# Anomaly-free scale symmetry and gravity

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Can conformal symmetry be an exact symmetry of Nature?

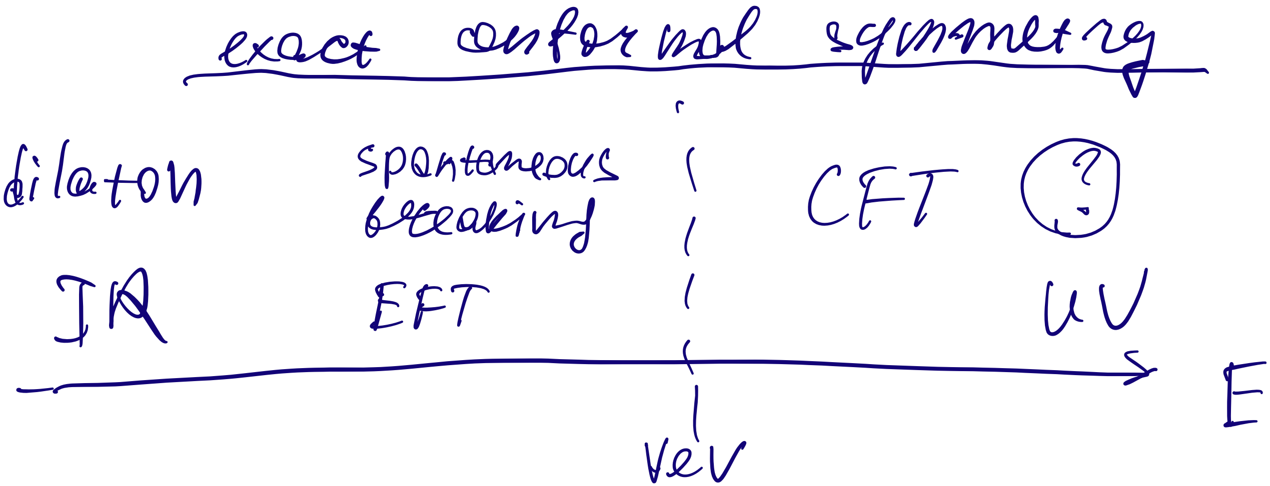
Can anomaly be avoided?

# Motivation for conformal symmetry

- Standard model: no dimensional parameters in the high energy limit – hint for the conformal invariance in the UV
- naturally small Higgs mass – easy to do with the scale invariance and without new heavy particles

  C. Wetterich, Phys. Lett. B 140 (1984), 215-222
  W. A. Bardeen, FERMILAB-CONF-95-391-T.
- Scale symmetry can be extended to conformal
- extra global symmetry can be helpful in fixing the UV limit of the fundamental theory
- *Spin quantisation of massless particles - no possibility to have continuous spin representations in the presence of the conformal symmetry. No continous spins were observed. G. Mack and I. Todorov, J. Math. Phys. 10 (1969), 2078-2085
  G. Mack and A. Salam, Annals Phys. 53 (1969), 174-202
  P. Schuster and N. Toro, Phys. Rev. D 91 (2015), 025023 [arXiv:1404.0675 [hep-th]]
  I. L. Buchbinder, V. A. Krykhtin and H. Takata, Phys. Lett. B 785 (2018), 315-319

# The setup




- Conformal symmetry is restored in the UV limit
- At low energies the symmetry is broken, dilaton is a Goldstone boson
- scale anomaly can be avoided within the special definition of the quantum theory

# Conformal transformations

Poincare transformations (ISO($D-1,1$))

$$\delta_T^{\sigma}\Phi(x) = \partial_\sigma \Phi(x)$$

$$\delta_L^{\sigma\tau}\Phi(x) = (x^\sigma \partial^\tau - x^\tau \partial^\sigma + \Sigma^{\sigma\tau})\,\Phi(x)$$

Dilatation

$$\delta_S\Phi(x) = (x_\tau\partial^\tau + \Delta)\Phi(x)$$

Special conformal transformation

$$\delta_C^{\sigma}\Phi(x) = (2x^\sigma x^\tau - g^{\sigma\tau}x^2)\partial_\tau\Phi(x) + 2x_\tau(g^{\sigma\tau}\Delta - \Sigma^{\sigma\tau})\Phi(x).$$

The finite transformations

$$x_\mu \to x'_\mu = \frac{x_\mu + c_\mu x^2}{1 + 2c_\mu x_\mu + c^2 x^2}. \tag{1}$$

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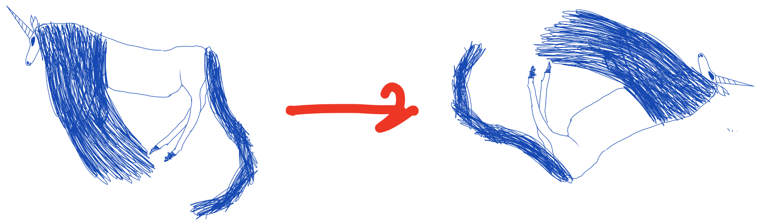
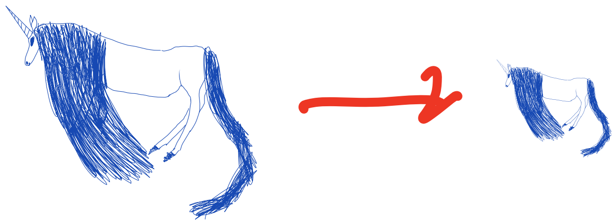
# Conformal transformations

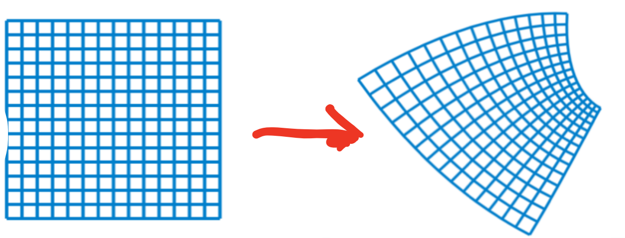
shift



rotation



dilatation



special con-
formal

# The Lagrangian

$$L = \frac{1}{2}(\partial\chi)^2 + \partial H^\dagger \partial H - \frac{\lambda}{4}\chi^{\frac{4-D}{D-2}}(H^\dagger H - \alpha^2\chi^2)^2 + L_{SM}$$

M. Shaposhnikov and D. Zenhausern, Phys. Lett. B 671 (2009), 187-192 [arXiv:0809.3395 [hep-th]]

With the use of the dimension regularization applied to this Lagrangian we obtain the model without the scale anomaly.

What about the conformal symmetry?

Toy model:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^q, \quad q = \frac{2D}{D-2}$$

The action is conformally invariant in $D$ dimensions. Does it guarantee the absence of the anomaly?

F. Englert, C. Truffin and R. Gastmans, Nucl. Phys. B 117 (1976), 407-432

# The effective potential: flat direction

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^{2D/(D-2)}$$

Is $\lambda = 0$ stable?

$$L_{int} = a\frac{(\Box\phi)^2}{\phi^{\frac{4}{D-2}}}.$$

$$S_1 = -\frac{i}{2}\log\det\left(\frac{\delta^2 L}{\delta\phi^2}\right).$$

$$S_1 = -\frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\log\left(\frac{\phi_0^{\frac{4}{D-2}}}{2a} + k^2\right)$$

$$S_1 = \int d^4x \frac{\Gamma(-D/2)}{2(4\pi)^{D/2}}\frac{\phi_0^{\frac{2D}{D-2}}}{(2a)^{D/2}} \to \frac{1}{8a^2(4\pi)^2}\frac{1}{D-4}\phi^4\phi^{4-D} + \textit{finite}$$

The divergence has the same form as the potential. It can be eliminated by the redefinition of $\lambda$.

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# The effective action

The effective action at the order $\lambda^2$ is,

$$\Gamma = \frac{9\lambda^2}{8\pi^2}\int d^D x d^D y \phi(x)^{\frac{4}{D-2}}\phi(y)^{\frac{4}{D-2}}\left(\frac{1}{(x-y)^2}\right)^{D-2}$$

This action has a local divergent part near $D = 4$, since

$$\langle\delta\phi(x)\delta\phi(y)\rangle^2 = \frac{1}{16\pi^2}\frac{2}{D-4}\delta^4(x-y) + (\text{finite at } D=4)$$

$$\mathcal{L}_{ct} = \frac{9}{4}\lambda^2\int d^D x\,\phi^{\frac{2D}{D-2}}(x).$$

The renormalized effective action

$$\Gamma_{ren} = \frac{9\lambda^2}{8\pi^2}\int d^D p\, e^{ip(x-y)}\int d^D x d^D y\left(\phi^{\frac{D}{D-2}}(x)\phi^{\frac{D}{D-2}}(y) - \phi^{\frac{4}{D-2}}(x)\phi^{\frac{4}{D-2}}(y)(p^2)^{\frac{D-4}{2}}\right)$$

$$\Gamma_{ren} = \frac{9\lambda^2}{8\pi^2}\int d^4 p e^{ip(x-y)}\int d^4 x d^4 y \phi^2(x)\phi^2(y)\log\frac{\phi(x)\phi(y)}{p^2}.$$

# The effective action

$$\Gamma_{ren} = \frac{9\lambda^2}{8\pi^2}\left(\int d^4x\,\phi^4(x)\log\frac{\phi^2(x)}{\mu^2} - \int d^4xd^4yd^4p\,\phi^2(y)\phi^2(x)\log\frac{p^2}{\mu^2}\right)$$

$$= 9\lambda^2/(8\pi^2)\left(\int d^4x\,\phi^4(x)\log\frac{\phi^2(x)}{\mu^2} -\right.$$
$$\left.- \int d^4x\,\phi^2(x)\log\frac{-\Box_x}{\mu^2}\phi^2(x)\right) = -\frac{9\lambda^2}{8\pi^2}\int d^4x\,\phi^2(x)\log\frac{-\Box}{\phi^2(x)}\phi^2(x).$$

Is it Conformal? Conformal operators

$$O_N = \phi^{\frac{2(N+1)}{2-D}}\Box^N(\phi^{\frac{D-2N}{D-2}})$$

$$\phi^{2-N}\Box^N\phi^{2-N} = \phi^4 + N\phi^2\left(\log\frac{\Box}{\phi^2}\right)\phi^2 + \ldots$$

$$\phi^2\left(\log\frac{\Box}{\phi^2}\right)\phi^2 = \lim_{N\to 0}\left(\frac{\phi^{2-N}\Box^N\phi^{2-N} - \phi^4}{N}\right)$$

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# Conformal operators ($4\partial$, $D = 4$)

All scale-invariant operators:

$$\frac{1}{\phi^2}(\Box\phi)^2, \quad \frac{1}{\phi^3}(\partial_\mu\phi)^2\Box\phi, \quad \frac{1}{\phi^4}(\partial_\mu\phi)^4$$

Conformal symmetry leaves

$$\phi^{\frac{D-4}{2}}\Box^2\phi^{\frac{D-4}{2}}, \quad \frac{1}{\phi^2}(\Box\phi)^2$$

$$\left(\Box \log\frac{\phi}{\mu}\right)^2 = \frac{1}{\phi^4}(\partial_\mu\phi)^4 - \frac{2}{\phi^3}(\partial_\mu\phi)^2\Box\phi$$

There are 2 conformal operators

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# Conformal operators ($6\partial$, $D = 4$)

All scale-invariant operators:

$$\mathcal{O}_1 = \frac{1}{\phi^4}\,\Box\phi\,\Box^2\phi, \quad \mathcal{O}_2 = \frac{1}{\phi^5}(\,\Box\phi)^3,$$

$$\mathcal{O}_3 = \frac{1}{\phi^5}\,\Box^2\phi\,(\,\partial_\mu\phi)^2,$$

$$\mathcal{O}_4 = \frac{1}{\phi^6}(\,\partial_\nu\phi)^2\,\Box(\,\partial_\mu\phi)^2, \quad \mathcal{O}_5 = \frac{1}{\phi^6}(\,\partial_\mu\phi)^2\,(\,\Box\phi)^2,$$

$$\mathcal{O}_6 = \frac{1}{\phi^7}(\,\partial_\mu\phi)^4\,\Box\phi, \quad \mathcal{O}_7 = \frac{1}{\phi^8}(\,\partial_\mu\phi)^6.$$

Conformal operators:

$$\frac{1}{\phi}\,\Box^3\frac{1}{\phi}, \quad \frac{1}{\phi^3}\,\Box\phi\,\Box^2\log\phi, \quad \frac{1}{\phi^5}(\,\Box\phi)^3, \quad \frac{\Box\phi}{\phi^2}\,\Box\left(\frac{\Box\phi}{\phi^2}\right)$$

4 operatrors of 7 are left

# Conformal anomaly

Does the invariance in arbitrary $D$ mean the absence of the anomaly?

**Theorem.** If the theory is defined in such a way that

- the action is invariant under the conformal transformations in an arbitrary (fractional) spacetime dimension
- the dimensional regularization is used for computations of divergent terms in the effective action in an entire number $D$ of dimensions
- the perturbative expansion can be applied

than the finite renormalized effective action $\Gamma_{ren}$ is invariant under the conformal transformations in $D$ dimensions.

The possible issue

- The counterterms should be extended to the conformal operators in $D - \epsilon$ dimensions by adding only finite at $\epsilon \to 0$ pieces

F. Gretsch and A. Monin, Phys. Rev. D 92 (2015) no.4, 045036 [arXiv:1308.3863 [hep-th]]

# Weyl anomlaly from $(\log\phi)\Box^2(\log\phi)$ term

$$S = \int d^D x\sqrt{-g}\,\phi^{\frac{4-D}{D-2}}\Box^2\left(\phi^{\frac{4-D}{D-2}}\right)$$

Weyl-invariant action:

$$S = \int d^D x\sqrt{-g}\,\varphi\Big(\Box^2\varphi + (A_1 R^{\alpha\beta} + A_2 g^{\alpha\beta}R)\nabla_\alpha\nabla_\beta\varphi +$$
$$+A_3\nabla_\beta R\nabla^\beta\varphi + (A_4\Box R + A_5 R^2 + A_6 R_{\alpha\beta}R^{\alpha\beta})\varphi\Big),\ \ \varphi = \phi^{\frac{4-D}{D-2}}$$

In the limit $D = 4$,

$$S = \int d^4 x\sqrt{-g}\left(\tau\Box^2\tau + \frac{1}{3(D-4)}\Big(R^2 - 3R_{\alpha\beta}R^{\alpha\beta} - 12R^{\alpha\beta}\nabla_\alpha\nabla_\beta\tau +\right.$$
$$\left.+4R\Box\tau - \Box R - 6\Box^2\tau - 2\nabla_\alpha\tau\nabla^\alpha R\Big) + (\text{finite}),\quad \tau = \log\phi\right.$$

No way to extend without a pole in $D = 4$

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# Weyl anomaly vs unitarity

$$T^{\mu}_{\mu} = aE_4 + cW^2$$

$a > 0$ -consequence of the unitarity

Question: if the Lagrangian is defined as Weyl invariant in $D$ dimensions, does it mean $a = c = 0$ and problems with unitarity?

The answer is NO

The Weyl anomaly is unavoidable even in this approach.

The theorem breaks because the Weyl-invariant counterterm to one of the operators, $(\log\phi)\Box^2(\log\phi)$, would give the extra $1/(D-4)$ pole.

Z. Komargodski and A. Schwimmer, JHEP 12 (2011), 099 [arXiv:1107.3987[hep-th]]

# Phenomenology: Higgs-dilaton Lagrangian

$$L = \frac{1}{2}\zeta_\chi \chi^2 R + \zeta_H H^\dagger H R + \frac{1}{2}(\partial\chi)^2 + \partial H^\dagger \partial H - \\ - \frac{\lambda}{4}\chi^{\frac{4-D}{D-2}}(H^\dagger H - \alpha^2\chi^2)^2 + L_{SM}$$

J. Garcia-Bellido, J. Rubio, M. Shaposhnikov and D. Zenhausern, Phys. Rev. D 84 (2011), 123504 [arXiv:1107.2163 [hep-ph]]

F. Bezrukov, G. K. Karananas, J. Rubio and M. Shaposhnikov, Phys. Rev. D 87 (2013) no.9, 096001

- Conformally-invariant without gravity
- What is the maximal symmetry of this lagrangian if gravity is included?

# Conformal symmetry in the curved space

Conformal=Diff-Weyl, $g_{\mu\nu} \to (1+\omega)g_{\mu\nu}$, $\omega = \partial_\mu \xi_\mu$

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D} g_{\mu\nu} \partial^\alpha \xi_\alpha$$

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{2}{D} g_{\mu\nu} \nabla^\alpha \xi_\alpha$$

Problems:

- Anomaly from $\Box^2$ term

$$\delta_\omega S = \int d^4x \sqrt{-g} \left( -\nabla_\alpha \nabla_\beta \omega \nabla^\alpha \tau \nabla^\beta \tau + \frac{1}{2} \Box\omega \Box(\tau^2) \right)$$

$$\nabla_\alpha \nabla_\beta \omega = \frac{D}{2-D} \left( \xi^\mu \nabla_\mu R_{\alpha\beta} + \frac{2}{D} \omega R_{\alpha\beta} + \frac{g_{\alpha\beta}}{2(1-D)} \left( \frac{2}{D} \omega R + \xi^\mu \nabla_\mu R \right) \right)$$

- It is not a symmetry of the Higgs-dilaton Lagrangian

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# Restricted Weyl symmetry

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$$

The condition for being the subgroup of Weyl transformations:

$$\Box \left( \Omega^{\frac{D-2}{2}} \right) = 0$$

- This is a symmetry of the Higgs-dilaton Lagrangian
- Anomaly will not cancel

A. Edery and Y. Nakayama, Phys. Rev. D 90 (2014), 043007 [arXiv:1406.0060 [hep-th]]

A. Edery and Y. Nakayama, Mod. Phys. Lett. A 30 (2015) no.30, 1550152 [arXiv:1502.05932 [hep-th]]

# Conditions for anomaly cancellation

$$L_{anom} = \frac{E_4}{2(D-4)}$$

- The infinitesimal Weyl transformation $\Omega = 1 + \omega,\ \omega \ll 1$ of the anomaly term is $\delta_\omega L_{anom} = \frac{1}{2} E_4 \omega$
- It is finite and non-zero when $D \to 4$
- Our main observation is that $E_\omega = \int d^4x \sqrt{-g} E_4 \omega$ can be rewritten as a surface integral in the space with arbitrary metric if and only if

$$\nabla_\alpha \nabla_\beta \omega = 0$$

$$\Sigma^{\mu\nu\alpha\beta} = 2R(g^{\alpha\mu} g^{\beta\sigma} - g^{\alpha\beta} g^{\mu\nu}) + 4R^{\mu\nu} g^{\alpha\beta} + 4g^{\mu\nu} R^{\alpha\beta} - 8g^{\mu\beta} R^{\alpha\nu} - 4R^{\mu\alpha\nu\beta}$$

$$\delta E_\omega = \int d^4x \sqrt{-g}\, h_{\mu\nu} \Sigma^{\mu\nu\alpha\beta} \nabla_\alpha \nabla_\beta \omega$$

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# Solution to the condition $\nabla_\mu \nabla_\nu \omega = 0$

- Flat space: conformal transformations
- Curved space:

$$[\nabla_\gamma \nabla_\beta] \nabla_\alpha \omega = R_{\beta\gamma\lambda\alpha} \nabla^\lambda \omega = 0$$

$$R_{\alpha\beta} \nabla^\beta \omega = 0$$

Only trivial solution $\omega =$const for the generic metric.

Only scale symmetry can be anomaly-free in all curved spaces with dynamical metric.

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# General Lagrangian for the dilaton and gravity

- Two derivatives

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}\zeta\phi^2 R + \frac{1}{2}(\partial_\mu\phi)^2\right)$$

- Four derivatives, $\tau = \log\phi$

$$S = \int d^4x\sqrt{-g}\left[AR\Box\tau + CR(\partial_\mu\tau)^2) + BG^{\mu\nu}\partial_\mu\tau\partial_\nu\tau + \right.$$
$$\left. + C\tau E_4 + ER^2 + FW^2_{\mu\nu\lambda\rho} + G((\partial_\mu\tau)^2)^2 + H(\Box\tau)^2 + J(\Box\tau + (\partial_\mu\tau)^2)^2\right]$$

Scale-invariant non-conformal operators are expected to be suppressed by the gravitational scale while the operators which are conformal in flat space can have lower cutoff scale.

# Conclusions

- The idea of conformal symmetry appearing at high energies and spontaneously broken al low energies can be realized in nature
- Not all scale invariant higher derivative operators are conformally invariant
- In flat space conformal symmetry can be made anomaly-free
- In theories with gravity conformal symmetry reduces to the scale symmetry due to the presence of the anomaly

Thank you

for your attention!