}}\Big)^{2j+1}$$

$$B_i(\lambda) = 4\,\big(\tfrac{\lambda}{8\pi^2}\big)^{i+1}\tfrac{(-1)^i}{i+1}\zeta_{2i+1}(1 - 2^{2i})$$

$$\Delta F(\lambda; N) = F_{\mathrm{FA}} - F_{\mathrm{SYM}} = N\,F_1(\lambda) + F_2(\lambda) + \mathcal{O}(\tfrac{1}{N})$$

- novel order N term due to $n_{\mathrm{F}} \neq 0$ from single Tr term
- WL defined by $\mathcal{N} = 2$ vector multiplet fields: $\langle \mathcal{W} \rangle = \langle \mathrm{Tr}\, e^{2\pi a} \rangle$

$$\langle \mathcal{W} \rangle = N\,W_0(\lambda) + W_1(\lambda) + \tfrac{1}{N}\big[W_{0,2}(\lambda) + W_2(\lambda)\big] + \mathcal{O}(\tfrac{1}{N^2})$$

$Sp(2N)$ FA model $(n_{\text{Adj}} = 0, n_{\text{A}} = 1, n_{\text{F}} = 4)$:

much simpler than $SU(N)$ one:

only single Tr term in S_{int} [Fiol, Martinez-Montoya, Fukelman 2000]

$$S_{\text{int}}(\hat{a}) = \sum_{i=1}^{\infty} B_i(\lambda)\,\text{Tr}\,\Big(\frac{\hat{a}}{\sqrt{N}}\Big)^{2i+2}$$

- $\Delta F = F_{\text{FA}} - F_{\text{SYM}}$ in terms of F_1 and its derivatives only (!): [Beccaria, Dunne, AT 21]

$$\Delta F(\lambda; N) = N\,\mathsf{F}_1(\lambda) + \mathsf{F}_2(\lambda) + \tfrac{1}{N}\mathsf{F}_3(\lambda) + \ldots$$

$$\mathsf{F}_1(\lambda) = 2F_1(\lambda)$$

$$\mathsf{F}_2(\lambda) = \tfrac{1}{2}\tfrac{d}{d\lambda}\big[\lambda F_1(\lambda)\big] + 2\,\widetilde{F}_2(\lambda)\,, \qquad \tfrac{d}{d\lambda}\widetilde{F}_2 = -\tfrac{\lambda}{2}\Big[\tfrac{d^2}{d\lambda^2}(\lambda\,F_1)\Big]^2$$

- $\langle \mathcal{W} \rangle = NW_0 + W_1 + \frac{1}{N}W_2 + \ldots$

also expressed in terms of derivatives of F_1 only

Aim: leading $\frac{1}{N^2}$ correction to WL

$$\frac{\langle \mathcal{W} \rangle}{\langle \mathcal{W} \rangle_0} = 1 + \frac{1}{N^2}\, q(\lambda) + \mathcal{O}\big(\frac{1}{N^4}\big), \qquad \langle \mathcal{W} \rangle_0 = \frac{2N}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

• normalized to planar one = $SU(N)$ SYM one (expansion of Laguerre)

$$q^{\rm SYM}(\lambda) = \frac{\lambda}{96}\Big[\frac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} - 12\Big] = \begin{cases} -\frac{1}{8}\lambda + \frac{1}{384}\lambda^2 + \mathcal{O}(\lambda^3), & \lambda \ll 1, \\ \frac{1}{96}\lambda^{3/2} - \frac{9}{64}\lambda + \frac{1}{256}\lambda^{1/2} + \mathcal{O}(1), & \lambda \gg 1\ . \end{cases}$$

$$\frac{1}{N^2}\, q_{\rm SYM}(\lambda) \ \stackrel{\lambda \gg 1}{=}\ k_0 \frac{\lambda^{3/2}}{N^2} \sim \frac{g_s^2}{T}\,, \qquad k_0 = \tfrac{1}{96}$$

• will confirm string-theory prediction

$$q(\lambda) \ \stackrel{\lambda \gg 1}{=}\ k_1\, \lambda^{3/2} + \mathcal{O}(\lambda)$$

$SU(N) \times SU(N)$ quiver (Z_2 orbifold of SYM)

• from matrix model : $Z_{\rm orb} = e^{-F}$ and $\langle \mathcal{W} \rangle_{\rm orb} \ \to$

$$\Delta q \equiv q_{\rm orb}(\lambda) - q_{\rm SYM}(\lambda) = -\tfrac{1}{8}\lambda^2 \tfrac{d}{d\lambda}\, \Delta F(\lambda)$$

$$\Delta F(\lambda) \equiv \big[F_{\rm orb}(\lambda; N) - 2\, F_{\rm SYM}(\lambda; N) \big]_{N\to\infty} = -\log \frac{Z_{\rm orb}(\lambda;N)}{[Z_{\rm SYM}(\lambda;N)]^2}\Big|_{N\to\infty}$$

Matrix model results:

• weak coupling

$$\Delta q(\lambda) \overset{\lambda \ll 1}{=} -\tfrac{3}{4}\zeta_3 \left(\tfrac{\lambda}{8\pi^2}\right)^3 + \tfrac{45}{8}\zeta_5 \left(\tfrac{\lambda}{8\pi^2}\right)^4 + \left[\tfrac{9}{2}\zeta_3^2 - \tfrac{315}{8}\zeta_7\right] \left(\tfrac{\lambda}{8\pi^2}\right)^5 + \mathcal{O}(\lambda^6)$$

• strong-coupling expansion of $q_{\rm orb}(\lambda)$ requires resummation

• numerical approach [Beccaria, AT 21]

$$q_{\rm orb}(\lambda) \overset{\lambda \gg 1}{=} k_1\, \lambda^\eta \left[1 + a_1 \lambda^{-1/2} + \dots\right],$$

$$\eta \sim 1.5\,, \qquad k_1 \simeq -0.005\,, \qquad a_1 \simeq 15$$

consistent with string theory prediction $q \sim \lambda^{3/2}$

• $\Delta q_{\rm orb}(\lambda) = q_{\rm orb} - q_{\rm SYM} = -\frac{\lambda^2}{8}\frac{d}{d\lambda}\Delta F_{\rm orb}(\lambda)$

$$\Delta F_{\rm orb}(\lambda) \overset{\lambda\gg 1}{=} c_1\,\lambda^{1/2} + \cdots\,, \qquad c_1 = -16 k_1 \simeq 0.08$$

• analytic approach to strong coupling expansion: [Beccaria, Korchemsky 22]
$\Delta F_{\rm orb}$ as det of semi-infinite matrix [Galvagno, Preti 2020; Billo, Frau, Galvagno, Lerda, Pini 2021]

$$\Delta F_{\rm orb}(\lambda) = \tfrac{1}{2}\log\det\Big[(1+M^+)(1+M^-)\Big]$$

$$M^+_{nm} = 8\,(-1)^{n+m}\sqrt{2n}\sqrt{2m}\int_0^\infty \frac{dt}{t}\,\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\,J_{2n}(t\sqrt{\lambda})\,J_{2m}(t\sqrt{\lambda}), \quad n,m>0$$

$$M^-_{nm} = 8\,(-1)^{n+m}\sqrt{2n+1}\sqrt{2m+1}\int_0^\infty \frac{dt}{t}\,\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\,J_{2n+1}(t\sqrt{\lambda})\,J_{2m+1}(t\sqrt{\lambda})$$

• such Bessel integrals appear also in BES equation for cusp anomaly and in octagon correlator $\langle {\rm Tr}\,(Z^{k/2}\bar{X}^{k/2})(x_1)\,{\rm Tr}\,X^k(x_2)\,{\rm Tr}\,(Z^{k/2}\bar{X}^{k/2})(x_3)\,{\rm Tr}\,\bar{Z}^k(x_4)\rangle$
• similar analysis of det as for octagon case [Belitsky, Korchemsky 20]

- exact analytic results for coefficients [Beccaria, Korchemsky 22]

$$\Delta F_{\rm orb}(\lambda) \overset{\lambda\gg 1}{=} \tfrac{1}{4}\sqrt{\lambda} - \log\sqrt{\lambda} + n_0 + \tfrac{1}{\sqrt{\lambda}} n_1 + \ldots$$

$$F_{\rm orb} = F_{{\rm SYM}\ N\to\infty} + \Delta F_{\rm orb} \overset{\lambda\gg 1}{=} -N^2 \log\lambda + \tfrac{1}{4}\sqrt{\lambda} - \tfrac{1}{2}\log\lambda + \ldots + \mathcal{O}(\tfrac{1}{N^2})$$

$$\Delta q_{\rm orb} = -\tfrac{1}{8}\lambda^2 \tfrac{d}{d\lambda}\Delta F_{\rm orb} \overset{\lambda\gg 1}{=} -\tfrac{1}{64}\lambda^{3/2} + \tfrac{1}{16}\lambda + \ldots$$

$$\langle \mathcal{W}\rangle_{\rm orb} = \langle \mathcal{W}\rangle_0 \big[1 + \tfrac{1}{N^2} q_{\rm orb}(\lambda) + \ldots\big]$$

$$q_{\rm orb} = q_{\rm SYM} + \Delta q_{\rm orb} \overset{\lambda\gg 1}{=} -\tfrac{1}{192}\lambda^{3/2} - \tfrac{5}{64}\lambda + \ldots$$

- $\mathcal{O}(N^0)$ terms: predictions for string theory coeffs.:

"sphere with handle" (torus) contribution to $F_{\rm orb}$

"disk with handle" contribution to $\langle \mathcal{W}\rangle_{\rm orb}$

String theory interpretation

• $F \sim Z_{\rm str}$: presumably related to type IIB string effective action

$$S = S_0 + S_1 + ... = \frac{1}{(2\pi)^7 g_s^2 \alpha'^4} \int d^{10}x \sqrt{G} \Big[(R + ...) + \alpha'^3 R^4 + ... \Big] + \mathcal{O}(g_s^0) + ...$$

• $\mathcal{N} = 4$ SYM dual to $AdS_5 \times S^5$ string: tree (sugra) term $\to \frac{1}{\pi^2} N^2 V_{AdS_5}$ in "AdS/CFT motivated" IR regularization $V_{AdS_5} \to -\pi^2 \log \sqrt{\lambda}$ same as in matrix model scheme $\to -\frac{1}{2} N^2 \log \lambda$ term in $F_{\rm SYM}$ [Russo, Zarembo 12]

• maximal susy case: tree level ($\alpha'^3 R^4$ etc.) and loop corrections to F should vanish apart from 1-loop (torus) contribution: $N^2 \to N^2 - 1$

• $\mathcal{N} = 2$ orbifold theory dual to $AdS_5 \times (S^5/\mathbb{Z}_2)$: from matrix model

$$F_{\rm orb}(\lambda; N) \stackrel{\lambda \gg 1}{=} -N^2 \log \lambda + \Big[c_1 \lambda^{1/2} + c_2 \log \lambda + \mathcal{O}(\tfrac{1}{\sqrt{\lambda}}) \Big] + \mathcal{O}\big(\tfrac{1}{N^2}\big)$$

• N^2 from planar equivalence: $F = -2{\rm a} \log \lambda + \mathcal{O}(N^0)$
$SU(2N)$ SYM $\to SU(N) \times SU(N)$ with 2 bi-fund hypers: ${\rm a} = \frac{1}{2} N^2 - \frac{5}{12}$

• extra 2 also from IIB supergravity action on $\mathrm{AdS}_5 \times (S^5/\mathbb{Z}_2)$:
$N^2 \to (2N)^2$ and $\mathrm{vol}(S^5/\mathbb{Z}_2) = \frac{1}{2}\mathrm{vol}(S^5)$
• planar equivalence: tree level α'-corrections vanish on $\mathrm{AdS}_5 \times (S^5/\mathbb{Z}_2)$

• subleading $\lambda^{1/2}$ term:
from 1-loop ($g_s^0 \sim N^0$) term in IIB eff. action?
$S_1 \sim \frac{1}{\alpha'} \int d^{10}x \sqrt{G}\, R^4 + ...$
on dimensional grounds $S_1 \sim \frac{L^2}{\alpha'} \sim \lambda^{1/2} \to \Delta F \sim \lambda^{1/2}$
$\neq 0$ on $\mathrm{AdS}_5 \times (S^5/\mathbb{Z}_2)$?
localized contribution due to curvature singularity?

• puzzle: why 1-loop R^4 contributes to F_{orb}
while tree-level R^4 does not (both have same structure in type IIB theory)
• possible resolution:
$\lambda^{1/2}$ term comes from $Z_{\mathrm{str}}(\mathrm{torus})$ and not from low-energy eff. action?

$SU(N)$ SA model [Beccaria, Dunne, AT 21]

• $\mathcal{N}=2$ SA: $n_{\rm S}=n_{\rm A}=1 \quad \to$ planar-equivalent to $\mathcal{N}=4$ SYM:
e.g. $\quad \mathrm{a}=\frac{1}{4}N^2-\frac{5}{24},\ \mathrm{c}=\frac{1}{4}N^2-\frac{1}{6} \quad$ vs $\quad \mathrm{a}=\mathrm{c}=\frac{1}{4}N^2-\frac{1}{4}$
"orientifold" of orbifold of $SU(N)\times SU(N)$ SYM:
$f_{ij'}$ vs $s_{(ij)}+a_{[ij]}$ [Billo et al 21]

• string dual of $\mathcal{N}=2$ SA model: [Park, Rabadan, Uranga 99; Ennes, Lozano, Naculich, Schnitzer 00]
IIB string on orientifold $AdS_5\times S^5/G_{\rm orient}, \quad G_{\rm orient}=(\mathbb{Z}_2)_{\rm orb}\times(\mathbb{Z}_2)_{\rm orient}$
$(\mathbb{Z}_2)_{\rm orient}$: inversions in 2 directions $\perp$ D3-branes $\times$ [w-sh parity Ω] $\times(-1)^{F_L}$
• $S'^5=S^5/G_{\rm orient}$: special identifications of angles

$$ds_5'^2=d\theta_1^2+\sin^2\theta_1\,d\phi_3^2+\cos^2\theta_1\,(d\theta_2^2+\sin^2\theta_2\,d\phi_2^2+\cos^2\theta_2\,d\phi_1^2)$$

$$\theta_1\equiv\theta_1+\tfrac{\pi}{2},\ \theta_2\equiv\theta_2+\tfrac{\pi}{2},\quad \phi_1\equiv\phi_1+\tfrac{\pi}{2},\quad \phi_2\equiv\phi_2-\tfrac{\pi}{2},\quad \phi_3\equiv\phi_3+\pi$$

• $Z_{\rm str}$ expanded near AdS$_2$ in AdS$_5$: UV ∞ of 1-loop dets not sensitive to global identifications S'^5 $\to$ universal $g_{\rm s}$ and $T \gg 1$ expansion of $\langle W \rangle$

$$\frac{\langle \mathcal{W} \rangle_{\rm SA}}{\langle \mathcal{W} \rangle_{\rm SYM}} = 1 + \frac{1}{N^2}\,\Delta q(\lambda) + \mathcal{O}\big(\tfrac{1}{N^4}\big)\,, \qquad \Delta q(\lambda \gg 1) \sim \lambda^{3/2}$$

Matrix model results

$$\Delta q = -\tfrac{1}{4}\lambda^2 \tfrac{d}{d\lambda}\Delta F(\lambda)\,, \qquad \Delta F(\lambda) = \lim_{N\to\infty} \Delta F(\lambda; N)$$

• like in orbifold model $\quad (M = M^-)$

$$\Delta F(\lambda) = \tfrac{1}{2} \log\det(1+M)$$

$$M_{mn} = 8\,(-1)^{m+n}\sqrt{2n+1}\sqrt{2m+1}\int_0^\infty \frac{dt}{t}\,\frac{e^{2\pi t}}{(e^{2\pi t}-1)^2}\,J_{2n+1}(t\sqrt{\lambda})\,J_{2m+1}(t\sqrt{\lambda})$$

Strong coupling expansion:

- approximate results confirmed $\Delta q \sim \lambda^{3/2}$ [Beccaria, Dunne, AT 2021]
- exact form of $\lambda \gg 1$ expansion [Beccaria, Korchemsky 2022]

$$\Delta F = \tfrac{1}{8}\sqrt{\lambda} - \tfrac{3}{8}\log\lambda + k_0 + k_1\lambda^{1/2} + \ldots$$

$$F_{\rm SA}(\lambda; N) \overset{N,\lambda\gg 1}{=} -\tfrac{1}{2}N^2\log\lambda + \tfrac{1}{8}\lambda^{1/2} - \tfrac{3}{8}\log\lambda + k_0 + \ldots + \mathcal{O}\big(\tfrac{1}{N^2}\big)$$

$$\Delta q = -\tfrac{1}{64}\lambda^{3/2} + \tfrac{3}{32}\lambda + \tfrac{1}{8}k_1\lambda^{1/2} + \ldots$$

$$q_{\rm SA} = q_{\rm SYM} + \Delta q_{\rm SA} \overset{\lambda\gg 1}{=} -\tfrac{1}{192}\lambda^{3/2} - \tfrac{3}{64}\lambda + \ldots$$

- Remarks:
  - coeff of $\log\lambda$ is not conf anomaly $\mathrm{a} = \frac{1}{4}N^2 - \frac{5}{24}$ beyond planar limit
  - leading term in $q_{\rm SA} = -\frac{1}{192}\lambda^{3/2} + \ldots$ as in $SU(N)\times SU(N)$ model
  - expansion in even powers of $1/N$

  (despite crosscup contributions expected for orientifold on string side?)

$\mathcal{N} = 2$ models with fundamental hypers [Beccaria, Dunne, AT 2021]

• $SU(N)$ FA ($n_\mathrm{F} = 4$, $n_\mathrm{A} = 2$) and $Sp(2N)$ FA ($n_\mathrm{F} = 4$, $n_\mathrm{A} = 1$)

• realised on N D3-branes with few D7-branes and O7-plane:
dual string theories – particular orbifolds/orientifolds of $\mathrm{AdS}_5 \times S^5$ superstring

• $1/N$ expansion of F and $\langle \mathcal{W} \rangle$ at large λ from matrix model:
structure of expansion now different: odd and even powers of $1/N$

• $Sp(2N)$ case much simpler – novel features:

- get resummed expressions for $\lambda \gg 1$ terms at each order in $1/N$
- find exponentially suppressed at $\lambda \gg 1$ terms in $1/N$ expansion

$SU(N)$ FA model

$$F(\lambda) = F_{\text{SYM}}(\lambda) + N\,F_1(\lambda) + F_2(\lambda) + \mathcal{O}(\tfrac{1}{N})\,, \qquad F_{\text{SYM}} = -\tfrac{1}{2}(N^2-1)\log\lambda$$

F_1 (absent in SA case) has explicit form:

$$F_1(\lambda) = \tfrac{2}{\sqrt{\lambda}}\int_0^\infty \frac{dt}{t^2}\,\frac{e^{2\pi t}}{(e^{2\pi t}+1)^2}\Big[J_1(2t\sqrt{\lambda}) - t\sqrt{\lambda} + \tfrac{1}{2}(t\sqrt{\lambda})^3\Big]$$

$$F_1 \overset{\lambda\gg 1}{=} f_1\lambda + f_2\log\lambda + f_3 + f_4\,\lambda^{-1} + \mathcal{O}\big(e^{-\sqrt{\lambda}}\big)$$

$$f_1 = \tfrac{1}{4\pi^2}\log 2\,, \quad f_2 = -\tfrac{1}{4}\,, \quad f_3 = \tfrac{1}{2}\log\pi + ..., \quad f_4 = -\tfrac{\pi^2}{4}, \;\; ...$$

f_1 from Dirichlet $\eta(1) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} = \log 2$

• large λ expansion:

finite set of "polynomial" terms + infinite set of $e^{-(2n+1)\sqrt{\lambda}}$

• $F_2 = \widetilde{F}_2 + \bar{F}_2$: $\widetilde{F}_2$ related to F_1 and $\bar{F}_2$= same as in SA case

$$F_2(\lambda) = \widetilde{F}_2(\lambda) + \bar{F}_2(\lambda)\,, \qquad \widetilde{F}_2' = -\tfrac{1}{2}\lambda\big[(\lambda\,F_1)''\big]^2, \qquad (...)' = \tfrac{d}{d\lambda}(...)$$

Sp(2N) FA model

$$F = F_{\text{SYM}} + N\,\mathsf{F}_1(\lambda) + \mathsf{F}_2(\lambda) + \tfrac{1}{N}\mathsf{F}_3(\lambda) + \tfrac{1}{N^2}\mathsf{F}_4(\lambda) + \mathcal{O}(\tfrac{1}{N^3})$$

$$F_{\text{SYM}} = -\tfrac{1}{2}N(2N+1)\log\lambda$$

- F_n expressed only in terms of F_1 of $SU(N)$ FA model

$$\mathsf{F}_1 = 2F_1\,, \qquad \mathsf{F}_2 = \tfrac{1}{2}(\lambda F_1)' - \lambda\big[(\lambda\, F_1)''\big]^2$$

$$\mathsf{F}_3 = \tfrac{\lambda^2}{24}(\lambda F_1)''' - \tfrac{\lambda^2}{4}\big[(\lambda F_1)''\big]^2 + \tfrac{\lambda^3}{3}\big[(\lambda F_1)''\big]^3$$

$$F = F_{\text{SYM}} + \Delta F \overset{\lambda\gg 1}{=} \Delta F_{\text{pol}} - (N^2 + N - \tfrac{3}{16})\log\lambda - \tfrac{\pi^2}{2}N\lambda^{-1} + \mathcal{O}(e^{-\sqrt{\lambda}})$$

- leading terms in ΔF_{pol} at each order in $1/N$ sum up to simple log

$$\begin{aligned}\Delta F_{\text{pol}} =& N\big(2f_1\lambda + ...\big) + \big(2f_1^2\lambda^2 + ...\big) + \tfrac{1}{N}\big(\tfrac{8}{3}f_1^3\lambda^3 + ...\big) + \mathcal{O}(\tfrac{1}{N^2}) \\ =& N^2\mathcal{F}(\tfrac{\lambda}{N}) + ...\,, \qquad \mathcal{F}(\tfrac{\lambda}{N}) = \log\big(1 + 2f_1\tfrac{\lambda}{N}\big)\,, \qquad f_1 = \tfrac{\log 2}{4\pi^2}\end{aligned}$$

- leading terms in strong-coupling expression for $F \quad (\lambda = N g_{\rm YM}^2)$

$$F \stackrel{\lambda\gg 1}{=} -N^2 \log\lambda + N^2 \mathcal{F}(\tfrac{\lambda}{N}) + \dots$$
$$= N^2 \log\big(\lambda^{-1} + 2 f_1 N^{-1}\big) + \dots = N^2 \log\big[N^{-1}(g_{\rm YM}^{-2} + 2f_1)\big] + \dots$$

- large N expansion of Wilson loop

$$\langle\mathcal{W}\rangle = \langle\mathcal{W}\rangle_{\rm SYM} + \Delta\langle\mathcal{W}\rangle\,, \qquad \Delta\langle\mathcal{W}\rangle = \langle\mathcal{W}\rangle_1 + \tfrac{1}{N}\langle\mathcal{W}\rangle_2 + \tfrac{1}{N^2}\langle\mathcal{W}\rangle_3 + \dots$$

- $\mathcal{N}=4$ SYM contribution in case of $Sp(2N)$ [Fiol, Garolera, Torrents 2014; Giombi, Offertaler 2000]

$$\langle\mathcal{W}\rangle_{\rm SYM} = 2\, e^{\frac{\lambda}{16N}} \sum_{k=0}^{N-1} L_{2k+1}\big(-\tfrac{\lambda}{8N}\big) = N\,\langle\mathcal{W}\rangle_0 + \langle\mathcal{W}\rangle_{0,1} + \tfrac{1}{N}\langle\mathcal{W}\rangle_{0,2} + \mathcal{O}(\tfrac{1}{N^2})$$

$$\langle\mathcal{W}\rangle_0 = \tfrac{4}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = 2W_0\,, \quad \langle\mathcal{W}\rangle_{0,1} = \tfrac12 I_0(\sqrt{\lambda}) - \tfrac12\,, \quad \langle\mathcal{W}\rangle_{0,2} = \tfrac{\lambda}{96} I_2(\sqrt{\lambda})$$

- $\langle\mathcal{W}\rangle_{1,2}$ expressed in terms of $\mathsf{F}_1 = 2F_1$

$$\frac{\langle\mathcal{W}\rangle_1'}{\langle\mathcal{W}\rangle_0} = -\tfrac{\lambda}{4}(\lambda F_1)''\,, \quad \frac{\langle\mathcal{W}\rangle_2}{\langle\mathcal{W}\rangle_0} = -\tfrac{\lambda^2}{8}\mathsf{F}_2' = -\tfrac{\lambda^2}{8}\Big[\tfrac12(\lambda F_1)'' - \lambda\big[(\lambda F_1)''\big]^2\Big]$$

• large λ expansion:

$$\frac{\langle\mathcal{W}\rangle_1}{\langle\mathcal{W}\rangle_0} \stackrel{\lambda\gg1}{=} \frac{W_1}{W_0} = -f_1\,\lambda^{3/2} + \tfrac{3}{2}\,f_1\,\lambda - (\tfrac{3}{8}f_1 + \tfrac{1}{2}f_2)\lambda^{1/2} + \mathcal{O}(\lambda^0)$$

$$\frac{\langle\mathcal{W}\rangle_2}{\langle\mathcal{W}\rangle_0} \stackrel{\lambda\gg1}{=} \tfrac{1}{2}f_1^2\lambda^3 - \tfrac{1}{8}f_1(1-4f_2)\lambda^2 - \tfrac{1}{16}f_2(1-2f_2)\lambda + \mathcal{O}(e^{-\sqrt{\lambda}})$$

• suggests that leading large λ terms at each order in $1/N$ exponentiate

$$\langle\mathcal{W}\rangle = (N\langle\mathcal{W}\rangle_0 + ...) + \Delta\langle\mathcal{W}\rangle \stackrel{\lambda\gg1}{=} N\langle\mathcal{W}\rangle_0 \exp\big[-f_1\frac{\lambda^{3/2}}{N}\big] + ...$$

• cf. similar exponentiation of large λ terms in $\mathcal{N}=4$ SYM cases:

$$SU(N): \quad \langle\mathcal{W}\rangle_{\rm SYM} \stackrel{\lambda\gg1}{=} NW_0 \exp\big[\frac{\lambda^{3/2}}{96\,N^2}\big] + ... ,$$

$$Sp(2N): \quad \langle\mathcal{W}\rangle_{\rm SYM} \stackrel{\lambda\gg1}{=} 2NW_0\,\big(1+\frac{\lambda^{1/2}}{8N}\big)\exp\big[\frac{\lambda^{3/2}}{96\,(2N)^2}\big] + ..$$

$Sp(2N)$: $(1+\frac{\lambda^{1/2}}{8N})$ that gives odd powers of $1/N$ in $\langle\mathcal{W}\rangle_{\rm SYM}$
can be absorbed into $e^{\sqrt{\lambda}}$ in W_0 by $N\to N+\frac{1}{4}$ in $\sqrt{\lambda}=g_{\rm YM}\sqrt{N}$

Comments on dual string theory interpretation of FA models

• $SU(N)$ FA ($n_{\mathrm{F}}=4,\ n_{\mathrm{A}}=2$) engineered in flat-space IIB string
as low-energy theory on N D3-branes + 4 D7-branes and 1 O7-plane
$\rightarrow$ modding out by $G_{\mathrm{ori}}=\mathbb{Z}_{2,\mathrm{orb}}\times\mathbb{Z}_{2,\mathrm{ori}}$
$\mathbb{Z}_{2,\mathrm{orb}}=\{1,\ I_{6789}\},\quad \mathbb{Z}_{2,\mathrm{ori}}=\{1,\ I_{45}\,\Omega\,(-1)^{F_L}\}$
$I_{n_1\ldots n_r}$ on $\mathbb{R}^6$ (4,...,9 $\perp$ D3): $\ \mathbb{Z}_{2,\mathrm{orb}}:\ x_{6,7,8,9}\rightarrow -x_{6,7,8,9}\,,\ \mathbb{Z}_{2,\mathrm{ori}}:\ x_{4,5}\rightarrow -x_{4,5}$
fixed-point set of $\mathbb{Z}_{2,\mathrm{ori}}$: $x_{4,5}=0$ – position of O7 and 4 D7
• large-N near-horizon limit $\rightarrow$ dual string is projection of $\mathrm{AdS}_5\times S^5$:
IIB on $\mathrm{AdS}_5\times S'^5,\ \ S'^5=S^5/G_{\mathrm{ori}},\ \ G_{\mathrm{ori}}=\mathbb{Z}_{2,\mathrm{orb}}\times\mathbb{Z}_{2,\mathrm{ori}}$ [Ennes et al 2000]

$$ds_5^2=d\theta_1^2+\cos^2\theta_1\,(d\theta_2^2+\cos^2\theta_2\,d\varphi_1^2+\sin^2\theta_2\,d\varphi_2^2)+\sin^2\theta_1\,d\varphi_3^2$$

$$\mathbb{Z}_{2,\mathrm{orb}}:\quad \varphi_1\rightarrow\varphi_1+\pi,\ \ \varphi_2\rightarrow\varphi_2+\pi;\ \ \mathbb{Z}_{2,\mathrm{ori}}:\quad \varphi_3\rightarrow\varphi_3+\pi$$

• $Sp(2N)$ FA ($n_{\mathrm{F}}=4,\ n_{\mathrm{A}}=1$): near-horizon limit of
N D3 + 8 D7 + O7-plane $\rightarrow$ IIB on $\mathrm{AdS}_5\times S'^5,\quad S'^5=S^5/\mathbb{Z}_{2,\mathrm{ori}}$
D7 wrapping $\mathrm{AdS}_5\times S^3$ (S^3 locus of $\mathbb{Z}_{2,\mathrm{ori}}$) [Fayyazuddin, Spalinski; Aharony, Fayyazuddin, Maldacena 98]

• $n_{\rm F} \neq 0 \;\to$ D3-D7 open string sector:
open-string ($g_s^{2n+1} \sim 1/N^{2n+1}$) + closed-string ($g_s^{2n} \sim 1/N^{2n}$) topologies

• $SU(N)$ case: orientable surfaces (2-sphere with holes and handles)

• $Sp(2N)$ $\mathcal{N} = 4$ SYM: $\;1/N^{2n+1}$ contributions from crosscups [Witten 98]
[orientifold projection of $U(2N)$ SYM dual to IIB on $\mathrm{AdS}_5 \times \mathbb{RP}^5$;
$N \to N + \frac{1}{4}$ and $L^4 = 4\pi g_s(2N + \frac{1}{2})\alpha'^2$ due to O3 or due to crosscups]
• $Sp(2N)$ $\mathcal{N} = 2$ FA: crosscups due to O7 + bndries due to D7

• $F(S^4) \sim Z_{\rm str}$ on $\mathrm{AdS}_5 \times S'^5$:
N^2 term from $Z_{\rm str}(S^2) \sim$ IIB tree eff. action
• type I (disk) term in string eff action (here as D7 w-vol action) $\to$
AdS/CFT interpretation of N-term in conf. anom. of FA model

[Aharony,Pawelczyk,Theisen,Yankielowicz; Blau,Narain,Gava 99]

• $SU(N)$ FA model: F in terms of string parameters T and g_s

$$F(T,g_s) \overset{T\gg 1}{=} -\frac{\pi^2 T^4}{g_s^2}\log(2\pi T) + \frac{\pi T^2}{g_s}\left(f_1' T^2 + f_2' \log T + f_3' + ...\right)$$
$$+ \left(p_1' T^4 + p_2' T^2 + k_1' T + k_2'' \log T + k_3'' + ...\right) + \mathcal{O}(g_s)$$

• $\frac{1}{g_s^2}$ term from sphere: $\frac{1}{g_s^2\alpha'^4}\int d^{10}x\sqrt{g}\,(R+...)$ on $AdS_5 \times S'^5$

• $\frac{1}{g_s}$ term from disk [in $Sp(2N)$ case also from crosscup]

$\frac{T^2}{g_s}\log T$ from $\frac{1}{g_s\alpha'^2}\int d^8x\sqrt{g}\,RR$ in D7 action (on $AdS_5 \times S^3$) $\sim$ vol(AdS)

$\rightarrow N$ term in conf. a-anomaly of FA model [Blau, Narain, Gava 99]

• $\mathcal{O}(g_s^0)$ terms may come from closed-string (torus) and open-string (annulus or disk with crosscup):

S'^5 not smooth (orbifold action has fixed points)

$\rightarrow$ from "localized" contributions

- WL in FA models

$$\langle \mathcal{W} \rangle \overset{\lambda \gg 1}{=} e^{\sqrt{\lambda}} \Big[N(b_0 \lambda^{-3/4} + b_{01} \lambda^{-1/4} + ...) + (b_1 \lambda^{3/4} + b_{12} \lambda^{1/4} + ...) + \tfrac{1}{N} \, (b_2 \lambda^{9/4} + b_{21} \lambda^{5/4} + ...) + \mathcal{O}\big(\tfrac{1}{N^2}\big) \Big]$$

$$= \frac{T^{1/2}}{g_s} e^{2\pi T} \big(b_0' + b_1' g_s T + b_2' g_s^2 T^2 + ... \big)$$

- $SU(N)$: expansion near AdS_2 minimal surface with extra "disk with holes" in addition to "disk with handles"

$\mathcal{O}(g_s^0)$ term – annulus contribution (with Dirichlet + Neumann bc)

- $Sp(2N)$: extra "disk with crosscups" contributions

leading large λ parts sum up to simple exp: $\left(f_1 = \frac{\log 2}{4\pi^2}\right)$

$$\langle \mathcal{W} \rangle \overset{T \gg 1}{=} \frac{T^{1/2}}{\pi \, g_s} \, e^{2\pi T} \, e^{-8\pi^2 f_1 g_s T} + ... = \frac{T^{1/2}}{\pi \, g_s} \, \exp \big[2\pi T - 2 \log 2 \, T g_s \big] + ...$$

Exponentially suppressed corrections in $SU(N)$ FA model

$$F_1 \overset{\lambda \gg 1}{=} F_1^{\rm pol} + F_1^{\rm exp}\,, \qquad\qquad F_1^{\rm pol} = f_1 \lambda + f_2 \log \lambda + f_3 + f_4\, \lambda^{-1}$$

$$F_1^{\rm exp}(\lambda) \overset{\lambda \gg 1}{=} \lambda^{-1/4} \sum_{n=0}^{\infty} b_n(\lambda)\, e^{-(2n+1)\sqrt{\lambda}}$$

$$b_n(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k [4k(k+4)+3]\, \Gamma\big(k+\frac{1}{2}\big)\Gamma\big(k-\frac{3}{2}\big)}{\pi^{5/2}\, 2^{k-5/2}\, \Gamma(k+1)(2n+1)^k}\, \frac{1}{(\sqrt{\lambda})^k}$$

• "instanton sum" $e^{-(2n+1)\sqrt{\lambda}}$ multiplied by asymptotic series in $\frac{1}{\sqrt{\lambda}}$:
$b_n(\lambda)$ factorially div, but resurgent (large k is encoded in low k) [Dunne, Unsal 16]

• $e^{-c\sqrt{\lambda}}$ in observables in CFT with AdS string dual:
$\frac{1}{\sqrt{\lambda}}$ expansion in 2d string sigma model is expected to be asymptotic
cf. similar terms in cusp anom. dim in $\mathcal{N} = 4$ SYM [Alday, Maldacena 07; Basso, Korchemsky 09]

• $e^{-(2k+1)\sqrt{\lambda}}$ in F_1 – instanton interpretation – wrapping of S^2 of S'^5

• similar exp terms in W_1 in $\langle \mathcal{W} \rangle = NW_0 + W_1 + \frac{1}{N} W_2 + ...$
related to F_1, etc.

• compare to WL in $\mathcal{N}=4$ SYM: $W_0 = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$

$$W_0 \overset{\lambda \gg 1}{=} \sqrt{\tfrac{2}{\pi}} \lambda^{-3/4} \Big[e^{\sqrt{\lambda}} \big(1 - \tfrac{3}{8\sqrt{\lambda}} + ...\big) \; - \; i \, e^{-\sqrt{\lambda}} \big(1 + \tfrac{3}{8\sqrt{\lambda}} + ...\big) \Big]$$

$e^{\sqrt{\lambda}}$: AdS_2 in AdS_5; $e^{-\sqrt{\lambda}}$: unstable surface wrapping S^2 of S^5 [Drukker 06]
[no $e^{-n\sqrt{\lambda}}$ terms – would correspond to multiply wrapped WL]

• F_1 in $\mathcal{N}=2$ FA: infinite exp series – multiple wraps allowed
as F_1 = string path integral over disk with free boundary

• real coeffs in F_1: w-sheet solutions stable due to orbifolding of S^5
no Im part: series Borel summable (coeffs. factorially div but sign-alternate)

• W_1 from both F_1 and W_0: 2 sources of exp corrections

$$\frac{d}{d\lambda} W_1 = -\frac{\lambda}{4} W_0 \frac{d^2}{d\lambda^2}(\lambda F_1) \sim \left[e^{\sqrt{\lambda}} \, w(\lambda) + i e^{-\sqrt{\lambda}} \, w(-\lambda) \right] \sum_{k=0}^{\infty} u_k(\lambda) \, e^{-(2k+1)\sqrt{\lambda}}$$

• W_1: annular w-sheets with one bndry fixed by WL circle and other free:
stability of wrappings of S^2 in S'^5 implying real F_1 may no longer apply

Comments and open questions

• related results for strong-coupling expansions of $1/N$ terms
in 2-point and 3-point correlation functions of chiral operators

[Beccaria, Billo, Galvagno, Hasan, Lerda 20; Billo, Frau, Galvgno, Lerda, Pini 21; Billo, Frau, Lerda, pini, Vallario 22]

• need further progress towards analytic control of $\mathcal{N} = 2$ matrix models:
generalization of differential relations between F and $\langle W \rangle$?
exact form of F and $\langle W \rangle$ in $Sp(2N)$ FA model?
direct proof of exponential resummations of leading strong-coupling terms?

• need more computations on string side (already for dual of $\mathcal{N} = 4$ SYM)
planar: subleading $\frac{1}{\sqrt{\lambda}}$ terms on a disk
non-planar: reproduce coefficients of 1-handle and 1-crosscup terms

• string-side understanding of resummation of leading crosscups in $Sp(2N)$?

• possible role of integrability?
all planar-equivalent models integrable at leading N^2 order
integrability determines string spectrum $\to$ should have some consequences in loops (handle operator, etc.)
planar integrability may be reflected in $1/N$ corrections?

• string interpretation of differential relations between the $1/N$ corrections to F and $\langle W \rangle$ follow from localization matrix model on gauge side
but unexpected on dual string theory side:
F and $\langle \mathcal{W} \rangle$ are computed using quite different procedures

• $1/N$ corrections in other planar-equivalent $\mathcal{N} = 2$ models?
in $\mathcal{N} = 1$ models?