

Jordan algebras and Conformal Dynamical Symmetries

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CONFORMAL ANOMALIES: THEORY AND APPLICATIONS.
TOURS 2022

- Newton-Hooke duality

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- H-atom Spectrum Generating Algebra $SO(2, 4)$

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- Majorana spinors and $Sp(4, \mathbb{R})$

- Tekin Dereli, Philippe Nounahon, Todor Popov
A Remarkable Dynamical Symmetry of
the Landau Problem.
IoP Tekin Dereli's Festschrift, Istanbul 2021
- Mariana Kirchbach, Todor Popov, José-Antonio Vallejo.
Color confinement at the boundary of AdS_5 .
Journal of High Energy Physics, 2021(9).

Kepler problem and hodographs

Super-integrable model

$$H = \frac{p^2}{2\mu} - \frac{k}{r} \quad E_n = -\frac{\mu k^2}{2\hbar^2} \frac{1}{n^2}$$

Angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Laplace-Runge-Lenz vector, Pauli (19)

$$\mathbf{M} = \frac{1}{2\mu}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - k \frac{\mathbf{r}}{r}$$

Closed orbits: Newton and Hooke potentials

$$-\frac{1}{r} \quad r^2$$

Trajectories in momentum space are (part) of circles!

Vladimir Fock (1935), Hamilton (1843)

Newton-Hooke duality

Hydrogen atom versus Harmonic Oscillators
2T-physics (Itzhak Bars)

e^- in electric field $\overset{dual}{\leftrightarrow}$ e^- in magnetic field .

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Kustaanheimo-Stiefel Conformal spinorial regularization

3D H-atom \leftrightarrow 4D Harmonic Oscillator
 $SO(2, 4)$ \cong $SU(2, 2)$

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Levi-Civita (Bohlin) regularization

2D H-atom \leftrightarrow 2D Harmonic Oscillator
 $SO(2, 3)$ \cong $Sp(4, \mathbb{R})$ (Landau problem)

Dirac "A Remarkable Representation of the 3 + 2 de Sitter Group"

H-atom Spectrum Generating Algebra $SO(2, 4)$

Conformal dynamical symmetry $SO(2, 4)$ of H-atom spectrum

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} & B_0 - A_0 &= r & B_0 + A_0 &= r\mathbf{p}^2 \\ \mathbf{\Gamma} &= r\mathbf{p} & \mathbf{B} - \mathbf{A} &= r & \mathbf{B} + \mathbf{A} &= r\mathbf{p}^2 - 2\mathbf{p}(r \cdot \mathbf{p}) \\ D &= \mathbf{r} \cdot \mathbf{p} - i \end{aligned}$$

Causal space-time automorphisms of the Minkowski space $\mathbb{R}^{1,3}$

$\text{co}(\mathfrak{J})$	space-time cone	hydrogen atom
\mathfrak{g}_0	$M_{\mu\nu}; D$	$L_{ij}, \Gamma_i; D$
\mathfrak{g}_{-1}	K_μ	$B_\mu + A_\mu$
\mathfrak{g}_{+1}	P_μ	$B_\mu - A_\mu$

A_i Laplace-Runge-Lenz vector

B_i dual Laplace-Runge-Lenz vector

Barut's H-atom SGA $SO(2, 4)$

Conformal algebra of Minkowski $\mathbb{R}^{1,3}$

$$L_{AB} \in \mathfrak{so}(2, 4), \quad \eta_{AB} = \text{diag}(+1, +1, -1, -1, -1, -1).$$

$$[L_{AB}, L_{CD}] = -i(\eta_{AC}L_{BD} + \eta_{BD}L_{AC} - \eta_{AD}L_{BC} - \eta_{BC}L_{AD}) \quad (1)$$

$$L_{AB} = \begin{pmatrix} 0 & B_0 & B_1 & B_2 & B_3 & D \\ & 0 & \Gamma_1 & \Gamma_2 & \Gamma_3 & A_0 \\ & & 0 & L_3 & -L_2 & A_1 \\ & & & 0 & L_1 & A_2 \\ & & & & 0 & A_3 \\ & & & & & 0 \end{pmatrix} \quad A, B \in \{-1, 0, 1, 2, 3, 5\}.$$

Fock's sphere and null cone

Null cone of isotropic rays in $\mathbb{R}^{2,4}$

$$\mathcal{N} = \{\vec{x} \in \mathbb{R}^{2,4} | x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_5^2 = 0; x \neq 0\}.$$

compactified null cone $\mathcal{M}_{1,3} = \mathcal{N}/\mathbb{R}^* \cong (\mathbb{S}^1 \times \mathbb{S}^3)/\mathbb{Z}_2$

$$p_0 = -2\mu E > 0$$

Bander Itzykson 1968
stereographic projection

$$\mathbf{u} = \frac{p^2 + p_0^2}{p^2 + p_0^2} \mathbf{n} + \frac{2p_0}{p^2 + p_0^2} \mathbf{p}$$

2D H-atom SGA $SO(2, 3)$

Conformal algebra of Minkowski $\mathbb{R}^{1,2}$

$$L_{ab} = \begin{pmatrix} 0 & B_0 & B_1 & B_2 & D \\ & 0 & \Gamma_1 & \Gamma_2 & A_0 \\ & & 0 & L_3 & A_1 \\ & & & 0 & A_2 \\ & & & & 0 \end{pmatrix} \quad a, b \in \{-1, 0, 1, 2, 3 = 5\}$$

Null Cone of isotropic rays in $\mathbb{R}^{2,3}$

$$\tilde{\mathcal{N}} = \{\vec{y} \in \mathbb{R}^{2,3} \mid y_{-1}^2 + y_0^2 - y_1^2 - y_2^2 - y_3^2 = 0; y \neq 0\}$$

Compactified null cone

$$\mathcal{M}_{1,2} = \tilde{\mathcal{N}}/\mathbb{R}^* \cong (\mathbf{S}^1 \times \mathbf{S}^2)/\mathbb{Z}_2$$

Euclidean Jordan algebra \mathfrak{J}

Algebra of observables in Quantum Mechanics
(Pascual Jordan)

$$x \circ y = y \circ x = \frac{1}{2}(xy + yx)$$

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x) \quad \forall x, y \in \mathfrak{J}.$$

\mathfrak{J} is commutative but is not associative

Euclidean Jordan algebra (Jordan, von Neumann, Wigner)

$$x^2 + y^2 = 0 \quad \Rightarrow \quad x = 0 \quad \text{and} \quad y = 0$$

it is power-associativity, $x^{m+n} = x^m \circ x^n$

conformal (Möbius) algebra

$$\mathrm{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \cong \mathfrak{J} \oplus \mathrm{str}(\mathfrak{J}) \oplus \mathfrak{J}^*$$

$\mathrm{co}(\mathfrak{J})$ is a 3-graded Lie algebra with involution \dagger

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \mathfrak{g}_k^\dagger = \mathfrak{g}_{-k}$$

$$(x, y, z) := [[x, y^\dagger], z] \quad x, y, z \in \mathfrak{g}_{+1} \cong \mathfrak{J}.$$

Jordan algebra and Jordan triple product

$$(abc) = a \circ (b \circ c) - b \circ (a \circ c) + (a \circ b) \circ c .$$

Jordan Triple System

$$\begin{aligned}(abc) &= (cba) & (2) \\ (ab(cdx)) - (cd(abx)) &= (a(dcb)x) - ((cda)bx) .\end{aligned}$$

a linear map $S_x^y : \mathfrak{J} \rightarrow \mathfrak{J}$ through

$$S_x^y(z) = (xyz) .$$

Structure algebra $\text{str}(\mathfrak{J})$

$$[S_a^b, S_c^d] = S_{(abc)}^d - S_c^{(dab)} = S_a^{(bcd)} - S_{(cda)}^b \quad (3)$$

TKK conformal algebra $\mathfrak{co}(\mathfrak{J})$

Structure constants $\Sigma_{\mu\rho}^{\nu\sigma}$

$$S_x^y(z) = (xyz) \quad (e_\mu, e_\nu, e_\rho) = \Sigma_{\mu\rho}^{\nu\sigma} e_\sigma .$$

determine conformal algebra $\mathfrak{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \cong \mathfrak{J}$

$$\begin{aligned} [U_a, U^b] &= -2S_a^b, & [U_a, U_b] &= 0, & U_a &\in \mathfrak{g}_{-1} \\ [S_a^b, U_c] &= U_{(abc)}, & [S_a^b, S_c^d] &= S_{(abc)}^d - S_c^{(bad)}, & S_a^b &\in \mathfrak{g}_0 \\ [S_a^b, U^c] &= -U^{(bac)}, & [U^a, U^b] &= 0, & U^b &\in \mathfrak{g}_{+1} \end{aligned}$$

many layer realization of the Erlangen program

$$Aut(\mathfrak{J}) \subset Str(\mathfrak{J}) \subset Co(\mathfrak{J}) \quad (4)$$

Group	Transformations	Space
$Aut(\mathfrak{J})$	rotations $x' = Ox$	eulidean
$Str_0(\mathfrak{J})$	Lorentz group $x' = \Lambda x$	pseudo-euclidean
$\mathcal{P}(\mathfrak{J})$	Poincaré group $x' = \Lambda x + a$	affine pseudo-euclidean
$\mathcal{P}_+(\mathfrak{J})$	Similitude group $x' = e^\rho(\Lambda x + a)$	rescaled affine pseudo-euclidean
$Co(\mathfrak{J})$	Möbius group $z' = \frac{az+b}{cz+d}$	projective

transformations acting on projective space of rays in the Hilbert space.

Jordan Triple Structure constants

$$x = x^\mu \sigma_\mu = x = x^\mu \sigma_\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \quad x \in \mathfrak{J}_2^{\mathbb{C}}$$

$$(\sigma_\alpha, \sigma_\beta, \sigma_\gamma) = \Sigma_{\alpha\gamma}^{\beta\rho} \sigma_\rho = \sigma_\alpha \circ (\sigma_\beta \circ \sigma_\gamma) - \sigma_\beta \circ (\sigma_\gamma \circ \sigma_\alpha) + (\sigma_\alpha \circ \sigma_\beta) \circ \sigma_\gamma$$

gives back the concise formula

$$\Sigma_{\alpha\gamma}^{\beta\rho} = \delta_\gamma^\rho \delta_\alpha^\beta + \delta_\alpha^\rho \delta_\gamma^\beta - g^{\beta\rho} g_{\alpha\gamma}. \quad (5)$$

Günaydın's Oscillator Realization of $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2, 4)$

$\mathfrak{co}(\mathfrak{J})$	operator	$\in \mathfrak{co}(\mathfrak{J})$	mapping	x -rep basis $\mathfrak{so}(2, 4)$
\mathfrak{J}	$U_a = -ia^\mu P_\mu$	$\in \mathfrak{g}_{-1}$	$x \mapsto a$	$P_\nu = i\partial_\nu$
$\mathfrak{str}(\mathfrak{J})$	$S_a^b = ia^\nu b_\mu S_\nu^\mu$	$\in \mathfrak{g}_0$	$x \mapsto (a, b, x)$	$S_\nu^\mu = -i\Sigma_{\nu\alpha}^{\mu\beta} x^\alpha \partial_\beta$
\mathfrak{J}^*	$U^b = ib_\mu K^\mu$	$\in \mathfrak{g}_{+1}$	$x \mapsto -(x, b, x)$	$K^\mu = i\Sigma_{\nu\alpha}^{\mu\beta} x^\nu x^\alpha \partial_\beta$

Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$ and 3-graded Lie algebra $\mathfrak{so}(2, 4)$

\mathfrak{g}_{-1}	$-iP_\nu = \partial_\nu$	translations
\mathfrak{g}_0	$iM^\mu_\nu = -x^\mu \partial_\nu + x_\nu \partial^\mu$	Lorentz transformations
\mathfrak{g}_0	$iD = x^\mu \partial_\mu$	dilatation
\mathfrak{g}_{+1}	$iK^\mu = -2x^\mu x^\nu \partial_\nu + x^\nu x_\nu \partial^\mu$	special conformal

$\mathfrak{co}(\hat{\mathfrak{J}}_2^{\mathbb{C}}) = \mathfrak{so}(2, 4)$ acting on $\mathcal{M}_{1,3} = \mathcal{N}/\mathbb{R}^* \cong (\mathcal{S}^1 \times \mathcal{S}^3)/\mathbb{Z}_2$

$$\mathfrak{co}(\hat{\mathfrak{J}}_2^{\mathbb{C}}) = \mathfrak{so}(2, 4) = \underbrace{(\hat{\mathfrak{J}}_2^{\mathbb{C}})^*}_{K^\mu} \oplus \overbrace{(\underbrace{\mathfrak{so}(1, 3)}_{M_\nu^\mu} \oplus \underbrace{\mathbb{R}}_D)}^{\mathfrak{st}(\hat{\mathfrak{J}}_2^{\mathbb{C}})} \oplus \underbrace{\hat{\mathfrak{J}}_2^{\mathbb{C}}}_{P_\nu} \quad \mu, \nu = 0, 1, 2, 3$$

$$x = x^\mu \sigma_\mu = x = x^\mu \sigma_\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \quad x \in \hat{\mathfrak{J}}_2^{\mathbb{C}}$$

$$\Sigma_{\alpha\gamma}^{\beta\rho} = \delta_\gamma^\rho \delta_\alpha^\beta + \delta_\alpha^\rho \delta_\gamma^\beta - g^{\beta\rho} g_{\alpha\gamma} . \quad (6)$$

real Pauli matrices with coordinates $\{y_0, y_1, y_2\}$

$$y = \sum_{\mu=0,1,3} x^\mu \sigma_\mu = \begin{pmatrix} y_0 + y_2 & y_1 \\ y_1 & y_0 - y_2 \end{pmatrix}, \quad y^T = y. \quad (7)$$

conformal algebra of Minkowski space $\mathbb{R}^{1,2}$ acting on $\mathcal{M}_{1,2} = \mathcal{N}/\mathbb{R}^* \cong (\mathcal{S}^1 \times \mathcal{S}^2)/\mathbb{Z}_2$

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) = \mathfrak{so}(2,3) = \underbrace{(\mathfrak{J}_2^{\mathbb{R}})^*}_{K_{\tilde{\mu}}} \oplus \underbrace{\overbrace{(\mathfrak{so}(1,2) \oplus \mathbb{R})}^{\text{stt}(\mathfrak{J}_2^{\mathbb{R}})}}_{M_{\tilde{\nu}} \oplus D} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{R}}}_{P_{\tilde{\nu}}} \quad \tilde{\mu}, \tilde{\nu} = 0, 1, 2$$

Landau Problem and Harmonic oscillator

Larmour frequency $\omega = \frac{eB}{mc}$

$$H = \frac{1}{2m} \left(\mathbf{p} - \frac{e}{c} \mathcal{A} \right)^2 =: \frac{1}{2m} \mathbf{P}^2. \quad (8)$$

$$\mathcal{A} = (\mathcal{A}_x, \mathcal{A}_y) = \frac{B}{2}(-y, x), \quad \mathcal{A}_i = -\frac{B}{2} \epsilon_{ij} x^j \quad (9)$$

$$H = \frac{P_x^2 + P_y^2}{2m} = \frac{\hbar\omega}{4} \left\{ p_\xi^2 + p_\eta^2 + (\xi^2 + \eta^2) \right\} - \frac{\hbar\omega}{2} (\xi p_\eta - \eta p_\xi).$$

$$H = \frac{\hbar\omega}{2} \{ a^+, a^- \}, \quad [H, a^\pm] = \pm a^\pm, \quad [H, b^\pm] = 0. \quad (10)$$

Dirac's realization of $\mathfrak{so}(2, 3)$

"A Remarkable Representation of the 3 + 2 de Sitter Group"

$$\begin{aligned}
 m_{12} &= \frac{1}{2}(z\partial - \bar{z}\bar{\partial}) &= \frac{1}{4}(\{b^-, b^+\} - \{a^-, a^+\}), \\
 m_{23} &= \frac{1}{16}(z^2 + \bar{z}^2) - (\partial^2 + \bar{\partial}^2) &= \frac{1}{4}(\{a^-, b^+\} + \{a^+, b^-\}), \\
 m_{31} &= \frac{i}{16}(z^2 - \bar{z}^2) + i(\partial^2 - \bar{\partial}^2) &= \frac{i}{4}(\{a^-, b^+\} - \{a^+, b^-\}), \\
 m_{1-1} &= \frac{-i}{16}(z^2 - \bar{z}^2) + i(\partial^2 - \bar{\partial}^2) &= \frac{i}{4}(a^+a^+ - a^-a^- + b^-b^- - b^+b^+), \\
 m_{2-1} &= \frac{1}{16}(z^2 + \bar{z}^2) + (\partial^2 + \bar{\partial}^2) &= \frac{1}{4}(a^-a^- + a^+a^+ + b^-b^- + b^+b^+), \\
 m_{3-1} &= -\frac{i}{2}(z\partial + \bar{z}\bar{\partial} - 1) &= -\frac{i}{4}(\{a^-, b^-\} - \{a^+, b^+\}), \\
 m_{01} &= -\frac{1}{2}(z\bar{\partial} - \bar{z}\partial) &= -\frac{1}{4}(a^-a^- + a^+a^+ - b^-b^- - b^+b^+), \\
 m_{02} &= -\frac{i}{2}(\bar{z}\partial + z\bar{\partial}) &= -\frac{i}{4}(a^-a^- - a^+a^+ + b^-b^- - b^+b^+), \\
 m_{03} &= \frac{1}{8}z\bar{z} + (\partial\bar{\partial} + \bar{\partial}\partial) &= -\frac{1}{4}(\{a^+, b^+\} + \{a^-, b^-\}), \\
 m_{-10} &= \frac{1}{8}z\bar{z} - (\partial\bar{\partial} + \bar{\partial}\partial) &= \frac{1}{4}(\{a^-, a^+\} + \{b^-, b^+\}).
 \end{aligned}$$

Landau Hamiltonian $\frac{H}{\hbar\omega} = m_{-10} - m_{12} = \frac{1}{2}\{a^-, a^+\}$
 Angular momentum $\frac{L_z}{\hbar} = m_{12} \quad \omega = \frac{eB}{mc}$

2D H-atom and Landau problem

$$\begin{array}{ccccc}
 \mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) & = & \mathfrak{so}(2, 4) & \xleftarrow{\text{Kustaanheimo-Stiefel}} & \mathfrak{su}(2, 2) \ . \\
 \uparrow x=x^t & & & & \uparrow \psi=\psi^c \\
 \mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) & = & \mathfrak{so}(2, 3) & \xleftarrow{\text{Levi-Civita}} & \mathfrak{sp}(4, \mathbb{R})
 \end{array}$$

2D e^- in electric field

$\text{Newton} \leftrightarrow \text{Hooke}$

2D e^- in magnetic field

Defining representation of $\mathfrak{su}(2, 2)$

matrices in $\mathfrak{sl}(4, \mathbb{C})$ preserving a pseudo-Hermitian form β with signature $(++--)$

$$(\phi, \psi) = \phi_1^* \psi_1 + \phi_2^* \psi_2 - \phi_3^* \psi_3 - \phi_4^* \psi_4 = \phi^\dagger \beta \psi = \bar{\phi} \psi .$$

The Hermitian matrix $\beta = \beta^\dagger$ depends on the basis, it fixes the choice of the Dirac matrix $\gamma_0 := \beta$ and the invariance of the form implies

$$\beta \sigma^{AB} \beta^{-1} = (\sigma^{AB})^\dagger .$$

the 4×4 Dirac gamma matrices γ^μ ,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3 .$$

The 15 $\mathfrak{su}(2, 2)$ -generators σ^{AB} $A, B \in \{-1, 0, 1, 2, 3, 5\}$

$$\begin{aligned} \sigma^{\mu\nu} &= \frac{i}{4} [\gamma^\mu, \gamma^\nu] , & \sigma^{-15} &= -\frac{1}{2} \gamma^5 , \\ \sigma^{\mu 5} &= \frac{i}{4} [\gamma^\mu, \gamma^5] , & \sigma^{-1\mu} &= -\frac{1}{2} \gamma^\mu . \end{aligned} \quad (11)$$

Here we denoted $\gamma^5 := \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

minimal (zero-mass) $U(2, 2)$ representation

2 complex variables z^α and derivatives $w_\alpha = -i \frac{\partial}{\partial z^\alpha}$

$$\mathbf{z} = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \quad \boldsymbol{\partial} = \begin{pmatrix} \frac{\partial}{\partial z^1} \\ \frac{\partial}{\partial z^2} \end{pmatrix} \quad \psi = \begin{pmatrix} \bar{z} \\ \partial \end{pmatrix} \quad \bar{\psi} = (-\bar{\partial}, z). \quad (12)$$

A canonical representation of a pair of 2D harmonic oscillators

$$[\psi^\alpha, \bar{\psi}_\beta] = \delta_\beta^\alpha, \quad [\psi^\alpha, \psi^\beta] = 0.$$

Mack & Todorov zero-mass $U(2, 2)$ representation

$$J^{AB} = \bar{\psi} \sigma^{AB} \psi, \quad C_1 = \bar{\psi} \psi$$

its restriction to Poincaré is irreducible

the helicities λ are labelling the zero-mass representations of $U(2, 2)$.

$$C_1 + 2 = -2\lambda = z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} \quad \lambda = 0, \pm \frac{1}{2}, \pm 1, \dots$$

The hydrogen atom is described by helicity $\lambda = 0$

Weyl Spinors and $Sp(4, \mathbb{R})/\mathbb{Z}_2 \cong SO(2, 3)$

Majorana reduction from $SU(2, 2)$ to $Sp(4, \mathbb{R})$ -rep
(Stoyanov & Todorov)

$$\psi^c = \psi$$

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} + \partial \\ z + \bar{\partial} \end{pmatrix}, \quad \chi_\alpha^* = (b^+ \ a^+).$$

2D H-atom $\mathfrak{so}(2, 3)$ -spinorial representation, Barut & Duru

$$\begin{aligned} m_{ij} &= \frac{1}{2} \epsilon_{ijk} \chi^* \sigma_k^T \chi, & m_{-1i} &= \frac{i}{4} (\chi^* \sigma_i^T \epsilon^T (\chi^*)^T - \chi^T \epsilon \sigma_i^T \chi), \\ m_{-10} &= \frac{1}{2} (\chi^* \chi + 1), & m_{0i} &= \frac{1}{4} (\chi^* \sigma_i^T \epsilon^T (\chi^*)^T + \chi^T \epsilon \sigma_i \chi) \end{aligned}$$

Dirac's realization of $\mathfrak{so}(2, 3)$

"A Remarkable Representation of the 3 + 2 de Sitter Group"

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 \end{aligned}$$

$\mathfrak{so}(2)$ generated by conformal Hamiltonian m_{-10}

$\mathfrak{so}(3)$: generated by 2D Runge-Lenz vector (m_{23}, m_{31}) and m_{12}

Reduction of Kustaanheimo-Steifel KS transform

In celestial mechanics KS removes the singular trajectories due to binary collisions from the phase space of 3D Kepler motion
Kustaanheimo-Stiefel Conformal spinorial regularization

$$\begin{array}{ccc} 3D \text{ H-atom} & \leftrightarrow & 4D \text{ Harmonic Oscillator} \\ SO(2, 4) & \cong & SU(2, 2) \end{array}$$

elliptic Kepler orbits $E < 0$: geodesic motion on sphere S^3
Hodograph: stereographic projection of the momentum space
(Fock 1935)

Kirchbach, M., Popov, T., & Vallejo, J.A. (2021). Color confinement at the boundary of the conformally compactified AdS_5 . *Journal of High Energy Physics*, 2021(9).

Conformal Time Transformations $SO(1, 2)$

TKK construction of the Jordan algebra $\mathfrak{J}_1^{\mathbb{R}} = \mathbb{R}$

$$\mathfrak{co}(\mathbb{R}) = \mathfrak{so}(1, 2) = (\mathfrak{J}_1^{\mathbb{R}})^* \oplus \mathfrak{str}(\mathfrak{J}_1^{\mathbb{R}}) \oplus \mathfrak{J}_1^{\mathbb{R}} = \mathbb{R}K_0 \oplus \mathbb{R}D \oplus \mathbb{R}P_0 .$$

different $SO(1, 2)$ -generator is a conformal hamiltonian:

B_0	$=$	$r(p^2 + 1)/2$	$E < 0$	bound states ,
A_0	$=$	$r(p^2 - 1)/2$	$E > 0$	scattering states ,
$A_0 + B_0$	$=$	rp^2	$E = 0$	free motion .

Hopf fibration and KS regularization

$$0 \rightarrow S^1 \hookrightarrow S^3 \rightarrow S^2 \rightarrow 0 ,$$

$$|\mathbf{x}| = |\mathbf{z}|^2 , \quad \mathbf{z} = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} , \quad |\mathbf{z}|^2 = \mathbf{z}^\dagger \mathbf{z} .$$

KS is cotangent extension of Hopf mapping (Bruno Cordani)

$$KS : T^+ S^3 \rightarrow T^+ S^2 \quad T^+ S^n = T^* S^n - \{0\}$$

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$$\begin{aligned} x_1 &= u_1 u_3 + u_2 u_4 , \\ x_2 &= u_2 u_3 - u_1 u_4 , \\ x_3 &= -u_1^2 - u_2^2 + u_3^2 + u_4^2 , \end{aligned}$$

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$$p_1 = -(u_1 w_3 + w_1 u_3 + u_2 w_4 + w_2 u_4) / |\mathbf{z}|^2 ,$$

$$p_2 = -(u_2 w_3 + w_2 u_3 - w_1 u_4 - u_1 w_4) / |\mathbf{z}|^2 ,$$

$$p_3 = (u_1 w_1 + u_2 w_2 - u_3 w_3 - u_4 w_4) / |\mathbf{z}|^2 .$$

The coordinates on $T^+ S^3$ are subject to the constraint

$$K = u_1 w_2 - u_2 w_1 + u_3 w_4 - u_4 w_3 = 0 .$$

Majorana Spinor Reduction

$$|\mathbf{x}| = |\mathbf{z}|^2, \quad \mathbf{z} = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}, \quad |\mathbf{z}|^2 = \mathbf{z}^\dagger \mathbf{z}.$$

Planar reduction

$$u_2 = u_4 = 0, \quad w_2 = w_4 = 0$$

Levi-Civita regularization $T^+ S^1 \rightarrow T^+ S^1$

$$\xi + i\eta = Z^2, \quad \begin{aligned} \xi &= u_1^2 - u_3^2, & p_\xi &= (u_1 w_1 - w_3 u_3)/|Z|^2, \\ \eta &= 2u_1 u_3, & p_\eta &= -(u_1 w_3 + w_1 u_3)/|Z|^2 \end{aligned}$$

real spinor $\psi = \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$ is written as $Z = u_1 + iu_3 \in \mathbb{C}$

$$0 \rightarrow S^0 \hookrightarrow S^1 \rightarrow S^1 \rightarrow 0,$$

Weyl Spinors and $Sp(4, \mathbb{R})/\mathbb{Z}_2 \cong SO(2, 3)$

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} + \partial \\ z + \bar{\partial} \end{pmatrix}, \quad \chi_\alpha^* = (b^+ \ a^+).$$

2D H-atom $\mathfrak{so}(2, 3)$ -spinorial representation, Barut & Duru

$$\begin{aligned} m_{ij} &= \frac{1}{2} \epsilon_{ijk} \chi^* \sigma_k^T \chi, & m_{-1i} &= \frac{i}{4} (\chi^* \sigma_i^T \epsilon^T (\chi^*)^T - \chi^T \epsilon \sigma_i^T \chi), \\ m_{-10} &= \frac{1}{2} (\chi^* \chi + 1), & m_{0i} &= \frac{1}{4} (\chi^* \sigma_i^T \epsilon^T (\chi^*)^T + \chi^T \epsilon \sigma_i \chi) \end{aligned}$$

4D Dirac spinor yields 3D H-atom $\mathfrak{so}(2, 4)$ -spinorial representation

$$[\psi^\alpha, \bar{\psi}_\beta] = \delta_\beta^\alpha, \quad [\psi^\alpha, \psi^\beta] = 0.$$

via Mack & Todorov ladder $U(2, 2)$ representation

$$J^{AB} = \bar{\psi} \sigma^{AB} \psi, \quad G_1 = \bar{\psi} \psi$$

its Majorana reduction is $Sp(4, \mathbb{R})$ -rep: Stoyanov & Todorov

$$\psi^c = \psi$$

Scalar Quantum Field Theory

induced representation

$$\begin{aligned}P_{\mu}\chi(x) &= \partial_{\mu}\chi(x) \\M_{\mu\nu} &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) + \Sigma_{\mu\nu}\chi(x) \\D\chi(x) &= \{-ix^{\nu}\partial_{\nu} + \delta\}\chi(x) \\K_{\mu}\chi(x) &= i(2x_{\mu}x_{\nu}\partial^{\nu} - x^2\partial_{\mu} + 2ix^{\nu}[g_{\mu\nu}\delta - \Sigma_{\mu\nu}] + k_{\mu})\chi(x)\end{aligned}$$

Little group

$$(SL(2, \mathbb{C}) \times D) \ltimes T_4 \quad \{\Sigma_{\mu\nu}, \delta, k_{\mu}\}$$

$$U(\Lambda, a)\chi_{\sigma}(x)U(\Lambda, a)^{-1} = \mathcal{D}_{\sigma\sigma'}(\lambda^{-1})\chi_{\sigma'}(\Lambda x + a)$$

$$k_{\nu} = 0 \quad \delta = i\Delta \mathbb{1}$$

Hydrogen atom $\Delta = -1$ (multiplicity $\lambda = 0$)

Others Hopf fibrations

Guowu Meng, Universal \mathfrak{J} Kepler problem

$$\begin{array}{cccccccc} \mathbb{R} & 0 & \rightarrow & S^0 & \hookrightarrow & S^1 & \rightarrow & S^1 & \rightarrow & 0, \\ \mathbb{C} & 0 & \rightarrow & S^1 & \hookrightarrow & S^3 & \rightarrow & S^2 & \rightarrow & 0, \\ \mathbb{H} & 0 & \rightarrow & S^3 & \hookrightarrow & S^7 & \rightarrow & S^4 & \rightarrow & 0, \\ \mathbb{O} & 0 & \rightarrow & S^7 & \hookrightarrow & S^{15} & \rightarrow & S^8 & \rightarrow & 0, \end{array}$$

Murat Günaydın & Feza Gürsey, "Quark structure and octonions" J. Math. Phys. **14**, 1651-1667

Dubois-Violette, Michel. "Exceptional quantum geometry and particle physics." Nuclear Physics B 912 (2016): 426-449.

Standard Model: exceptional Jordan algebra $\mathfrak{J} = H_3(\mathbb{O})$

$$F_4 \subset E_6 \subset E_7$$