Double scaling limits for field theory defects

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Based on 2202.03471 $\oplus$ work in progress with Jorge Russo

- Studying defects in QFT is interesting for a number of reasons
    - Explore all operators in a QFT: extended operators may be sensible to finer details (e.g. topology of space, global properties of gauge group…)
    - If topological, they give rise to generalized symmetries
    - May serve as a diagnose the phases of the theory
    - Describe impurities coupled to the system
    - …

- However QFT (with/without defects) is hard…

- One strategy which has proved very successful is to look for small parameters on which one can expand. Celebrated examples include

    - the semiclassical approximation

    - large N

    - large spin

- In the recent past one new item added to the list

    - large charge sectors

- Inspired by this: can we access new information about defects in QFT???

Contents

Correlation functions in N=2 and large charge

- N=2 theories are interesting playgrounds to tinker with QFT: they have SUSY enough so as to constrain dynamics to accessible limits but not too much so as to “trivialize”

- Some of them have holographic duals

- In particular, one can exploit SUSY to compute observables exactly

LOCALIZATION

This includes correlators, defect operators and even the partition function itself (meaningful for 4d N=2)

- The 4d superconformal algebra contains

$$\{\overline{Q}^a_{\dot\alpha}, \overline{S}^b_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\epsilon^{ab}\Big(\Delta - \frac{R}{2}\Big) + \epsilon^{ab}M_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}J^{ab}$$

- Hence an interesting shortening condition is

$$[\overline{Q}^a_{\dot\alpha}, O] = 0 \rightsquigarrow \Delta_O = \frac{R_O}{2},\, j_L = s = 0,\, (\text{and } \mathrm{j_R} = 0) \longrightarrow$$ **Chiral Primary Operator (CPO)**

- CPO's have a non-singular OPE (not to violate the BPS bound). As a consequence, they form a ring: the chiral ring

- Their 2-point functions are

$$\langle O_I(0)\overline{O}_{\overline{J}}(x)\rangle = \frac{g_{I\overline{J}}(\tau^i, \overline{\tau}^i)}{|x|^{2\Delta_I}}\,\delta_{\Delta_I,\Delta_{\overline{J}}}$$

Endowes the Coulomb branch of a very interesting geometry…but that's another story. See Papadodimas; Baggio, Niarchos & Papadodimas

- The 2-point functions can be mapped to the sphere

$$\langle A(x)\overline{B}(0)\rangle = \frac{C_{AB}}{|x|^{2\Delta_A}}\,\delta_{\Delta_A\Delta_B} \rightsquigarrow \langle |x|^{2\Delta_A} A(x)\overline{B}(0)\rangle = C_{AB}\delta_{\Delta_A\Delta_B}$$

- To extract C, we can take the large x limit

$$\lim_{|x|\to\infty} |x|^{2\Delta_A} A(x) = 4^{\Delta_A} \lim_{|x|\to\infty} \left(1+\frac{|x|^2}{4}\right)^{\Delta_A} A(x)$$

- Since

$$ds^2_{\mathbb{R}^4} = \left(1+\frac{|x|^2}{4}\right)^4 ds^2_{\mathbb{S}^4}$$

- …it follows that $4^{\Delta_A}\langle A(N)\overline{B}(S)\rangle_{\mathbb{S}^4} = C_{AB}$

Computed through a matrix integral thanks to localization

- There is one subtlety, though: due to the conformal anomaly there can be mixing

$$O_{\Delta}^{\mathbb{R}^4} \to O_{\Delta}^{\mathbb{S}^4} + \frac{\alpha_1}{R^2}\, O_{\Delta-2}^{\mathbb{S}^4} + \frac{\alpha_2}{R^4}\, O_{\Delta-4}^{\mathbb{S}^4} + \cdots$$

- Remove this mixing by Gram-Schmidt orthogonalization!

Gerchkovitz, Gomis, Ishtiaque, Komargodski & Pufu, 1602.05971

- Let us look to the correlators of the simplest operators $\mathcal{O}_n = (\mathrm{Tr}\phi^2)^n$ in SU(N) SQCD

$$\begin{aligned} \frac{G_{2n}^{\mathrm{QCD}}}{G_{2n}^{\mathcal{N}=4}} &= 1 - \frac{9\, n\, (N^2+2n-1)\, \zeta(3)}{4\pi^2\, (\mathrm{Im}\tau)^2} \\ &+ \frac{5\, n\, (2N^2-1)\, (3N^4+(15n-3)N^2+(20n^2-15n+4))\, \zeta(5)}{4\pi^3\, N\, (N^2+3)\, (\mathrm{Im}\tau)^3} + \cdots . \end{aligned}$$

$$G_{2n}^{\mathcal{N}=4,\mathfrak{g}} = \frac{n!\, 2^{2n}}{(\mathrm{Im}\tau)^{2n}} \alpha\, (1+\alpha)_{n-1}\,, \qquad \alpha = \frac{1}{2}\dim(\mathfrak{g})\,.$$

- The polynomial in n multiplying each order in the coupling is just the appropriate so as to define the double scaling limit (at FIXED N!)

$$n \to \infty\,, \qquad g \to 0\,, \qquad \lambda \equiv g^2 n = \text{fixed}\,,$$

(Gauge instantons truly supressed!)

Bourget, R-G & Russo, 1803.00580

- Going beyond this tower by explicit computation is hard. The next simplest case is SU(3): there is only one more CPO. Explicitly computing the correlators shows that the limit continues to exist

Beccaria, 1809.06280
Beccaria, 1810.10483

- It turns out that the existence of the limit is rooted in the structure of the correlators: the GS can be recasted as a matrix model

  - Very sketchy: for SU(2) there is only one CPO, whose sphere correlators are derivatives of Z wrt. the coupling. The flat space correlators are ratios of subdeterminants of the matrix of derivatives

  - It turns out that each such subdeterminant can be written as a matrix integral: convert the computation of correlators into a matrix model!

$$\det \mathcal{M}_{(n)} = \frac{1}{n!} \int_0^\infty \prod_{j=0}^{n-1} dx_j e^{-4\pi \operatorname{Im} \tau x_j} x_j^{\frac{1}{2}} Z_{\text{1-Loop}} \left(\sqrt{x_j}\right) \prod_{j<k} (x_j - x_k)^2 \ .$$

  - The ’t Hooft limit of this matrix model is well defined: it is our double scaling limit (strictly speaking, the latter is the weak ’t Hooft coupling regime)

Grassi, Komargodski & Tizziano, 1908.10306
Beccaria, Galvagno & Hasan, 2001.06645

↓

(note that in any case, gauge instantons are safely supressed in this regime)

Wilson loops in the k-fold symmetrized product

- Consider now circular Wilson loops in the k-fold symmetrized representation. The exact formula is

$$\langle W_k \rangle = \frac{1}{N}\,\frac{1}{Z_N}\int d^N a\,\prod_{i<j}(a_i - a_j)^2\,Z_{1-\text{loop}}\,Z_{\text{inst}}\,e^{-\frac{8\pi^2}{g^2}\sum_{i=1}^N a_i^2}\,W_k\,,$$

- For N=4 both the instanton and 1-loop contributions are 1

- The insertion is the character of the k-fold symm rep (of U(N)/SU(N)). This is easy to compute: the generating function is by definition the PE of the fundamental. Then

$$W_k = (-1)^{N-1} \sum_{i=1}^{N} \frac{e^{2\pi(N-1)\,a_i + 2k\pi a_i}}{\prod_{j\neq i} e^{2\pi a_j} - e^{2\pi a_i}}\,. \qquad \longleftrightarrow \qquad W_k = \sum_{1\leq i_1 \leq i_2 \cdots \leq i_k \leq N} e^{2\pi a_{i_1} + 2\pi a_{i_2} + \cdots + 2\pi a_{i_k}}\,.$$

- Hence in the end

$$\langle W_k \rangle = \frac{(-1)^{N-1}}{Z_{U(N)}} \int d^N a \prod_{k<l} (a_k - a_l)^2 \, Z_{1-\text{loop}} \, Z_{\text{inst}} \, e^{-\frac{8\pi^2}{g^2} \sum_{m=1}^N a_m^2} \frac{e^{2\pi(N-1)\,a_N + 2k\pi a_N}}{\prod_{j\neq N} e^{2\pi a_j} - e^{2\pi a_N}}\,.$$

- For SU(N) impose

$$\textstyle\sum_{i=1}^N a_i = 0$$

- Introduce now $\kappa = g^2 k$
- Then (we specify to U(N) N=4)

$$\langle W_k \rangle = \frac{(-1)^{N-1}}{Z_{U(N)}} e^{\frac{k\kappa}{8}(1+\frac{N-1}{k})^2} \int d^N a \prod_{k<l} (a_k - a_l)^2 \, e^{-k\frac{8\pi^2}{\kappa}\sum_{m=1}^{N-1} a_m^2} \left(\frac{e^{-k\frac{8\pi^2}{\kappa}\left(a_N - a_N^\star\right)^2}}{\prod_{j\neq N} e^{2\pi a_j} - e^{2\pi a_N}} \right) .$$

$$a_N^\star \equiv \frac{\kappa}{8\pi}(1 + \frac{N-1}{k})$$

- This suggests the limit FOR FIXED N

$$g \to 0 \,, \quad k \to \infty \,, \qquad g^2 k = \kappa = \text{fixed} \,.$$

- Note that in this limit gauge instantons are completely supressed (just like in the large charge limit)

- The N-th eigenvalue gets stabilized at a much larger scale than the rest… so the integral breaks in two pieces

$$\langle W_k \rangle = \frac{(-1)^{N-1}}{Z_{U(N)}}\, e^{\frac{k\kappa}{8}(1+\frac{N-1}{k})^2} \int d^N a \prod_{k<l}(a_k - a_l)^2\, e^{-k\frac{8\pi^2}{\kappa}\sum_{m=1}^{N-1} a_m^2} \left(\frac{e^{-k\frac{8\pi^2}{\kappa}\left(a_N - a_N^\star\right)^2}}{\prod_{j\neq N} e^{2\pi a_j} - e^{2\pi a_N}} \right).$$

$e^{2\pi a_j} \to \sim 0$

$$a_N^\star \equiv \frac{\kappa}{8\pi}\left(1 + \frac{N-1}{k}\right)$$

$$\langle W_k \rangle = \frac{Z_{U(N-1)}}{Z_{U(N)}}\, e^{\frac{k\kappa}{8}(1+\frac{N-1}{k})^2} \int da_N \left(\frac{a_N^2}{e^{2\pi a_N} - 1} \right)^{N-1} e^{-k\frac{8\pi^2}{\kappa}(a_N - \frac{\kappa}{8\pi})^2}$$

- Doing the last integral (saddle) and putting all factors

$$\langle W_k \rangle = \frac{1}{N!}\left(\frac{k\,\kappa}{4}\right)^{N-1} e^{\frac{k\kappa}{8}(1+\frac{N-1}{k})^2} \left(e^{\frac{\kappa}{4}} - 1\right)^{1-N}$$

- Note that

We could take N large provided it is much smaller than k

$$\langle W_k \rangle = \frac{(-1)^{N-1}}{Z_{U(N)}} e^{\frac{k\kappa}{8}(1+\frac{N-1}{k})^2} \int d^N a \prod_{k<l} (a_k - a_l)^2 \, e^{-k\frac{8\pi^2}{\kappa}\sum_{m=1}^{N-1} a_m^2} \left(\frac{e^{-k\frac{8\pi^2}{\kappa}\left(a_N - a_N^\star\right)^2}}{\prod_{j\neq N} e^{2\pi a_j} - e^{2\pi a_N}} \right).$$

$$a_N^\star \equiv \frac{\kappa}{8\pi}(1 + \frac{N-1}{k})$$

- Our result becomes then

$$\langle W \rangle = \frac{e^{-S}}{\sqrt{2\pi N}}, \qquad S = -\frac{k\kappa}{8} - N \log\left(\frac{k\,\kappa}{4N}\right) - N + N\,\log\left(1 - e^{-\frac{\kappa}{4}}\right)$$

$$S = -\frac{k\kappa}{8} - N\log\left(\frac{k\,\kappa}{4N}\right) - N$$

- Lets compare with the holographic/matrix model@large N result

Drukker & Fiol

$$S_{\rm DF} = -2N\left[\frac{k\sqrt{\lambda}}{4N}\sqrt{1+\frac{k^2\lambda}{16N^2}} + \operatorname{arcsinh}\left(\frac{k\sqrt{\lambda}}{4N}\right)\right] \longrightarrow S_{DF} \sim -\frac{k\kappa}{8} - N\log\left(\frac{k\,\kappa}{4N}\right) - N$$

- What about SU(N)? Do

$$\sum_{i=1}^{N} a_i^2 = \sum_{i=1}^{N} \hat{a}_i^2 + \frac{1}{N}(\sum_{i=1}^{N} a_i)^2\,, \qquad \hat{a}_i = a_i - \frac{1}{N}\sum_{i=1}^{N} a_i$$

- Then

$$\langle W_k\rangle = \frac{(-1)^{N-1}}{Z_N}\int d^N a \prod_{k<l}(\hat{a}_k - \hat{a}_l)^2\, e^{-\frac{8\pi^2}{g^2}\sum_{m=1}^{N}\hat{a}_m^2}\, e^{-\frac{8\pi^2 N}{g^2}x^2 - 2\pi k x}\,\frac{e^{2\pi(N-1)\,\hat{a}_N + 2k\pi\hat{a}_N}}{\prod_{j\neq N} e^{2\pi\hat{a}_j} - e^{2\pi\hat{a}_N}}\,, \qquad x = \tfrac{1}{N}\textstyle\sum_{i=1}^{N} a_i$$

- The a’s sum zero: relax this by introducing a delta

$$\langle W_k\rangle = \frac{(-1)^{N-1}}{Z_N}\int d^N \hat{a} \prod_{k<l}(\hat{a}_k - \hat{a}_l)^2\, e^{-\frac{8\pi^2}{g^2}\sum_{m=1}^{N}\hat{a}_m^2}\,\frac{e^{2\pi(N-1)\,\hat{a}_N + 2k\pi\hat{a}_N}}{\prod_{j\neq N} e^{2\pi\hat{a}_j} - e^{2\pi\hat{a}_N}}\,\delta(\sum_{i=1}^{N}\hat{a}_i)\left(\int dx\, e^{-\frac{8\pi^2 N}{g^2}x^2 - 2\pi k x}\right)$$

- One recognizes the SU(N) result

$$\langle W_k \rangle_{U(N)} = \left(\frac{Z_{SU(N)}}{Z_{U(N)}} \int dx\, e^{-\frac{8\pi^2 N}{g^2}\, x^2 - 2\pi k x} \right) \langle W_k \rangle_{SU(N)} \longrightarrow \langle W_k \rangle_{U(N)} = e^{\frac{g^2 k^2}{8\,N}} \langle W_k \rangle_{SU(N)}$$

- The prefactor is a loop for the U(1) part

$$\frac{Z_{SU(N)}}{Z_{U(N)}} \int dx\, e^{-\frac{8\pi^2 N}{g^2}\, x^2 - 2\pi k x} = \frac{\int da\, e^{-\frac{8\pi^2}{g^2}\, a^2 - 2\pi \frac{k}{N} a}}{\int da\, e^{-\frac{8\pi^2}{g^2}\, a^2}}$$

- If k>N this is a (leading) contribution: this observable is sensible to U vs SU!!!

Defects in WF

- Using the same strategy we can also study lines in the WF fixed point near d=4, 6d.

  D.R-G.

  - Also a similar double-scaling limit exists

    Arias-Tamargo, Russo & R-G, Watanabe, Badel, Cuomo, Monin & Rattazzi, Hellerman, Orlando, Reffert *et al.*

  - Can be interpreted as effective description of large spin impurities in magnets

    Cuomo, Komargodski, Mezei & Raviv-Moshe

- For instance, consider O(2N+1) near d=4

$$S = \int \frac{1}{2}|\partial\vec{\varphi}|^2 + \frac{g}{4}(\vec{\varphi}^2)^2\,,$$

- One may imagine the trivial line along one direction. It admits a deformation

$$\mathcal{D}(\vec{z}) = e^{-h\int d\tau\,\varphi^{2N+1}(\tau,\vec{z})} = e^{-h\int dx\,\varphi^{2N+1}\,\delta_T(\vec{x}-\vec{z})} \longrightarrow \langle\mathcal{D}(\vec{z})\rangle = \int e^{-\int \frac{1}{2}|\partial\vec{\varphi}|^2 + \frac{g}{4}(\vec{\varphi}^2)^2 + h\varphi^{2N+1}\,\delta_T(\vec{x}-\vec{z})}\,.$$

- Assume $g = \frac{\lambda}{n}\,, \quad h = \nu\, n$

- Then

$$\langle \mathcal{D}(\vec{z}) \rangle = \int e^{-n S_{\text{eff}}}, \qquad S_{\text{eff}} = \int \frac{1}{2} |\partial \vec{\phi}|^2 + \frac{\lambda}{4} (\vec{\phi}^2)^2 + \nu \phi^{2N+1}\, \delta_T(\vec{x} - \vec{z})\,.$$

- So imagine taking large n with everything else fixed: use saddle point. The only relevant eq. can be easily solved

$$\partial^2 \phi^{2N+1} - \lambda (\phi^{2N+1})^3 - \nu\, \delta_T(\vec{x} - \vec{z}) = 0\,.$$

- Finally

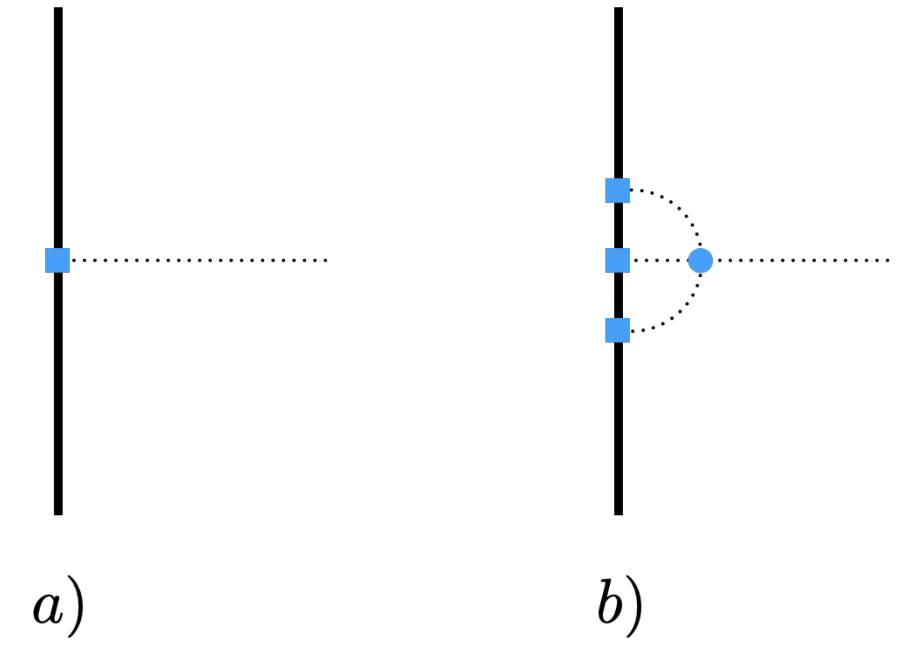
$$\begin{aligned} S_{\text{eff}} \quad &= \quad \Big(-\frac{\nu^2}{2} + \frac{\lambda \nu^4}{128 \pi^2 \epsilon} + \frac{\lambda \nu^4}{128 \pi^2} (3 - \gamma_E + \log(4\pi)) \Big)\, T \int \frac{d^{d-1} \vec{p}}{(2\pi)^{d-1}} \frac{1}{\vec{p}^2} \\ &\quad - \frac{\lambda \nu^4}{128 \pi^2}\, T \int \frac{d^{d-1} \vec{p}}{(2\pi)^{d-1}} \frac{\log |p|^2}{\vec{p}^2}\,. \end{aligned}$$

- Since $\frac{1}{\langle \mathcal{D} \rangle} \frac{d}{d\nu} \langle \mathcal{D} \rangle = -n \int \langle \phi^{2N+1} \rangle \, \delta_T(\vec{x} - \vec{z})$.
- We can use this to define the renormalized coupling

$$\nu = \mu^{\frac{\epsilon}{2}} \left(\nu_R + \frac{\lambda \nu_R^3}{2(4\pi)^2 \epsilon} \right) ,$$

- We can compute the beta function, which shows a fixed point

$$\mu \frac{d\nu_R}{d\mu} = -\frac{\epsilon}{2} \nu_R + \frac{\lambda \nu_R^3}{(4\pi)^2} \, . \quad \longrightarrow \quad \nu_R^2 = \frac{8\pi^2}{\lambda} \epsilon \, , \quad \longleftrightarrow$$



Allais & Sachdev,
Cuomo, Komargodski & Mezei

- …where

$$\langle \phi^{2N+1} \rangle = -\nu_R \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{e^{-i\vec{p}\cdot\vec{x}}}{|\vec{p}|^{2-\frac{\epsilon}{2}}} \sim \frac{1}{|\vec{x}_T|^{\frac{d-2}{2}}}$$

- One can also compute correlators of defect fields as well as correlators of defects themselves. For instance, for the latter

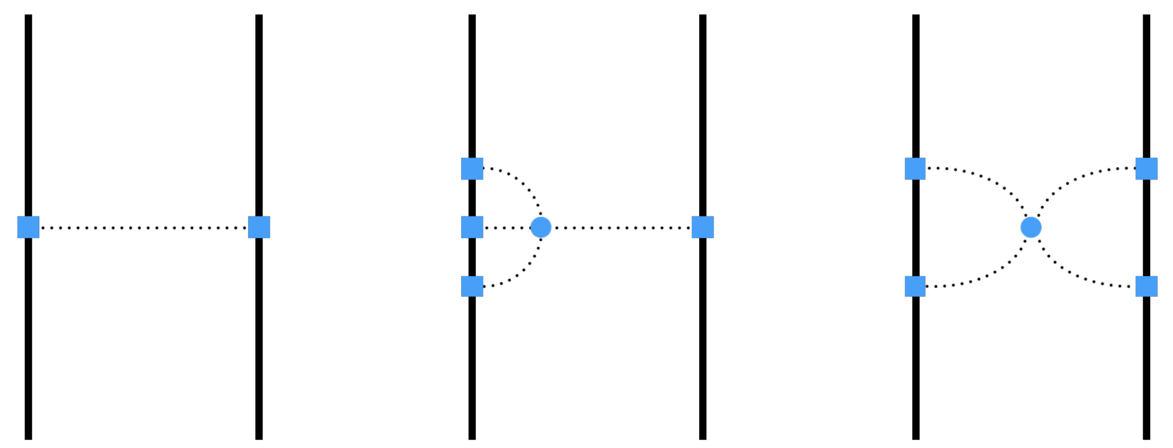
$$\langle \mathcal{D}(z_1)\mathcal{D}(z_2)\rangle = \int e^{-\int \frac{1}{2}|\partial\vec{\varphi}|^2+\frac{g}{4}(\vec{\varphi}^2)^2+h\varphi^{2N+1}\,\delta_T(\vec{x}-\vec{z}_1)+h\varphi^{2N+1}\,\delta_T(\vec{x}-\vec{z}_2)}\,.$$

- Assuming the same scaling

$$\langle \mathcal{D}(z_1)\mathcal{D}(z_2)\rangle = \int e^{-nS_{\rm eff}}\,, \qquad S_{\rm eff} = \int \frac{1}{2}|\partial\vec{\phi}|^2+\frac{\lambda}{4}(\vec{\phi}^2)^2+\nu\phi^{2N+1}\,\delta_T(\vec{x}-\vec{z}_1)+\nu\phi^{2N+1}\,\delta_T(\vec{x}-\vec{z}_2)\,.$$

- The saddle point eqs. are

$$\partial^2\phi^{2N+1} - \nu\,\delta_T(\vec{x}-\vec{z}_1) - \nu\,\delta_T(\vec{x}-\vec{z}_2) = 0\,. \longrightarrow \phi^{2N+1} = \rho_1(\vec{x}) + \rho_2(\vec{x})\,, \qquad \rho_i(\vec{x}) = -\nu\int dy\, G(x-y)\delta_T(\vec{y}-\vec{z}_i)\,.$$



$a)$ $b)$ $c)$

- So finally

$$\langle \mathcal{D}(z_1)\mathcal{D}(z_2)\rangle = \langle \mathcal{D}(z_1)\rangle\langle \mathcal{D}(z_2)\rangle e^{-nS_{\rm I}} \longrightarrow S_{\rm I} = -\Big(\frac{\nu_R^2}{4\pi}+\frac{3\lambda\nu_R^4}{512\pi}-\frac{\lambda\nu_R^4}{64\pi^3}(3+\gamma_E+\log(4\pi))\Big)\frac{T}{|\vec{z}_1-\vec{z}_2|^{1+(-\epsilon+\frac{\lambda\nu_R^2}{8\pi^2})}}\,. \longrightarrow S_{\rm I} = -\Big(\frac{\nu_R^2}{4\pi}+\frac{3\lambda\nu_R^4}{512\pi}-\frac{\lambda\nu_R^4}{64\pi^3}(3+\gamma_E+\log(4\pi))\Big)\frac{T}{|\vec{z}_1-\vec{z}_2|}\,.$$

c.f. Soderberg (free case)

Final comments

- Inspired by the large charge developments, we introduced a double scaling limit for defects
- In the WF fixed near 4/6d we considered lines/surfaces: a fixed point leading to a dCFT exists
- One can compute correlators of defect operators/defects themselves
- We introduced a novel double scaling limit for the k-fold symmetrized Wilson loop

Allows to compute exact observables for finite N in gauge theories (free of gauge instantons!)

- One could imagine many interesting applications (some in progress)
    - U(N) vs SU(N)
    - Other correlators: e.g. CPO-Wilson loop
    - Other theories: e.g. SQCD, N=2*…
    - Going beyond conformality
    - …

Many thanks!!!