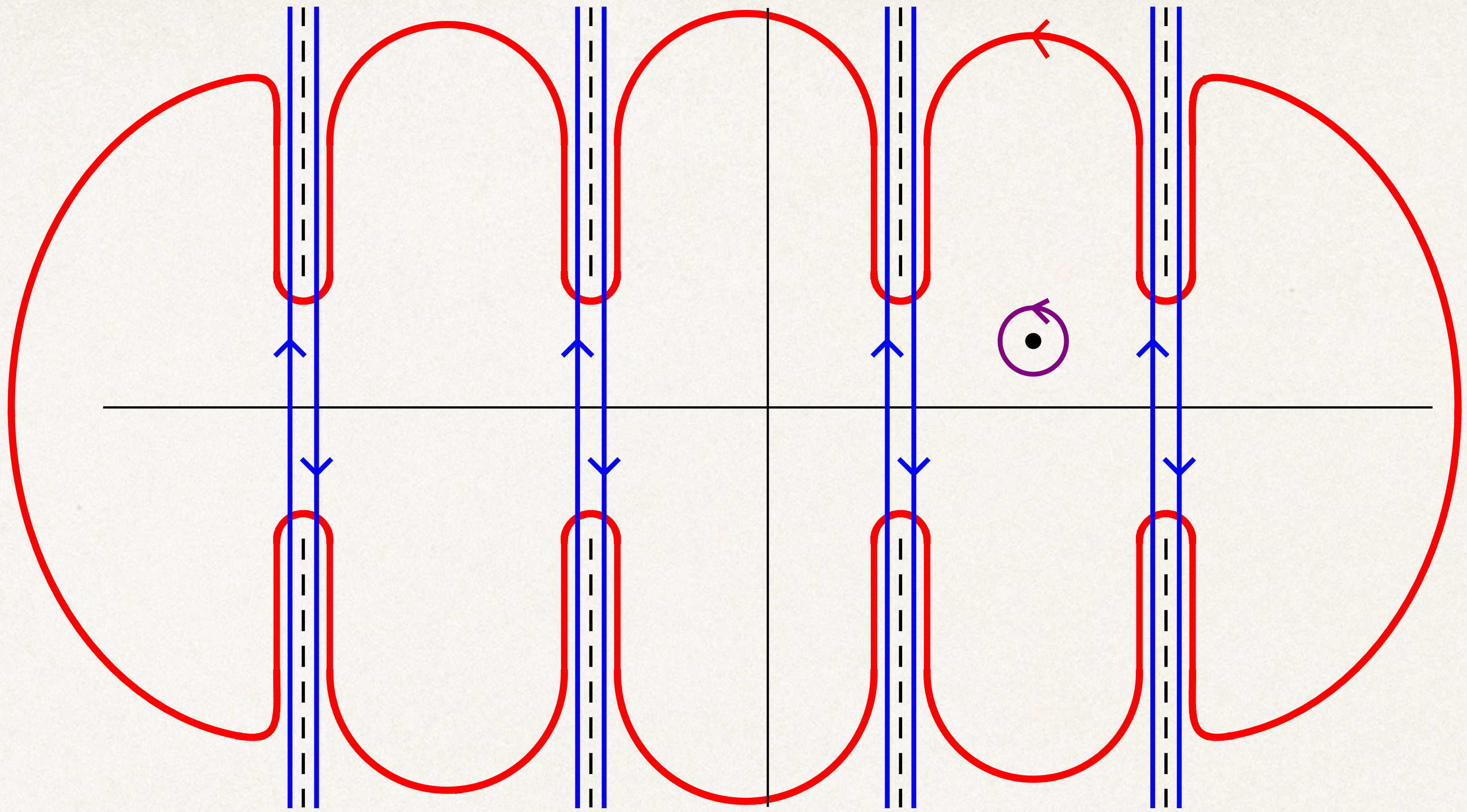

# Conformal Surface Defects in 4d Maxwell Theory are Trivial

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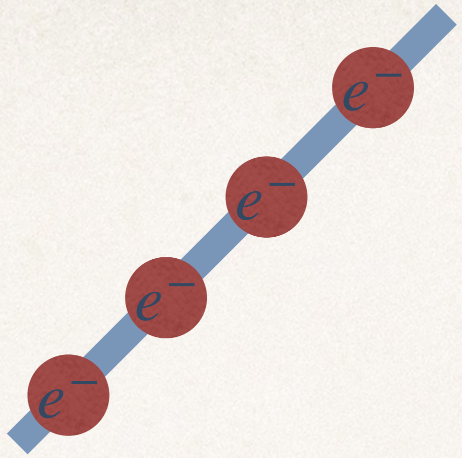
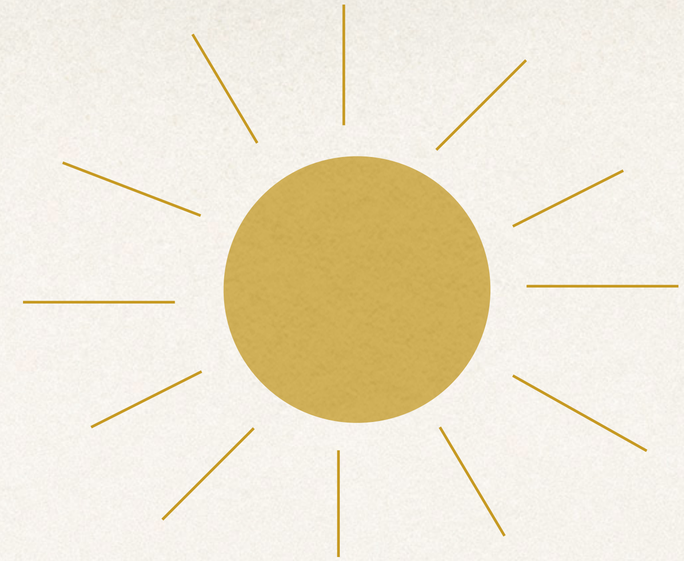
*May 31, 2022*

2202.09180 with Abhay Shrestha




$$\int A_\mu J^\mu \, d^2x$$



If I couple a photon to charged matter on a wire, can the system flow to a nontrivial conformal fixed point under the renormalization group?

*No*

NB: The surface of the title includes time.

# Symmetry Breaking

Recall that the photon has
full conformal symmetry = Poincare × special conformal × dilatations

$$= SO(4,2)$$

The wire must break this at least to $SO(2,2) \times SO(2)$

conformal group on the wire

rotations around wire

NB: We are looking for a fixed point theory with this symmetry.

Definition of Trivial: Only generalized free fields (obey Wick's Theorem) can exist on the wire and "interact" with the photon.

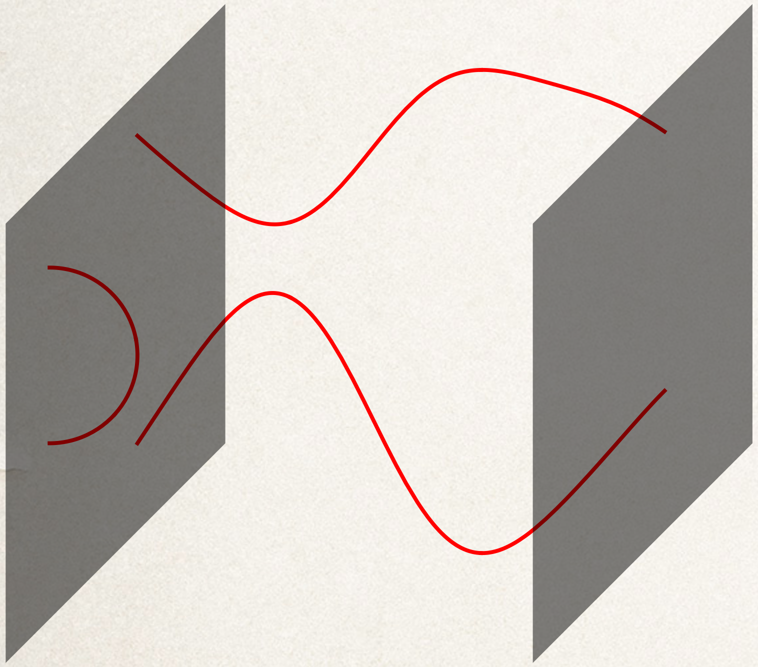
Why you may care:

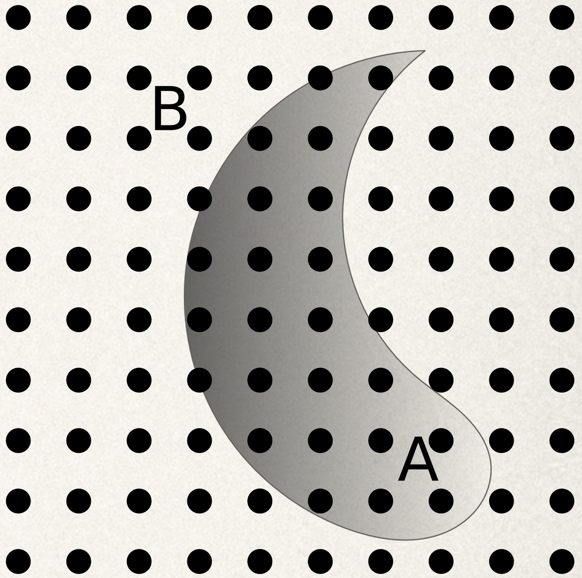
Basic Theory Question: interesting to know that the coupling $\int A_\mu J^\mu \, d^2x$ will never produce anything interesting.

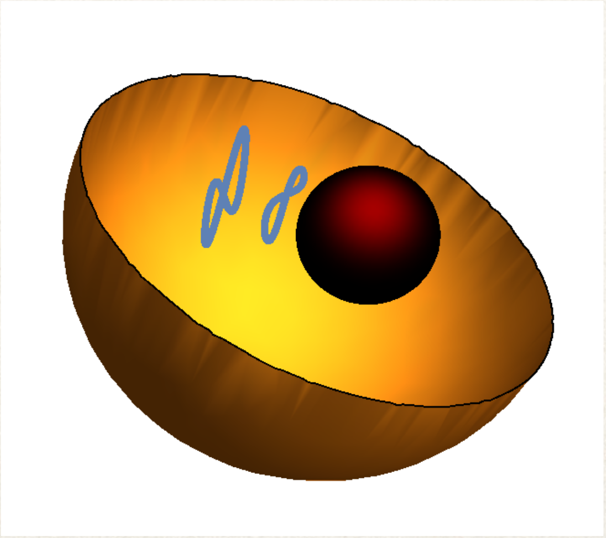
Applications: May be useful for people interested in quantum wires in condensed matter and cosmic strings in cosmology.

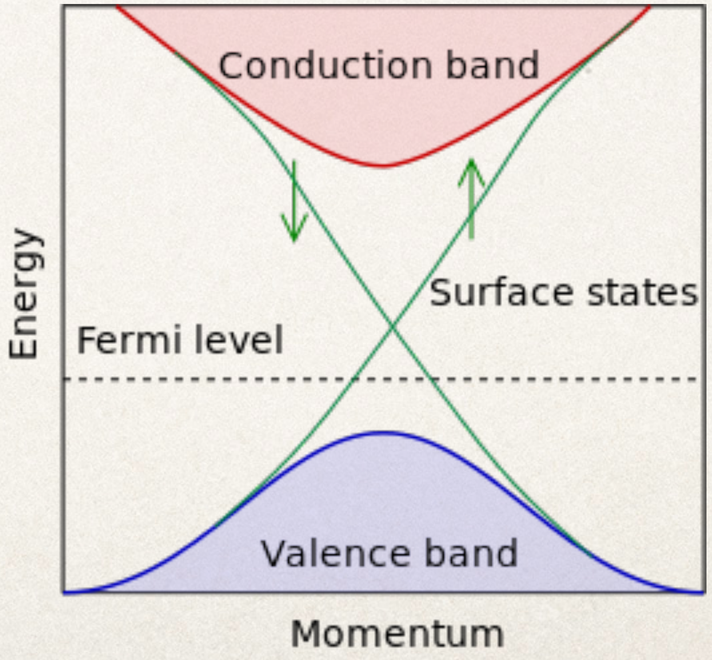
…. But there's a larger story that I would like to tell before getting to the proof.

# Areas in physics where defects and boundaries are prominent







# A Lesson from AdS/CFT

extra dimensions give insight into strong coupling physics

strongly interacting $\mathcal{N} = 4$ super Yang Mills in 4d $\sim$ classical type IIB supergravity on $AdS_5 \times S^5$

can we get rid of gravity on the right hand side and replace with a defect or boundary QFT?

remark: defect conformal field theory in 10d with 4d defects is Weyl equivalent to the QFT on $AdS_5 \times S^5$ (think D3-branes)

# A purely QFT analog of the AdS/CFT correspondence

charged, massless fermions
on a 2+1d surface
coupled to a 4d photon

$\sim$ 3d QED in a large $N_\mathrm{f}$ limit

$$S = -\frac{1}{4g^2}\int_{\mathcal{M}} \mathrm{d}^4x\, F^{\mu\nu}F_{\mu\nu} + \int_{\partial\mathcal{M}} \mathrm{d}^3x(i\bar{\psi}_i \not{D}\psi_i)$$

boundary conditions: $F_{nA} = g^2\bar{\psi}\gamma_A\psi$

vs.

$$S = \int_{\mathcal{M}} \mathrm{d}^3x\left[-\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}_i\not{D}\psi_i\right]$$

where $D_\mu = \nabla_\mu - iA_\mu$

photon propagator for mixed dimensional QED
(don’t FT the normal direction $y$)

$$-i\frac{e^{-py}}{p}\eta^{AB} \qquad \text{(Feynman gauge)}$$

propagator for large $N_f$ QED$_3$, resummed

$$-i\frac{\eta^{AB}}{p^2(1+\Pi(p))} \qquad \text{where} \qquad \Pi(p)=\frac{N_f e^2}{8|p|}+O(N_f^0)$$

Compensated by vertices, 3d $e$ drops out of the amplitudes.
For scattering processes on the boundary ($y$=0),
the Feynman rules are the same in the IR with the identification $\frac{1}{N_f}\sim g^2$

(Kotikov-Teber ’13)

# Mixed dimensional QED has something for everyone

$$S = -\frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^4x\, F^{\mu\nu}F_{\mu\nu} + \int_{\partial\mathcal{M}} \mathrm{d}^3x (i\bar{\psi}\not{D}\psi)$$

where $D_\mu = \nabla_\mu - igA_\mu$ boundary conditions: $F_{nA} = g\bar{\psi}\gamma_A\psi$

- relation to graphene
- relation to large $N_f$ $QED_3$ (Kotikov-Teber '13; Gaiotto '14)
- behavior under electric-magnetic duality (Son, Hsiao '17)
- example of a bCFT with an exactly marginal coupling
- conformal symmetry anomalies
- supersymmetric versions →
  - localization, exact results for transport

our work

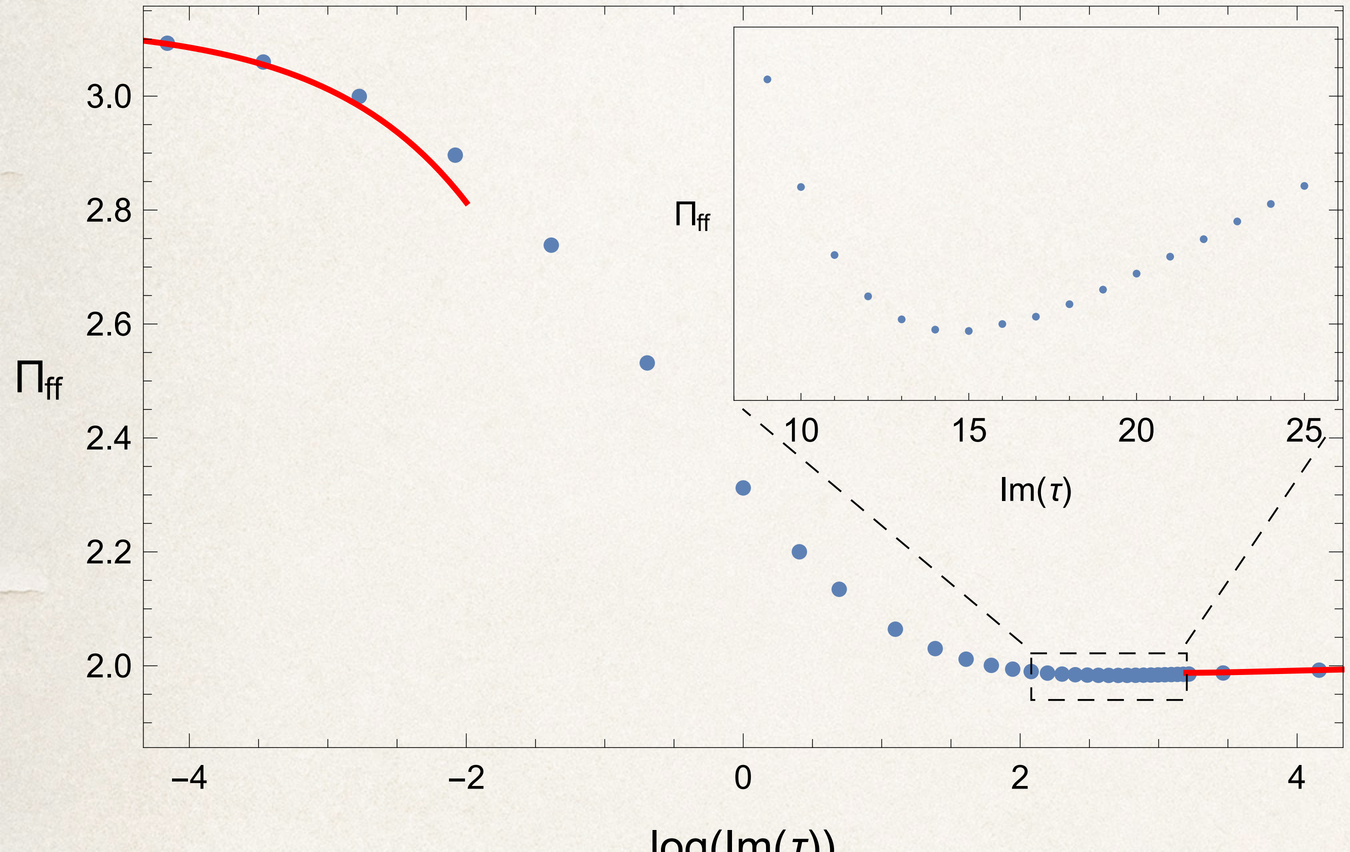
# An Exact Conductivity

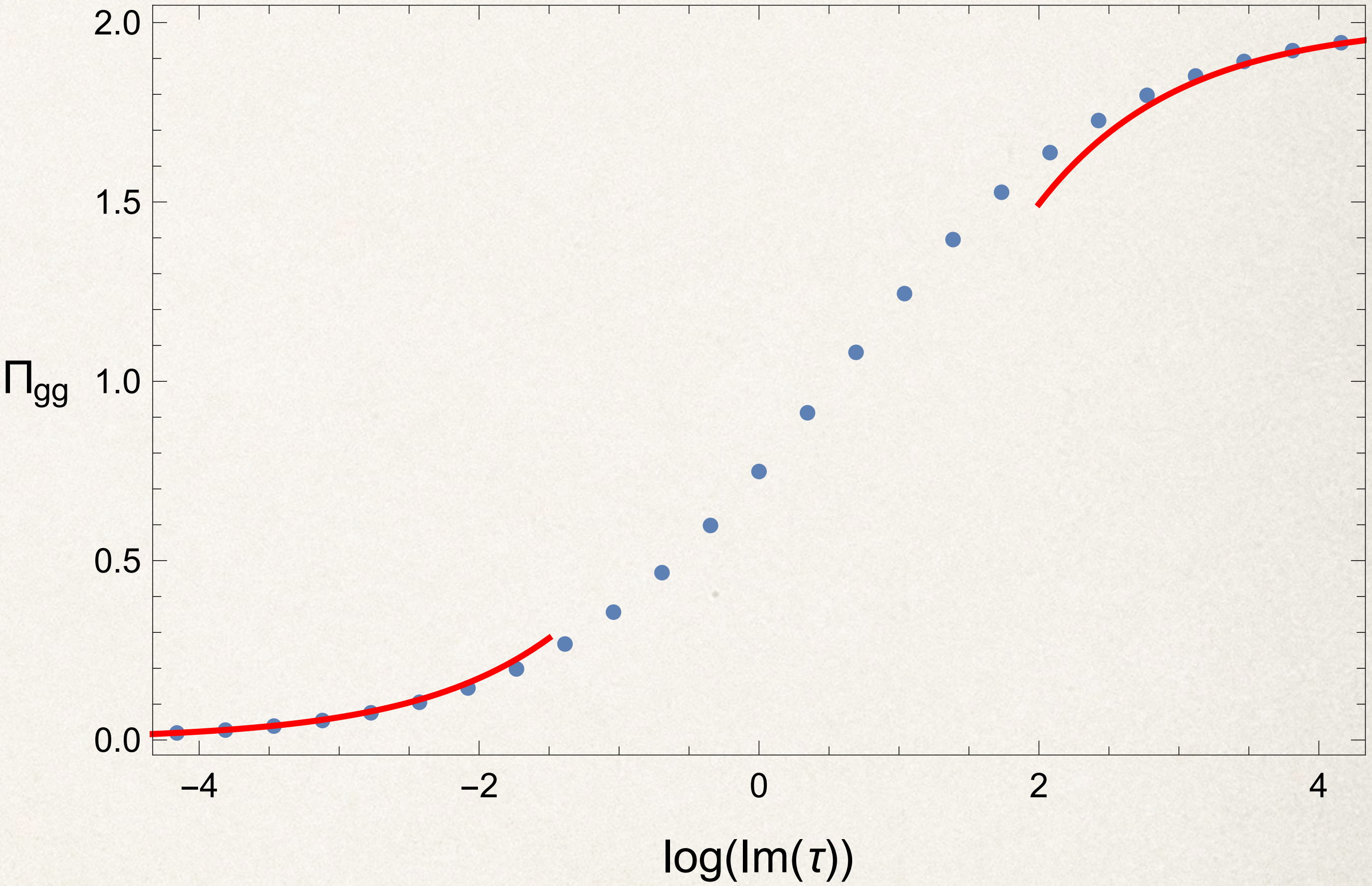
$$\tau = \frac{2\pi i}{g^2} + \frac{\theta}{2\pi}$$

(Gupta, H, Jeon '19)

$n_+ = 1\ ,\ \ n_- = 1$ theory at $\theta = 0$

$q_\pm = q_f \pm q_g$




points numerical

Very special that we can do this!

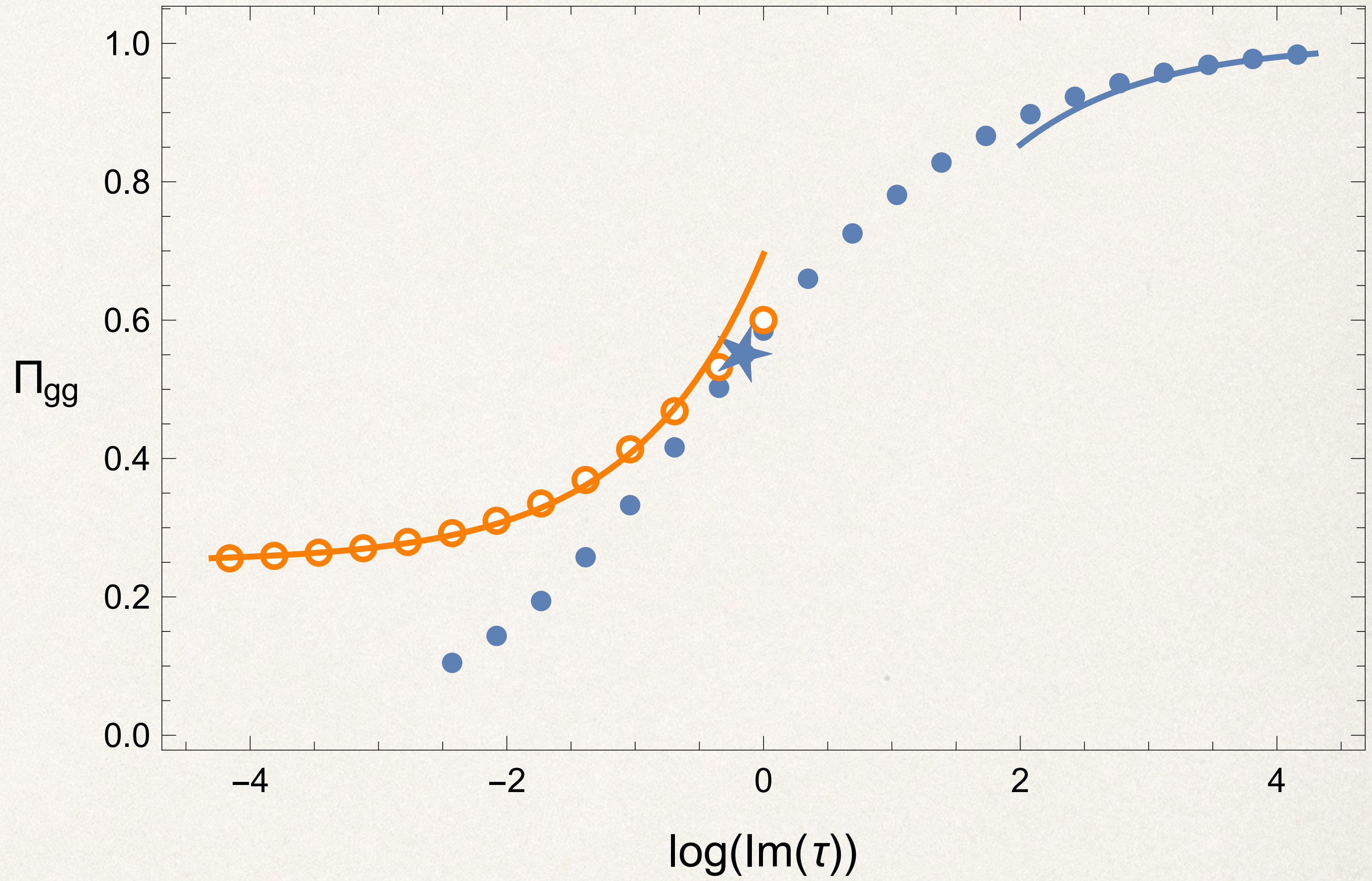
curves are saddle point approximations

# Another Exact Conductivity

$n_+ = 1\,, \;\; n_- = 0$ theory

orange $\mathrm{Re}(\tau) = \dfrac{1}{2}$

blue $\mathrm{Re}(\tau) = 0$



# free scalar coupled to boundary stuff

(H, Huang '17;
Behan, di Pietro, Lauria, van Rees '22)

$$\int \mathrm{d}^d x\, (\partial\phi)^2 + g \int \mathrm{d}^{d-1} x\, (\partial_n \phi)\mathcal{O}$$

Dirichlet boundary conditions

$$\int \mathrm{d}^d x\, (\partial\phi)^2 + g \int \mathrm{d}^{d-1} x\, \phi\mathcal{O}$$

Neumann boundary conditions

want $(\partial_n\phi)\mathcal{O}$ or $\phi\mathcal{O}$ to be slightly relevant so we have control in perturbation theory over endpoint of the RG flow

# 2d bry stuff = minimal model

(Behan, di Pietro, Lauria, van Rees ’22)

$\mathscr{M}_{p,q}$ diagonal minimal model has operators $\Phi_{r,s}$ with dimension

$$\Delta_{r,s} = \frac{(pr-qs)^2-(p-q)^2}{2pq} \qquad \begin{array}{l} 1 \leq r \leq q-1 \\ 1 \leq s \leq p-1 \end{array}$$

consider $\mathscr{M}_{m,m+1}$ for large $m$:

$$\Delta_{r,s} = \frac{1}{2}(r-s)^2 + \frac{s^2-r^2}{2m} + O(m^{-2})$$

| | | | |
|---|---|---|---|
| $\Phi_{1,3}$ <br> $\Phi_{1,2}\partial_n\phi$ | slightly irrelevant | $\Phi_{3,1}$ <br> $\Phi_{2,1}\partial_n\phi$ | slightly relevant |

# Dirichlet Scalar Coupled to $\mathscr{M}_{m,m+1}$

$$\delta S_{\text{bry}} = h \int \mathrm{d}^2 x \, \Phi_{3,1} + g \int \mathrm{d}^2 x \, \Phi_{2,1} \partial_n \phi$$

conformal perturbation gives a fixed point theory

$$h_* = -\frac{\sqrt{3}}{2\pi m}$$

$$g_* = \pm \frac{2}{\sqrt{\pi m}}$$

can compute anomalous dimensions…

match to a numerical bootstrap calculation for general $m$…

# Concluding Motivational Remarks

- unlike AdS/CFT where moving away from classical gravity can entail much pain and suffering, there is a clearer path here

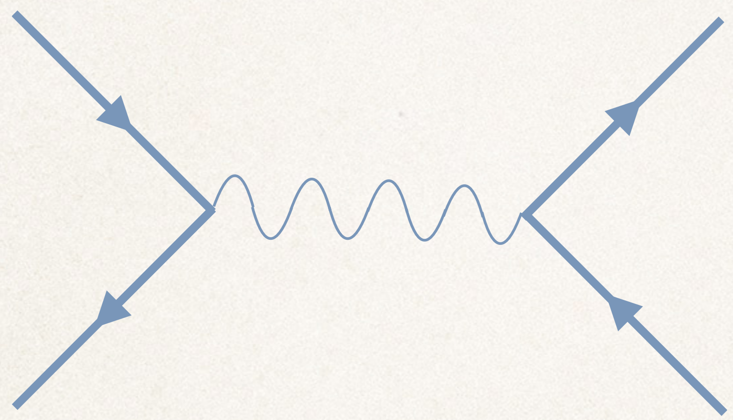
- localization: make for example mixed QED supersymmetric and compute certain observables exactly

- bootstrap: used recently in the free-in-the-bulk scalar case to great effect (Lauria et al.)

# What about higher codimension?

- Conformal defect for a free scalar with codimension $\geq 2$ is trivial (Lauria, Liendo, van Rees, Zhao ’21)
- This work is an analogous claim for surface defects in 4d Maxwell theory.
- Likely that a similar claim can be made for line defects in Maxwell, codimension $\geq 2$ defects for free fermions and for $p$ forms in $2p+2$ dimensions.
- Curious that the boundary case is special…

# Perturbative argument for triviality



What is the effective photon propagator experienced by the charged matter?

For the boundary case, we saw that it was $\frac{1}{p}$

A surface means Fourier transforming two fewer space-time coordinates

For the surface defect we Fourier transform only 2 of the 4 coordinates $\Longrightarrow \log p$

# Proof of Triviality

$s$: transverse spin
$\ell$: parallel spin
$\Delta$: scaling dimension
$\hat{}$ : defect operator

1. defect OPE of $F_{\mu\nu}(x)$: expand the Maxwell field as a sum over defect modes $\hat{\mathcal{O}}$; we find two scalars ($\ell = 0$) and a tower of vectors ($\ell = \pm 1$)

2. compute $\langle F_{\mu\nu}(x_1)\hat{\mathscr{W}}_2(x_2)\hat{\mathscr{W}}_3(x_3)\rangle$: regularity means a ``double twist'' condition, $h_3 = h_1 + h_2 + n$ and $h_3' = h_1' + h_2' + n'$ where $n$ and $n'$ are non-negative integers and $h_1 + h_1' = \Delta_1$ and $h_1' - h_1 = \ell_1$ are in the defect OPE of $F_{\mu\nu}(x)$

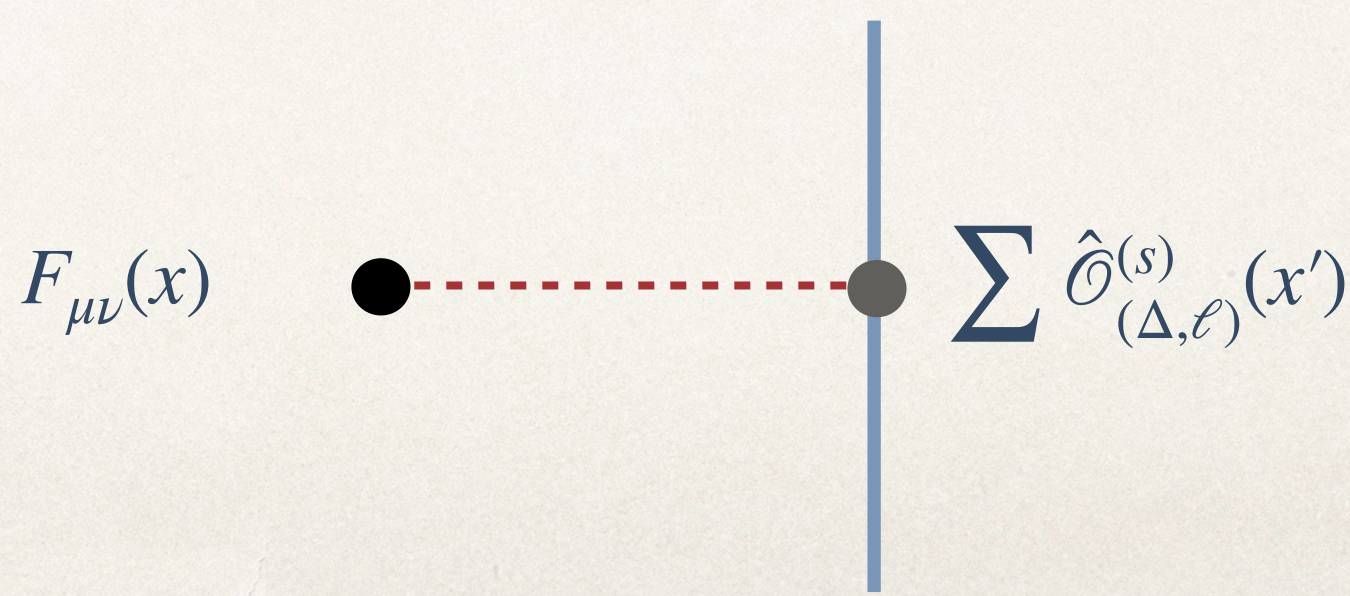
3. contour integral argument: double twist implies $\langle \hat{\mathcal{O}}_1(x_1)\cdots\hat{\mathcal{O}}_n(x_n)\rangle$ can be computed by Wick's Theorem.

# Step 1: Defect OPE

Conformal invariance fixes the form of $\langle F_{\mu\nu}(x)\hat{\mathscr{W}}(x')\rangle$ and $\langle \hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)\rangle$ up to constants. (Note also a nonzero result requires $s_1 + s_2 = 0$ by rotational invariance.)

The free field constraints $\partial_\mu F^{\mu\nu} = 0 = \partial_\mu \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$ further restrict $\langle F_{\mu\nu}(x)\hat{\mathscr{W}}(x')\rangle$.

$$\implies \quad F_{\mu\nu}(x) = \text{ two defect scalars + tower of defect vectors}$$




$s$: transverse spin
$\ell$: parallel spin
$\Delta$: scaling dimension

$$F_{\mu\nu}(x) = \mathscr{D}^{\parallel}_{\mu\nu}\hat{\pi}(w,\bar{w}) + \mathscr{D}^{\perp}_{\mu\nu}\hat{\tau}(w,\bar{w}) +$$

$$+ \sum_{s\geq 0}\left(\mathscr{A}^{(s)}_{\mu\nu}\hat{\psi}^{(s)}(w,\bar{w}) + \mathscr{B}^{(s)}_{\mu\nu}\hat{\psi}^{(-s)}(w,\bar{w}) + \mathscr{C}^{(s)}_{\mu\nu}\hat{\phi}^{(s)}(w,\bar{w}) + \mathscr{D}^{(s)}_{\mu\nu}\hat{\phi}^{(-s)}(w,\bar{w})\right)$$

defect coordinates

$$\hat{\mathscr{O}}^{(s)}_{(\Delta,\ell)} \in (\hat{\pi}, \hat{\tau}, \hat{\psi}^{(s)}, \hat{\phi}^{(s)})$$

$\hat{\pi}$ and $\hat{\tau}$ are defect scalars with $s = 0$, $\ell = 0$, and $\Delta = 2$

$\hat{\psi}^{(s)}$ are defect vectors with $\ell = 1$ and $\Delta = 1 + |s|$

$\hat{\phi}^{(s)}$ are defect vectors with $\ell = -1$ and $\Delta = 1 + |s|$

The sum is organized into conformal multiplets. The $\hat{\mathscr{O}}^{(s)}_{(\Delta,\ell)}$ are conformal primaries and the differential operators $(\mathscr{D}^{\parallel}_{\mu\nu}, \mathscr{D}^{\perp}_{\mu\nu}, \mathscr{A}^{(s)}_{\mu\nu}, \mathscr{B}^{(s)}_{\mu\nu}, \mathscr{C}^{(s)}_{\mu\nu}, \mathscr{D}^{(s)}_{\mu\nu})$ implement the sum over descendants

One more remark about the defect OPE:

$$F_{\mu\nu}(x) = \mathscr{D}^{(\pi)}_{\mu\nu}\hat{\pi}(w,\bar{w}) + \mathscr{D}^{(\tau)}_{\mu\nu}\hat{\tau}(w,\bar{w}) +$$

$$+\sum_{s\geq 0}\left(\mathscr{A}^{(s)}_{\mu\nu}\hat{\psi}^{(s)}(w,\bar{w}) + \mathscr{B}^{(s)}_{\mu\nu}\hat{\psi}^{(-s)}(w,\bar{w}) + \mathscr{C}^{(s)}_{\mu\nu}\hat{\phi}^{(s)}(w,\bar{w}) + \mathscr{D}^{(s)}_{\mu\nu}\hat{\phi}^{(-s)}(w,\bar{w})\right)$$

form of the differential operators: $\mathscr{A}^{(s)}_{z\bar{z}} = \dfrac{\bar{z}^s}{2}\displaystyle\sum_{m=0}^{\infty}\frac{(-|z|^2)^m}{m!(s)_{m+1}}\partial^m_{\bar{w}}\partial^{m+1}_{w}$

$z, \bar{z}$ are the orthogonal directions

To recap: form is fixed by consistency between the bulk-defect and defect-defect two point functions, conformal invariance, free field equation for $F_{\mu\nu}$

# Differences from the Boundary Case

- we are using an additional condition here, reflection positivity, to constrain the form of the defect OPE
- without reflection positivity, we would have found $\Delta = 1 \pm s$
- there is however a reflection positivity bound $\Delta \geq 1$, saturated by defect conserved currents
- at a similar stage for the scalar boundary case, one finds the analog of $\Delta = 1 - |s|$ is not below the reflection positivity bound
- for mixed QED, one finds a pair of boundary conserved currents
- relation to "alternate quantization" for fields in AdS

# Step 2: bulk-defect-defect 3 pt fn

$$\langle F_{\mu\nu}(x_1)\hat{\mathscr{W}}_2(x_2)\hat{\mathscr{W}}_3(x_3)\rangle = \mathcal{N}\sum_{A=1}^{6} f_A(u) S^A_{\mu\nu}(x_1, x_2, x_3)$$

prefactor encodes correct $s, \ell_2, \ell_3$ dependence and also dependence under dilatations

6 weight zero tensor structures fixed by conformal invariance

6 functions of an invariant cross ratio

$$u = \frac{(x_2 - x_1)^2(x_3 - x_1)^2}{(x_2 - x_3)^2 z_1 \bar{z}_1}$$

$|z_1|$ distance of $F_{\mu\nu}(x_1)$ from defect

$$\langle F_{\mu\nu}(x_1)\hat{\mathscr{W}}_2(x_2)\hat{\mathscr{W}}_3(x_3)\rangle = \mathscr{N}\sum_{A=1}^{6} f_A(u) S^A_{\mu\nu}(x_1, x_2, x_3)$$

two strategies:

1. Substitute the defect OPE into $\langle F_{\mu\nu}\hat{\mathscr{W}}_2\hat{\mathscr{W}}_3\rangle$ and use the fact that $\langle\hat{\mathscr{O}}\hat{\mathscr{W}}_2\hat{\mathscr{W}}_3\rangle$ is fixed up to a number: this procedure fixes the form of $\langle F_{\mu\nu}\hat{\mathscr{W}}_2\hat{\mathscr{W}}_3\rangle$.

2. Free field equations applied to $\langle F_{\mu\nu}\hat{\mathscr{W}}_2\hat{\mathscr{W}}_3\rangle$ fix its form up to some integration constants

difficult
(1) to compute sums
(2) to understand the boundary conditions

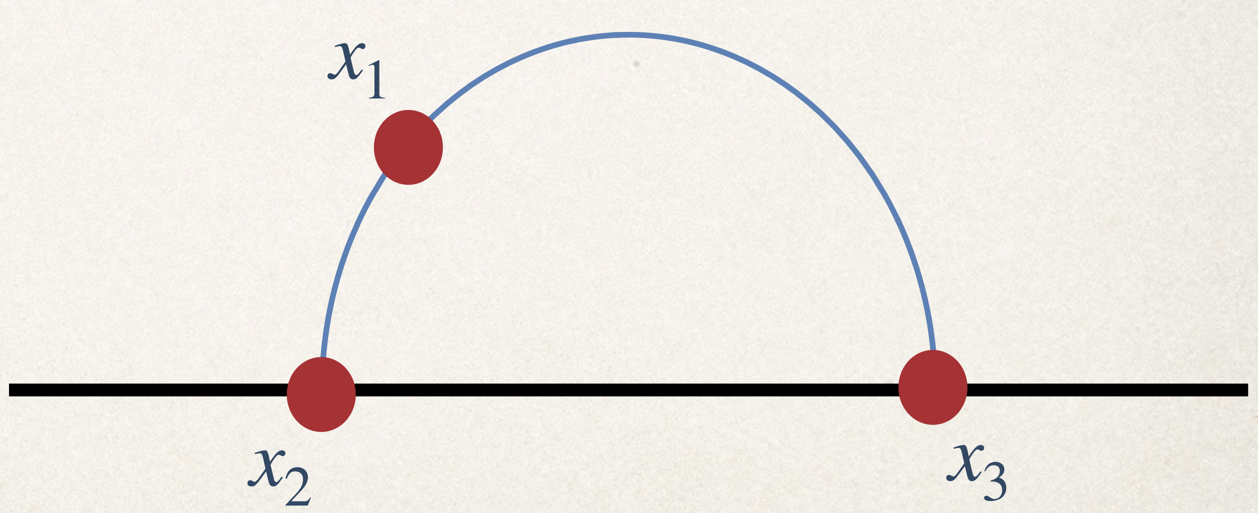
combining the two $\longrightarrow$ get everything consider $f_2(u)$ in the case $s \neq 0$ and $\ell_2 \neq \ell_3$

$$f_2(u) = c_2 \frac{u^{\Delta_2+1}}{(u-1)^{-h_3+h_2'+2+\frac{|s|}{2}}} {}_2F_1\left(1+h_2-h_3+\frac{|s|}{2}, 1+h_2'-h_3'+\frac{|s|}{2}, 1+|s|, \frac{1}{1-u}\right)$$

$$u = \frac{(x_2-x_1)^2(x_3-x_1)^2}{(x_2-x_3)^2 z_1 \bar{z}_1}$$

$c_2$ is a linear combination of the 3-pt fn coeffs $c_{\psi 23}$ and $c_{\phi 23}$

there is a potentially singular behavior at $u = 1$, which corresponds to the semi-circular configuration below — no obvious physical reason for a singularity here




only way to avoid singularity — restrict the spectrum

$$f_2(u) = c_2 \frac{u^{\Delta_2+1}}{(u-1)^{-h_3+h_2'+2+\frac{|s|}{2}}} {}_2F_1\left(1+h_2-h_3+\frac{|s|}{2}, 1+h_2'-h_3'+\frac{|s|}{2}, 1+|s|, \frac{1}{1-u}\right)$$

we can remove the singularity by turning the hypergeometric function into a polynomial

$$1+h_2-h_3+\frac{|s|}{2} = -n$$

or

$$1+h_2'-h_3'+\frac{|s|}{2} = -n'$$

$n, n'$ non-negative integers

that $\ell_2 = h_2' - h_2$ and $\ell_3 = h_3' - h_3$ are both integer means one condition implies the other

finally $h_1, h_1' = 1 + \frac{|s|}{2}$ or $\frac{|s|}{2}$ when $\ell = \pm 1$

# Differences from the boundary case

- In the boundary case, we have this extra mode with $\Delta = 1 - |s|$.
- By tuning the three point coefficients $c_{12+}$ and $c_{12-}$, we have an alternate method for getting rid of the singularity at $u = 1$.
- We would not find this double twist condition that the third operator should have the form $\hat{\mathcal{O}}\partial^n\hat{\mathcal{W}}_2$.

# Step 3: Contour Integral Argument

strategy of proof: induction

we know $\langle \hat{\mathscr{W}}_k(w, \bar{w}) \rangle = 0$ and $\langle \hat{\mathscr{W}}_i(w, \bar{w}) \hat{\mathscr{W}}_j(0,0) \rangle = \dfrac{\delta_{h_i+h_j} \delta_{h'_i+h'_j}}{w^{h_i+h_j} \bar{w}^{h'_i+h'_j}}$

assume we know how to compute a $n-2$ point function

show that we can compute an $n$ point function includes descendants

relevance of double twist condition:

$$\hat{\mathscr{W}}_1(w,\bar{w})\hat{\mathscr{W}}_2(0,0) = \frac{\delta_{h_1,h_1'}\delta_{h_2,h_2'}}{w^{h_1+h_2}\bar{w}^{h_1'+h_2'}}\mathbf{1} + \sum_k \frac{\lambda_{12k}}{w^{h_1+h_2-h_k}\bar{w}^{h_1'+h_2'-h_k'}}\hat{\mathscr{W}}_k$$

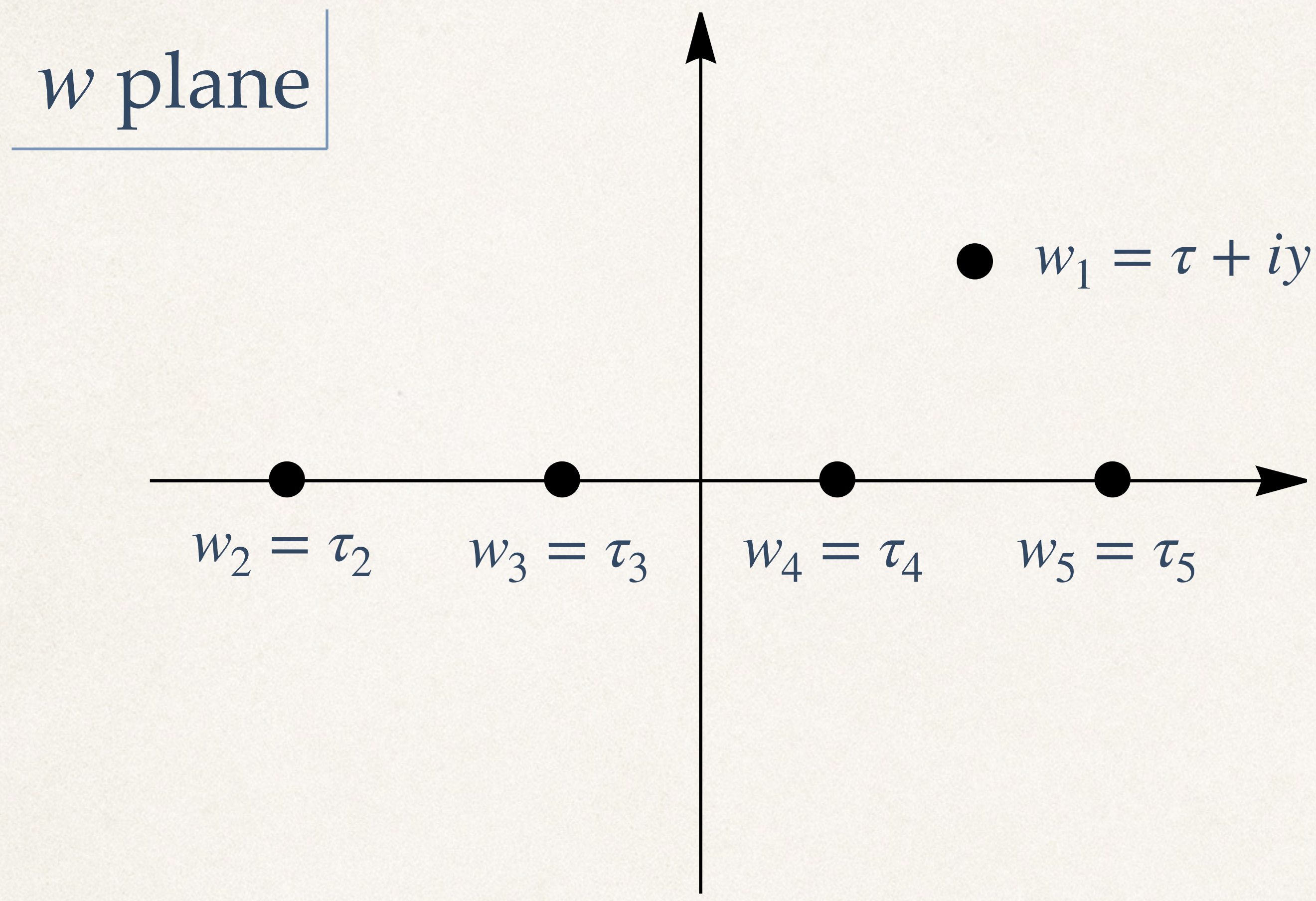
double twist condition $\quad h_1 + h_2 - h_k = -n$

$$h_1' + h_2' - h_k' = -n'$$

removes branch cuts in the sum

$$[\hat{\mathscr{W}}_1(w,\bar{w}), \hat{\mathscr{W}}_2(0,0)] = \text{discontinuity}\left(\frac{\delta_{h_1,h_2}\delta_{h_1',h_2'}}{w^{h_1+h_2}\bar{w}^{h_1'+h_2'}}\right)$$

introduce $G_n(\tau) = \langle \hat{\mathscr{W}}_1(w_1, \bar{w}_1)\hat{\mathscr{W}}_2(w_2, \bar{w}_2)\cdots\hat{\mathscr{W}}_n(w_n, \bar{w}_n)\rangle$




note we will take

$$\bar{w}_1 = \tau - iy$$

even though $\tau$ is complex,

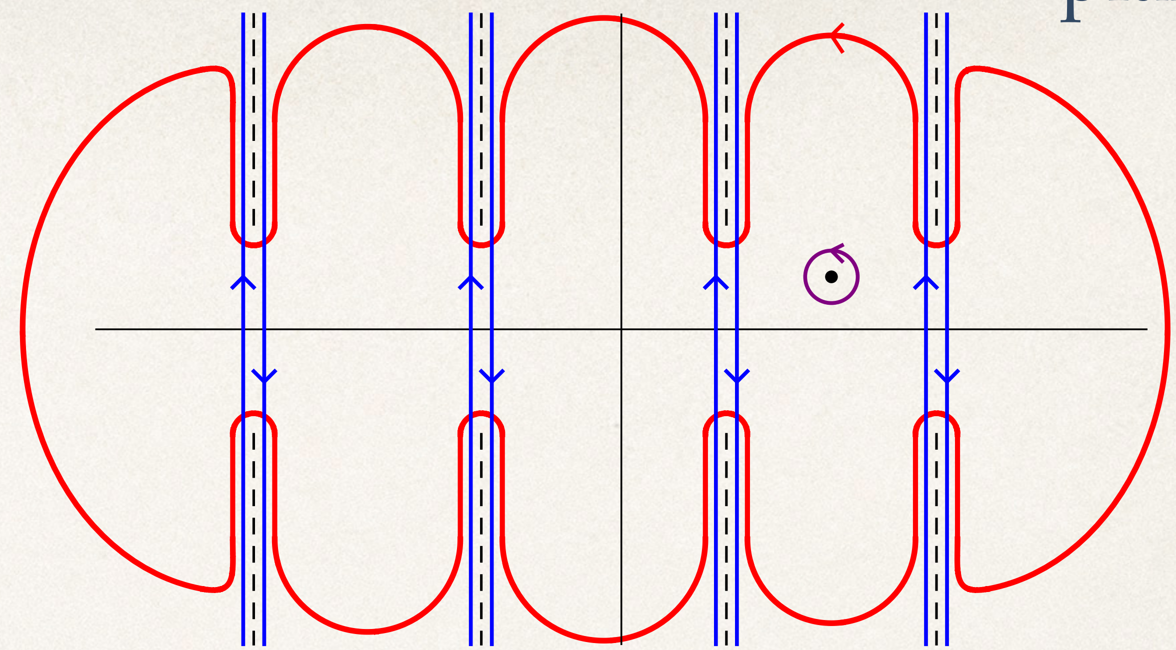
complexifying time

branch point conditions:

$$w_1 - w_k = \tau + iy - \tau_k = 0$$

$$\bar{w}_1 - \bar{w}_k = \tau - iy - \tau_k = 0$$

branch cuts start at $\tau = \tau_k \pm iy$




$$G_n(\tau) = \oint \frac{d\tau'}{2\pi i}\frac{G_n(\tau')}{\tau' - \tau} = \int_{-\infty}^{\infty}\frac{dt'}{2\pi} \times$$

$$\times \left( \frac{1}{\tau - \tau_2 - it'} \langle [\hat{\mathscr{W}}_1(\tau_2 + it' + iy, \tau_2 + it' - iy), \hat{\mathscr{W}}_2(\tau_2, \tau_2)] \hat{\mathscr{W}}_3(\tau_3, \tau_3) \cdots \hat{\mathscr{W}}_n(\tau_n, \tau_n) \rangle \right.$$

$$+ \frac{1}{\tau - \tau_3 - it'} \langle \hat{\mathscr{W}}_2(\tau_2, \tau_2) [\hat{\mathscr{W}}_1(\tau_3 + it' + iy, \tau_3 + it' - iy), \hat{\mathscr{W}}_3(\tau_3, \tau_3)] \cdots \hat{\mathscr{W}}_n(\tau_n, \tau_n) \rangle$$

$$+ \cdots$$

$$\left. + \frac{1}{\tau - \tau_n - it'} \langle \hat{\mathscr{W}}_2(\tau_2, \tau_2) \hat{\mathscr{W}}_3(\tau_3, \tau_3) \cdots [\hat{\mathscr{W}}_1(\tau_n + it' + iy, \tau_n + it' - iy), \hat{\mathscr{W}}_n(\tau_n, \tau_n)] \rangle \right)$$

$$G_n(\tau) = \frac{\delta_{h_1,h_2}\delta_{h_1',h_2'}}{z_{12}^{h_1+h_2}\bar{z}_{12}^{h_1'+h_2'}}\langle \hat{\mathscr{W}}_3(\tau_3,\tau_3)\hat{\mathscr{W}}_4(\tau_4,\tau_4)\cdots\hat{\mathscr{W}}_n(\tau_n,\tau_n)\rangle$$

$$+\frac{\delta_{h_1,h_3}\delta_{h_1',h_3'}}{z_{13}^{h_1+h_3}\bar{z}_{13}^{h_1'+h_3'}}\langle \hat{\mathscr{W}}_2(\tau_2,\tau_2)\hat{\mathscr{W}}_4(\tau_4,\tau_4)\cdots\hat{\mathscr{W}}_n(\tau_n,\tau_n)\rangle + \ldots$$

$$+\cdots$$

$$+\frac{\delta_{h_1,h_n}\delta_{h_1',h_n'}}{z_{1n}^{h_1+h_n}\bar{z}_{1n}^{h_1'+h_n'}}\langle \hat{\mathscr{W}}_2(\tau_2,\tau_2)\hat{\mathscr{W}}_3(\tau_3,\tau_3)\cdots\hat{\mathscr{W}}_{n-1}(\tau_{n-1},\tau_{n-1})\rangle$$

Remarks: proof extends to operators in arbitrary position by considering descendants — think of an arbitrary position as a Taylor series implemented by summing over descendants.

So what does happen….

# Fermions in different dimensions

- 4d: Weakly coupled in the IR. Strongly coupled in the UV.
- 3d: There may be a critical value of $N_f$ below which the theory confines and the flavor symmetry is broken spontaneously to a subgroup.
- 2d Schwinger model: Confinement and spontaneous symmetry breaking (allowed because EM forces are long range in 2d)
- 2d Thirring model: BKT transition for positive coupling

What happens when the photon lives in a different dimension?

Gorbar, Gusynin, Miransky (2001)

see also more recent work by Kotikov and Teber

# Fermions on branes

Solve Schwinger-Dyson equations in the rainbow approximation

3d/4d case: $$e_{\rm cr}^2 = \frac{16}{N_{\rm max} - N_f} \qquad N_{\rm max} = \frac{128}{3\pi^2} \approx 4.3$$

spontaneous chiral symmetry breaking takes place for
$e > e_{\rm cr}$ and $N_f < N_{\rm max}$

2d/4d case: there is always spontaneous chiral symmetry breaking but its character depends on coupling $\frac{N_f^2 e^2}{2\pi^2}$

$$m^2 \sim \Lambda_{\rm UV}^2 \, e^{-{\rm const}/e^2}$$

strong coupling resembles Thirring model

# Concluding Remarks

- Theories that are free-in-the-bulk and interacting on the boundary/defect can be tractable examples of strongly interacting QFT.
- Conjecture that free-in-the-bulk theories in codimension $\geq 2$ are always trivial. Proven for scalars and the surface defect in 4d Maxwell.
- Future projects: line defect in Maxwell, free-in-the-bulk fermions, bootstrapping mixed dimensional QED