Limiting curvature gravity and problem of singularities

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V.F. and A. Zelnikov:

(1) "Two-dimensional black holes in the limiting curvature theory of gravity", JHEP 08 (2021) 154; eprint: 2105.12808.

(2) "Bouncing cosmology in the limiting curvature theory of gravity", Phys.Rev.D104(2021)10,104060; eprint:2108.09927.

(3) "Spherically symmetric black holes in the limiting curvature theory of gravity", Phys.Rev.D105(2022)2,024041; eprint:2111.12846.

According to famous Penrose and Hawking theorems the General Relativity is an incomplete theory.
Its solutions have inevitable singularities both in cosmology and inside black holes.

- In cosmology, where $C^2 = 0$, in order to restrict infinite growing of rhe curvature it is sufficient to modify the equation of state ($T_{\alpha\beta}$);
- For black holes this might not be sufficient: main problem is fast growing anisotropy (C^2).

Markov's limiting curvature principle (1982):

There exists a fundamental length scale ℓ such that $|R| \le B\ell^{-2}$. R is a curvature invariant and B depends on its choice, but not of a choice of a solution.

- One or more new universes formation inside a BH;
- Bouncing solutions in cosmology;
- Solution of the mass inflation problem.

In this talk I shall present a recently proposed approach which allows one to control growth of the curvature in gravity equations. This is achieved by imposing inequality constraints on the curvature invariants. In what follows I describe this method and its application to two problems: contracting universe and black hole interior.

Dynamical theory with inequality constraints.

Main idea: Let $L(q,\dot{q})$ be a Lagrangian and $\Phi(q,\dot{q}) \le 0$ be an inequality constraint. Denote $\tilde{L}(q,\dot{q},\chi,\zeta) = L(q,\dot{q}) + \chi[\Phi(q,\dot{q}) + \zeta^2]$.

Variation of $\tilde{L}$ over the Lagrange multipliers χ and ζ gives $\Phi(q,\dot{q}) + \zeta^2 = 0, \quad \chi\,\zeta = 0.$

Subcritcal regime: $\Phi < 0 \rightarrow \zeta^2 = -\Phi,\ \chi = 0;$

Eqns of motion: $\dfrac{\delta L}{\delta q} \equiv \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = 0.$

Supercritcal regime: $\Phi = 0 \rightarrow \zeta = 0,\ \chi \neq 0.$

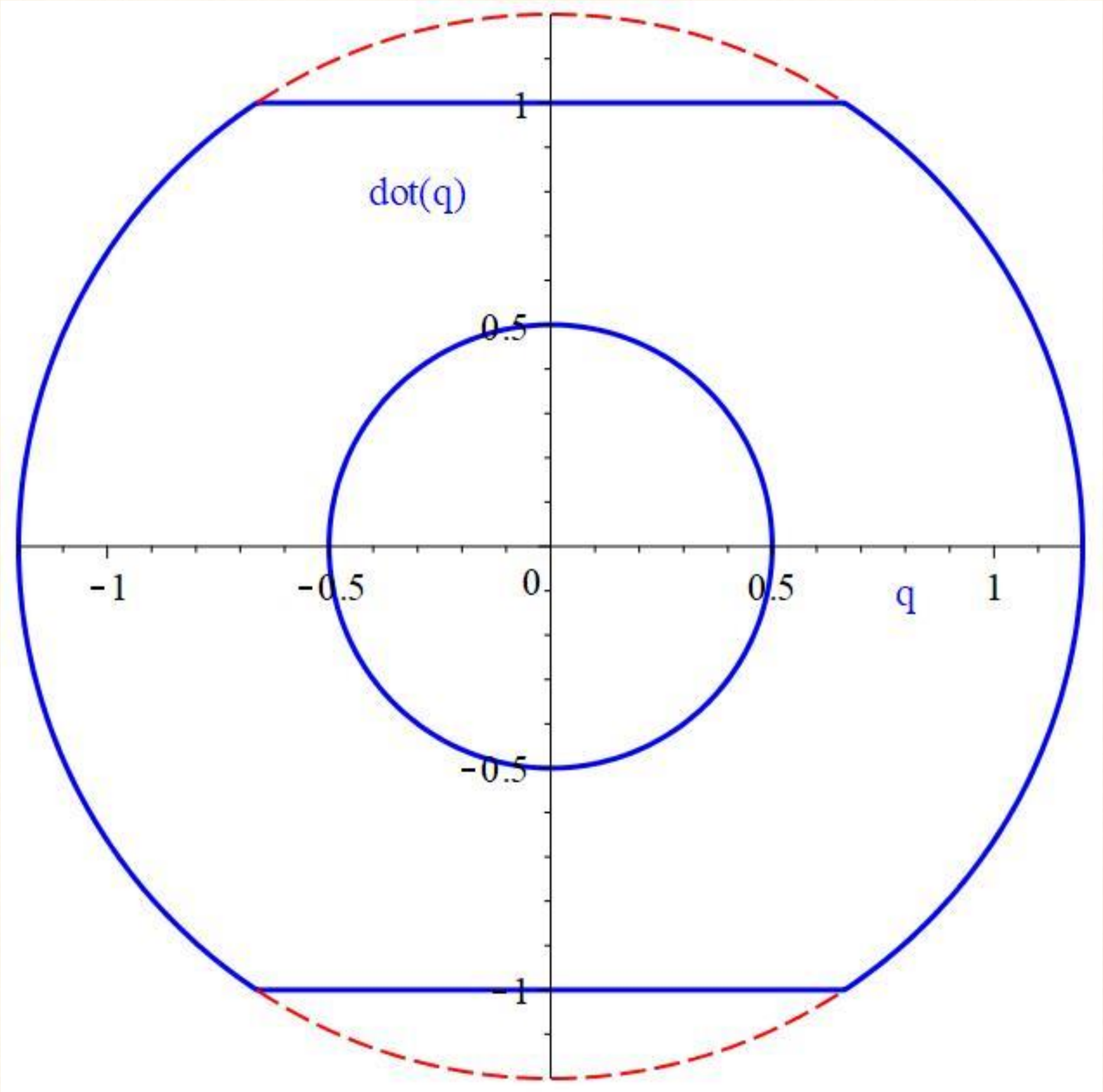
Eqns of motion: $\dfrac{\delta L}{\delta q} = -\dfrac{\delta(\chi\Phi)}{\delta q}.$

Harmonic oscillator with limiting velocity

$$S = \frac{1}{2}\int dt\,[\dot{q}^2 - q^2 + \chi(\dot{q}^2 - 1 + \zeta^2)]$$

$$\ddot{q} + q = -(\chi\dot{q})^{\cdot};$$

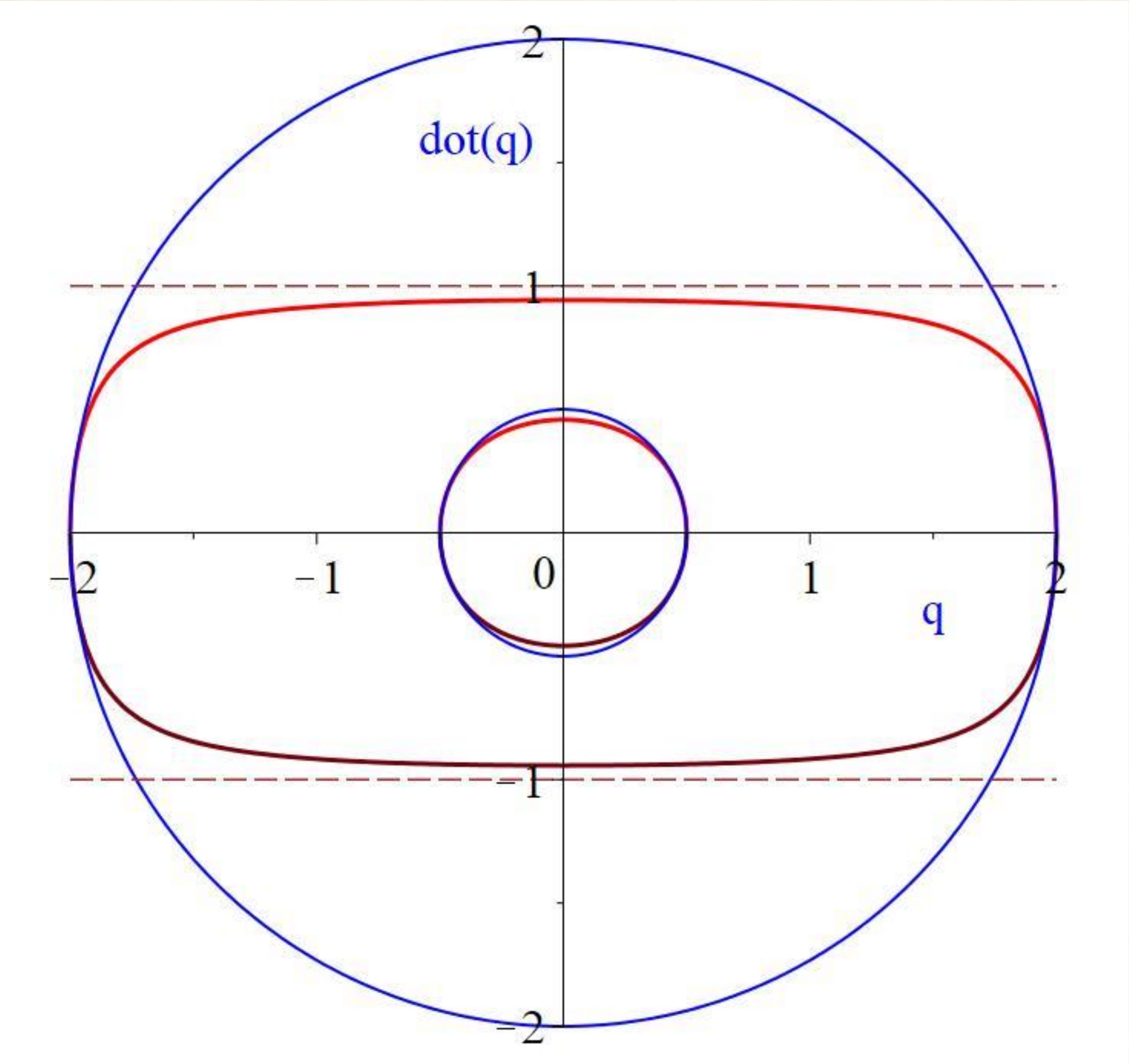
$$\dot{q}^2 - 1 + \zeta^2 = 0; \qquad \chi\zeta = 0.$$


dot(q)
1
0.5
-0.5
-1
-1
-0.5
0
0.5
q
1

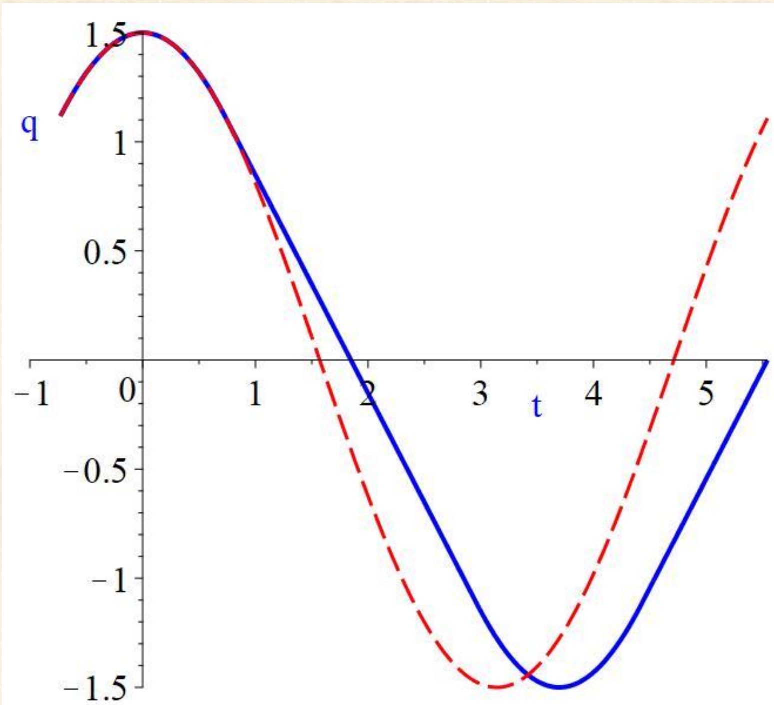
Relativistic oscillator:

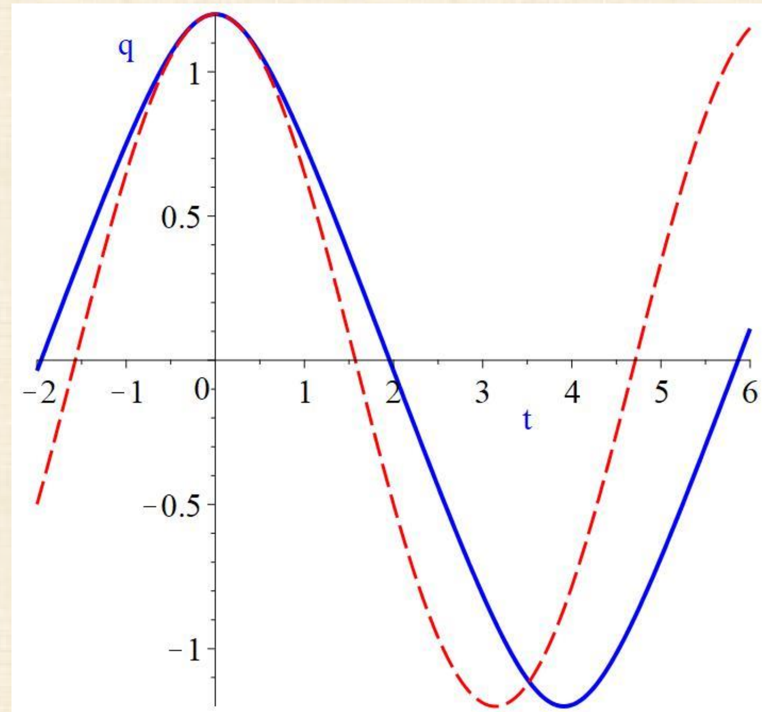
$$S=-\int dt\,[\sqrt{1-\dot{q}^2}+q^2]$$

$$\frac{d}{dt}\left(\frac{\dot{q}}{\sqrt{1-\dot{q}^2}}\right)+q=0.$$

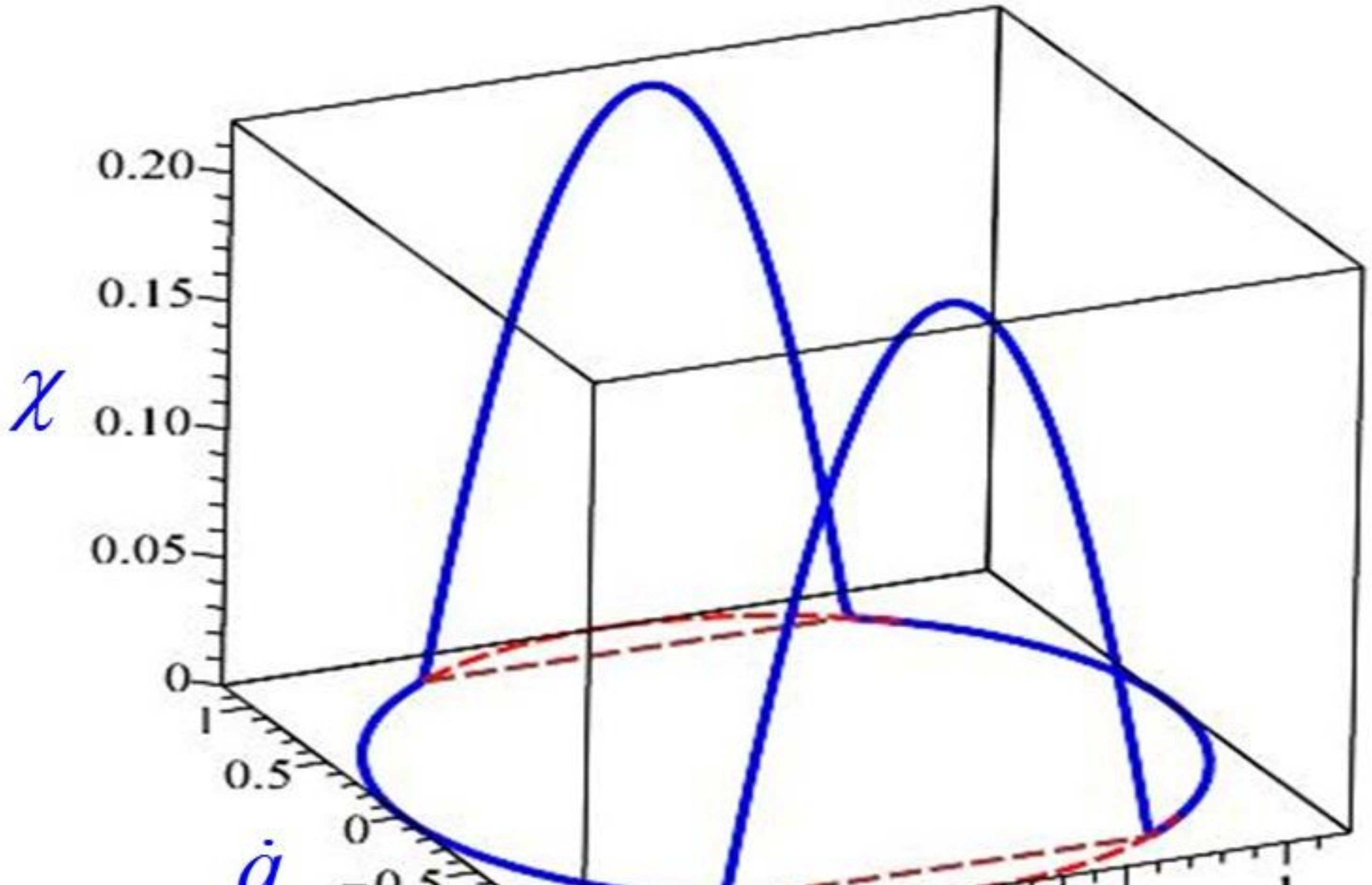


2
dot(q)
1
−2
−1
0
1
2
q
−1
−2





Displacement q as a function of time. Left: For limiting velocity case. Right: For the relativistic oscillator.



0.20
0.15
0.10
0.05
0
χ
1
0.5
0
−0.5
−1
$\dot{q}$
−1
−0.5
0
0.5
1
q

Oscillator with limiting velocity as a "pure man" relativistic oscillator.

Main idea of the LCG models: Modify Einstein-Hilbert action by adding terms which provide inequality constraints imposed on the curvature and control its growth.

Two examples will be discussed:

- Contracting cosmology,
- Interior of a spherically symmetric black hole.

Cosmology in the LCG theory

Let us consider a closed collapsing homogeneous isotropic universe filled with the thermal radiation. The equation of state is $P = \varepsilon / 3$. We denote by S its (conserved) entropy.

The metric is $ds^2 = -d\tau^2 + a^2(\tau)d\Omega^2$,

$d\Omega^2$ is a metric on a 3D unit sphere S^3.

Subcritical solution

$$a(\tau) = \sqrt{-2a_m\tau - \tau^2}, \quad \tau < 0,$$

$$a_m = 0.35\ell_{Pl}(S/k_B)^{2/3}, \quad S/k_B \sim 5.4\times 10^{89}.$$

$$\sqrt{R_{\mu\nu}R^{\mu\nu}} = \Lambda \sim \ell^{-2},$$

$$a_0 \sim \sqrt{l_{Pl}\ell}\left(\frac{S}{k_B}\right)^{1/3};\ \frac{S}{k_B} \sim 5.4\times 10^{89},$$

$$(a_0 / \ell \gg 1).$$

When the scale factor a reaches its critical value a_0 the supercritical regime stars.

$$ds^2 = -b^2(t)dt^2 + a^2(t)d\Omega^2,$$

$d\Omega^2$ is a metric on a 3D unit sphere S^3.

Any symmetric tensor $A_{\mu\nu}$, which respects the spacetime symmetries has a form

$$A_\mu^\nu = \mathrm{diag}(A(t), \hat{A}(t), \hat{A}(t), \hat{A}(t)).$$

Weyl tensor vanishes.

Eigenvalues of the Ricci tensor are linear combinations of two invariants

$$p = \frac{\dot{a}^2 + \dot{b}^2}{a^2 b^2}, \quad q = \frac{1}{b^2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{b}}{a^2 b^2}\right).$$

$$R_\mu^\nu = \mathrm{diag}(\mathrm{R}(t), \hat{\mathrm{R}}(t), \hat{\mathrm{R}}(t), \hat{\mathrm{R}}(t)).$$

$$\mathrm{R} = 3q, \quad \hat{\mathrm{R}} = q + 2p; \; R = p + q.$$

Mini-superspace approach. Reduced action:

$$S = 2\pi^2 \int dt\, a^3 b L, \quad L = L_g + L_m + L_c,$$

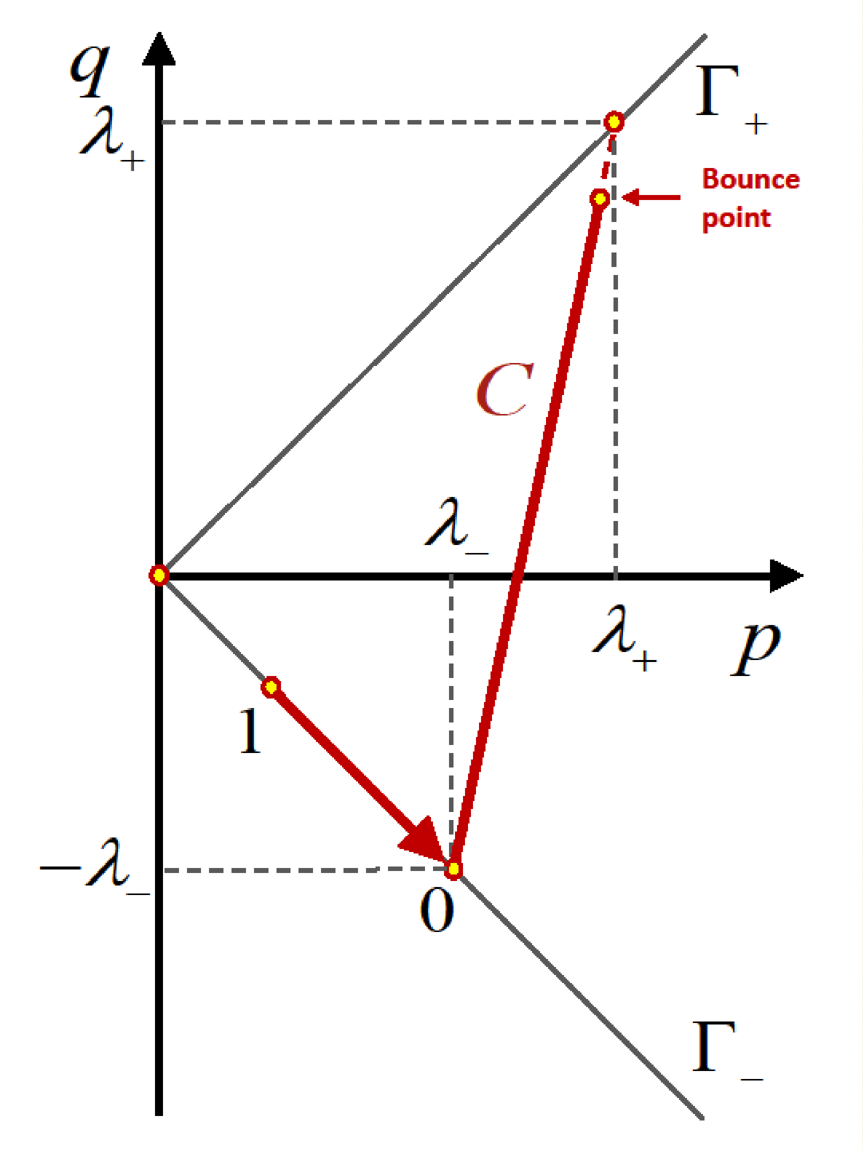
$$L_g = \frac{3}{8\pi G}(p+q), \quad L_m = -\frac{C}{a^4},$$

$$L_c = -\chi(p - \mu q - \Lambda + \zeta^2); \quad \mu \in (1/2, 1).$$

$$C = \nu\hbar c(S/k_B)^{4/3}, \quad \nu = \frac{3}{16\pi^3}(90/n\pi)^{1/3},$$

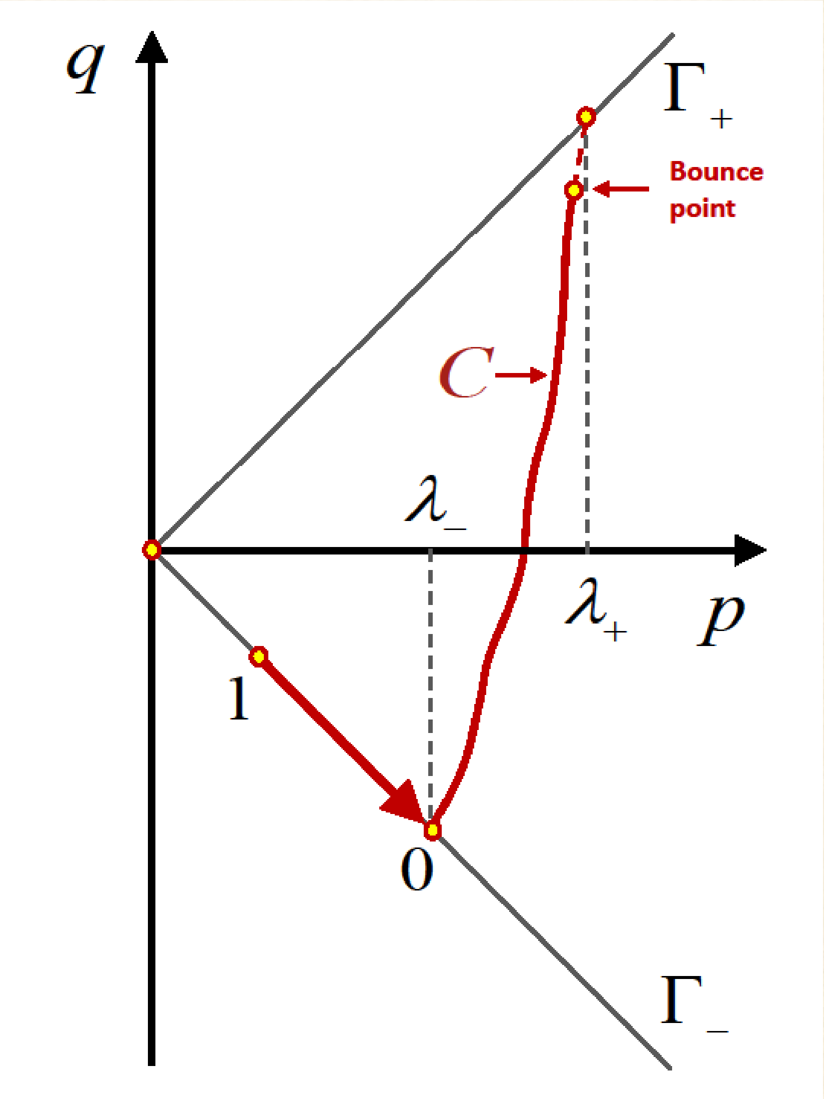
$$n = 2 \;\rightarrow \nu \approx 0.014686.$$

Linear constraint $\Phi = p - \mu q - \Lambda,\ \mu \in (1/2, 1)$.



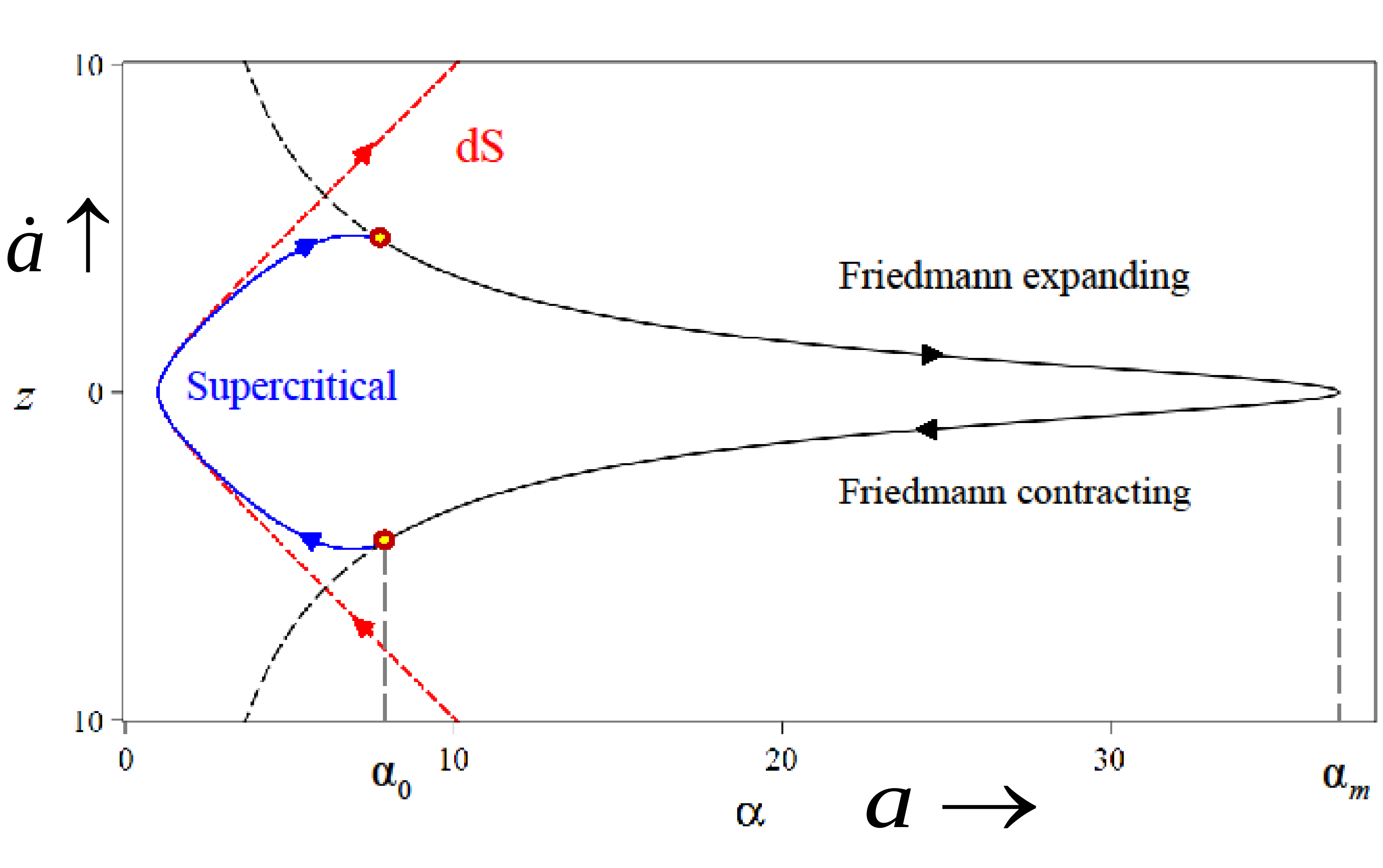
General case

Any scalar invariant constructed from the Ricci tensor can be written in the form of a function of its eigenvalues p and q. We write the curvature constraint in the form $f(p,q,\Lambda)=0$. Λ is a parameter defining the limiting curvature.



The constraint curve C intersects lines $\Gamma_{\pm}$, and

$q = q(p)$ along C is monotonically growing function



$\dot{a}$ ↑
10
dS
Friedmann expanding
Supercritical
z
0
Friedmann contracting
10
0
10
20
30
α_0
α_m
α
a →

$$\alpha = \frac{1}{\sqrt{1-\mu}}\frac{a}{\ell}, \quad \Lambda = \frac{1}{\ell^2},$$

$$a_m = 0.35\ell_{Pl}(S/k_B)^{2/3},$$

$$a_0 \sim \sqrt{l_{Pl}\ell}\left(\frac{S}{k_B}\right)^{1/3}, \quad a_b \sim \ell.$$

“Big Picture”: Bouncing oscillating cosmology

After the contracting universe reaches the point where its curvature becomes critical, the solution evolves along the constraint.

During this supercritical phase it reaches a point of bounce after which the scale function grows.

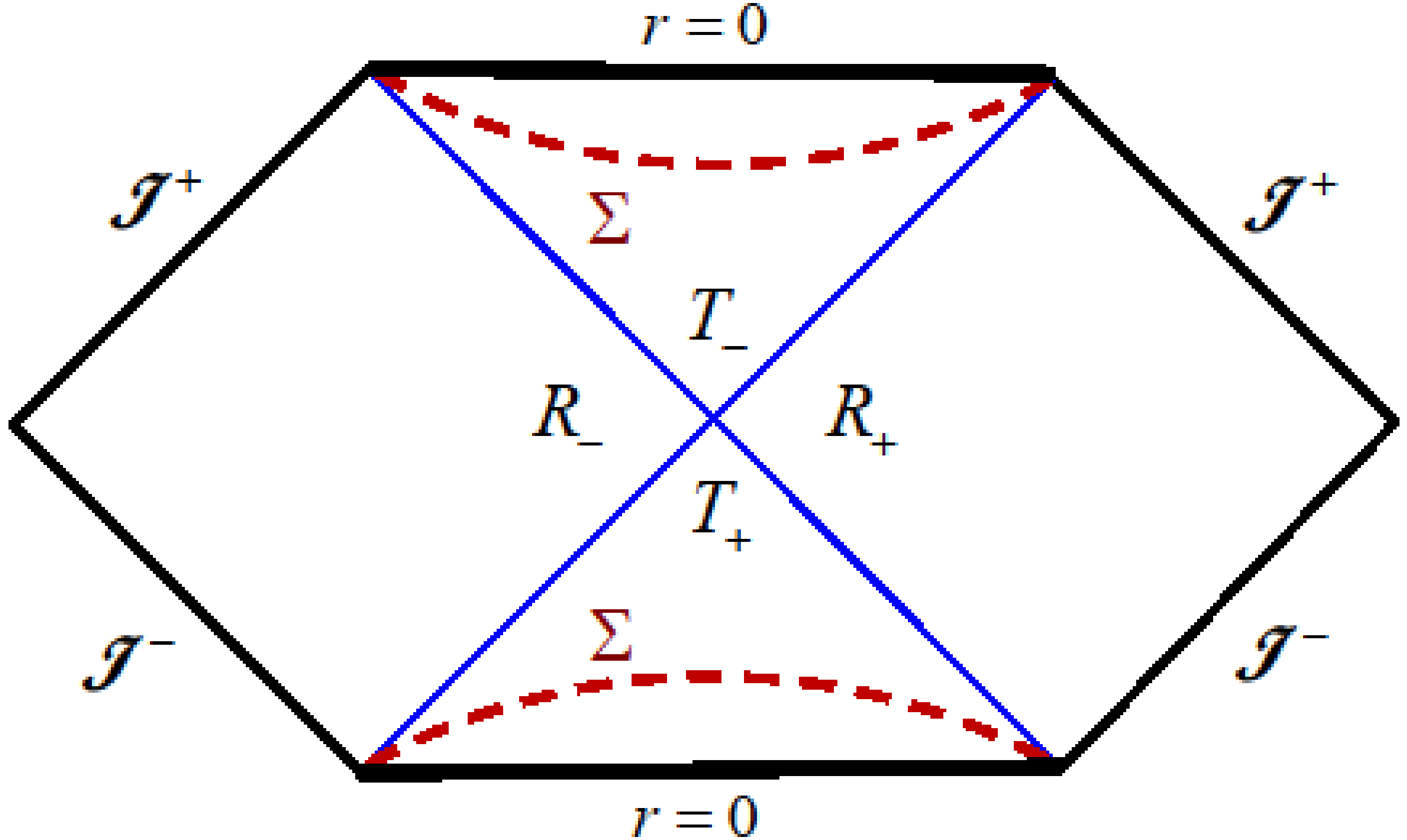
The control function can become zero again at this phase and the universe can leave its supercritical regime.

After this, one has an expanding universe filled with thermal radiation which follows the corresponding solution of the Einstein equations.

Black hole interior in the limiting curvature gravity

E.g. stationary BH solutions of the Einstein equations have a curvature singularity in their interior.

For Schwarzschild BH $\mathcal{K} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \dfrac{48M^2}{r^6}.$

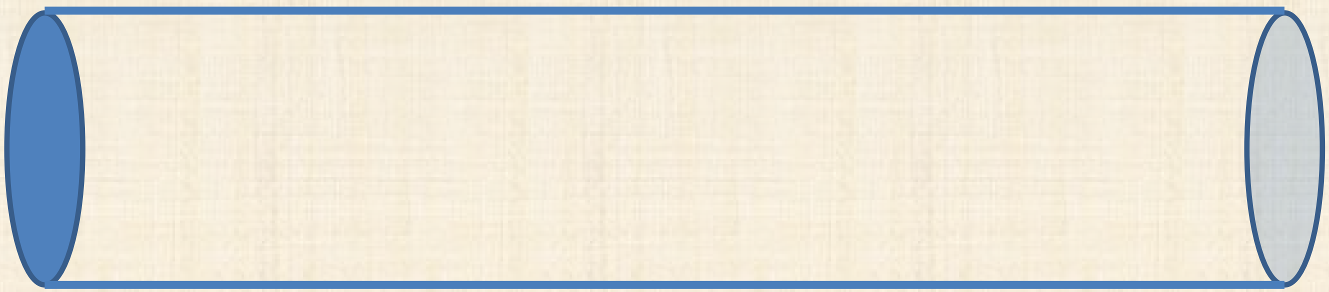


$r = 0$
$\mathscr{J}^+$
$\mathscr{J}^+$
Σ
T_-
R_-
R_+
T_+
Σ
$\mathscr{J}^-$
$\mathscr{J}^-$
$r = 0$

Proper time: $\tau \approx -\frac{2}{3\sqrt{2M}} r^{3/2}.$

$ds^2 \sim -d\tau^2 + a\,\tau^{-2/3} dt^2 + b\,\tau^{4/3} d\omega^2$

Contracting Kasner-type anisotropic universe.



$S^2 \times R$

Metric inside black hole

$$ds^2 = -b^2(\tau)d\tau^2 + B^2(\tau)dt^2 + a^2(\tau)d\omega^2.$$

Non-vanishing components of the Riemann tensor

$$p = R_{\theta\varphi\theta\varphi} = \frac{\dot{a}^2 + \dot{b}^2}{a^2 b^2}, \quad q = -R_{\tau\theta\tau\theta} = \frac{\ddot{a}}{ab^2} - \frac{\dot{a}\dot{b}}{ab^3},$$

$$v = -R_{\tau t \tau t} = \frac{\ddot{B}}{Bb^2} - \frac{\dot{B}\dot{b}}{Bb^3}, \quad u = R_{t\theta t\theta} = \frac{\dot{a}\dot{B}}{aBb^2}.$$

(1) Any scalar invariant constructed from the curvature can be wrirren as a function of p, q, v and u.

(2) The curvature eigen-values p, q, v and u can be written as functions of 4 basic curvature invariants.

(3) To restrict the values of the curvature invariants it is sufficient to impose bounds on p, q, v and u.

$$ds^2 = -b^2(\tau)d\tau^2 + B^2(\tau)dt^2 + a^2(\tau)d\omega^2.$$

$2D$ slice: $ds^2 = -d\tau^2 + B^2(\tau)dt^2$. This is a metric for the interior of 2D black hole and $v = \frac{1}{2}R^{(2)}$.

$3D$ slice: $ds^2 = -d\tau^2 + a^2(\tau)d\omega^2$. This is a metric of a collapsing 3D cosmology. $G^{(3)\nu}_{\mu} = -\mathrm{diag}(q, p, p)$.

Imposing curvature constraints:

(1) $\Phi_1 = v - \Lambda_1$. As a result of first constraint in the supercritical regime the $2D$ slice: $ds^2 = -d\tau^2 + B^2(\tau)dt^2$ is expanding 2D deSitter universe with $\frac{1}{2}R^{(2)} = \Lambda_1$.

(2) $\Phi_2 = p - \mu q - \Lambda_2, 0 < \mu < 1$. As a result of second constraint the $3D$ slice has a metric of a contracting 3D cosmology with a bounce.

Mini-superspace approach.

Reduced action :

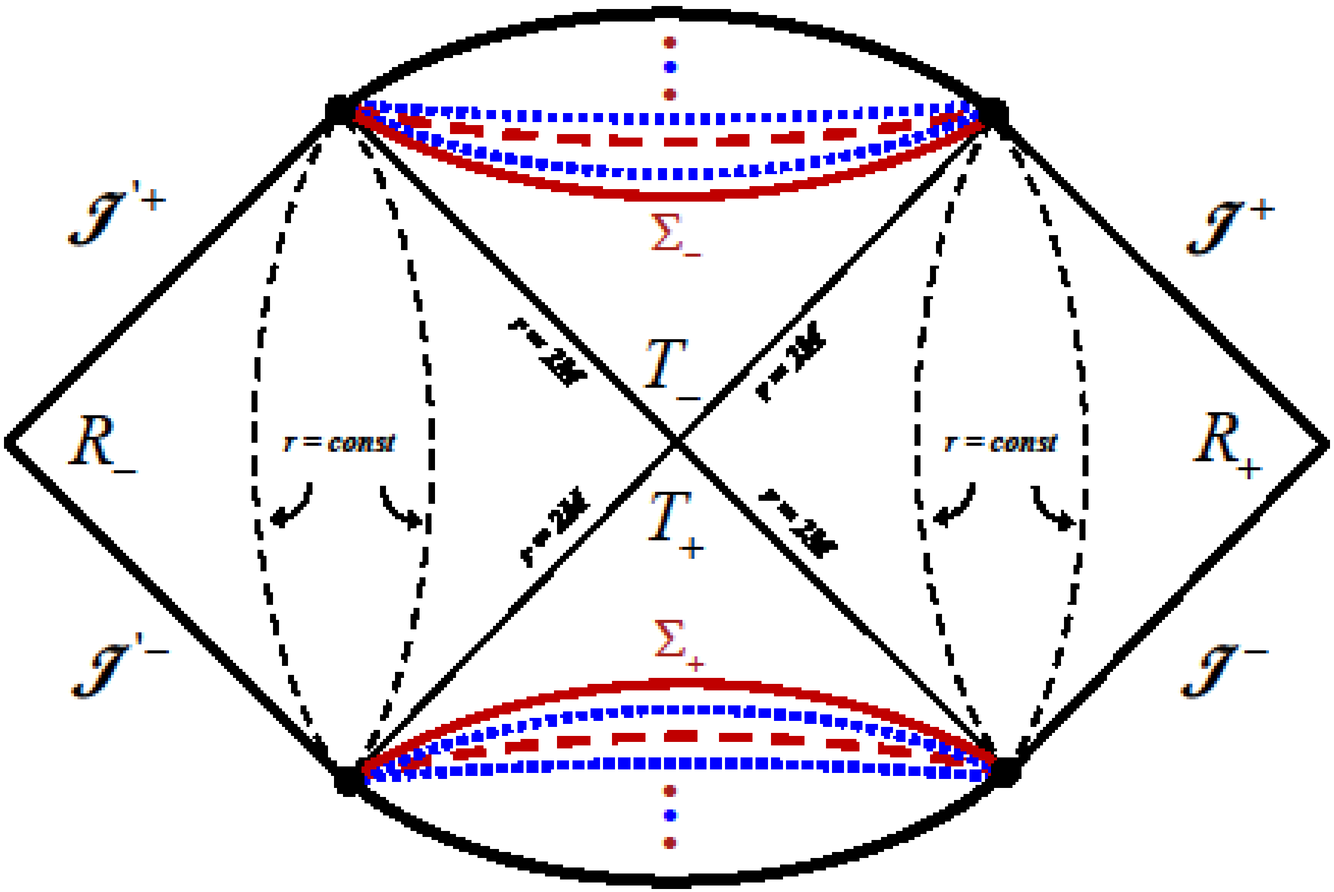
$$\mathbf{S} = \frac{V}{8\pi G} S, \quad V = 4\pi \int dt,$$

$$S = \int d\tau \; a^2 bB \, L, \quad L = L_g + L_c,$$

$$L_g = \frac{1}{2} R^{(4)} = p + \; 2q + \; 2u + v,$$

$$S_g = \int d\tau \; B[b - \frac{\dot{a}^2}{b} - 2 \frac{a\dot{a}\dot{B}}{Bb}],$$

$$L_c = \chi_1(\Phi_1 + \zeta_1^2) + \chi_2(\Phi_2 + \zeta_2^2).$$



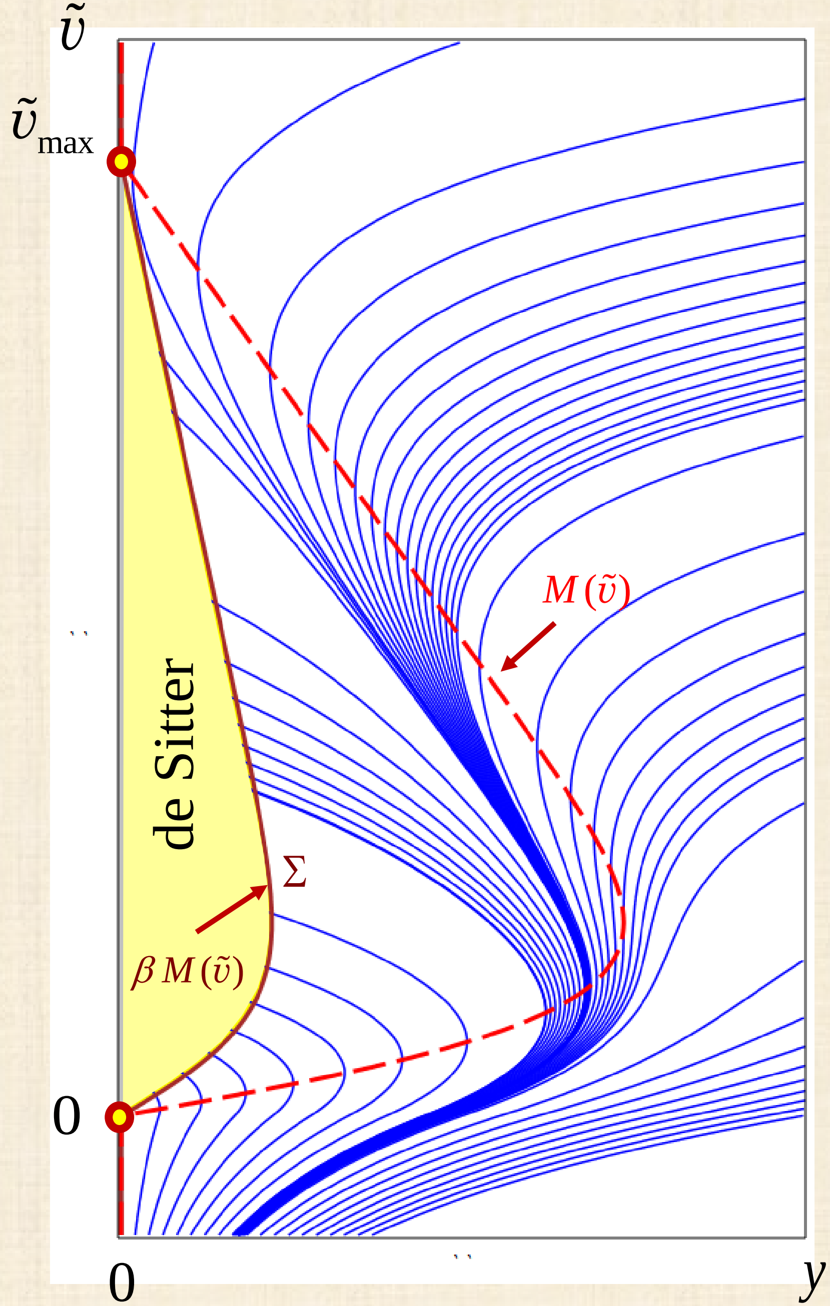
$\mathcal{J}'^{+}$
$\mathcal{J}^{+}$
Σ_{-}
T_{-}
R_{-}
$r = const$
$r = 2M$
$r = const$
R_{+}
T_{+}
Σ_{+}
$\mathcal{J}'^{-}$
$\mathcal{J}^{-}$

$$S^2 \times R$$

For $\Lambda_2 \sim \frac{1}{\ell^2}$ the radius $a(\tau)$ oscillates between

its maximum value $a_{\max} \sim \ell \left(\frac{r_g}{\ell} \right)^{1/3}$ and its minimal

value $a_{\min} \sim \ell$.



$\tilde{v}$
$\tilde{v}_{\max}$
$M(\tilde{v})$
de Sitter
Σ
$\beta M(\tilde{v})$
0
0
y

Summary

- Limiting curvature gravity theory is obtained by adding terms to Einstein-Hilbert action which provide inequality constrains restricting curvature invariants;
- Sub- and supercritical regimes;
- Solutions are modified in the supercritical regime.
- Control functions guarantee consistency of the equations.

Application of the LCG theory to cosmology:

In the contracting homogeneous isotropic cosmology no singularity is formed. Instead of this one has a Bouncing cosmology where after passing through the Big Crunch phase the universe scale factor reaches its turning point and begins its inflationary expansion. At this super-critical phase its evolution is close to the deSitter solution. After the curvature reduces to its critical value, a new subcritical phase starts, in which one has an expanding FRW universe filled by the thermal radiation. This does not require a reheating phase, and happens because of the "memory" of the entropy during the supercritical regime. The number of the e-folds is controled by the fundamental length ℓ.

Application of the LCG theory to black holes:

- An eternal black hole has two (oscillating) deSitter cores.
- Oscillating solution for BH interior of a charged BH (Novikov and Starobinsky, 1980)
- What happens if the BH is formed as a result of the matter collapse?
- Role of particle creation during supercritical phase?

- New Universe formation inside a black hole (V.F., Markov and Mukhanov, 1989,90).
- Timelike wormholes vs spacelike wormholes.

Thank you.