Five Decades of the Gravitational Weyl Anomaly

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Classical Weyl invariance

- Classically, Weyl invariance

$$S(g,\phi) = S(g',\phi')$$

under

$$g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x) \quad \Phi'(x) = \Omega(x)^\alpha \Phi(x)$$

implies

$$g^{\mu\nu} T_{\mu\nu} = 0$$

- To accommodate fermions

$$e'_\mu{}^a(x) = \Omega(x) e_\mu{}^a(x)$$

where $g_{\mu\nu}(x) = e_\mu{}^a(x) e_{\nu a}(x)$

Examples

- Conformally coupled scalar, $\phi'(x) = \Omega^{-1}(x)\phi(x)$, Penrose

$$S[\phi] = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2\right)$$

- Massless fermion, $\psi'(x) = \Omega^{-3/2}(x)\psi(x)$, Dirac

$$S[\psi] = \int d^4xe\left(\bar{\psi}\gamma^\mu\nabla_\mu\psi\right)$$

- Electromagnetic field, $A'_\mu(x) = A_\mu(x)$, Maxwell

$$S[A] = \int d^4x\sqrt{-g}\left(-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\right)$$

Quantum Weyl anomalies

- But in the quantum theory

$$g^{\mu\nu} < T_{\mu\nu} > \neq 0$$

Over the period 1973-2022 this Weyl anomaly has found a variety of applications in quantum gravity, black hole physics, inflationary cosmology, string theory and statistical mechanics.

- Note that in generic curved space

$$g^{\mu\nu} T_{\mu\nu}$$

not associated with a Noether current

Recall flat space ancestry 1970

- For example $SO(D,2)$ in the case of flat Minkowski space. Coleman and Jackiw Callan, Coleman and Jackiw
- More generally, for D-dimensional spacetimes admitting conformal Killing vectors $\xi^i_\mu(x)$

$$\nabla_\mu \xi^i_\nu + \nabla_\nu \xi^i_\mu = \frac{2}{D} g_{\mu\nu} \nabla^\rho \xi^i_\rho$$

there is a classically conserved dilatation current

$$J^{i\nu} = \xi^i_\mu T^{\mu\nu}$$

- Anomaly appears in the quantum theory

$$\nabla_\nu < J^{i\nu} >= \frac{1}{D} \nabla^\rho \xi^i_\rho g^{\mu\nu} < T_{\mu\nu} > \neq 0$$

but this is not an anomaly in local Weyl symmetry

$$g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x)$$

Timeline 1973

- Corrections to graviton propagator from closed loops of spin $s = 0, 1/2/, 1$ using dimensional regularization.
  Capper Duff and Halpern $s = 1$
  Capper and Duff $s = 1/2$
  Geist et al
  Capper$s = 0$
- Discovery of the Weyl anomaly using dimensional regularization
  Capper and Duff

Diffeomophism Ward identity

- This involved the self-energy insertion

$$\Pi_{\mu\nu\rho\sigma}(p) = \int d^D x e^{ipx} < T_{\mu\nu}(x) T_{\rho\sigma}(0) >|_{g_{\mu\nu}=\eta\mu\nu}$$

where D is the spacetime dimension and $T_{\mu\nu}(x)$ the energy-momentum tensor of the massless particles.

- One of our goals was to verify that dimensional regularization correctly preserved the Ward identity

$$p^\mu \Pi_{\mu\nu\rho\sigma}(p) = 0$$

that follows as a consequence of general covariance.

Diffeomophism Ward identity

- Capper and I were able to show that $\Pi_{\mu\nu\rho\sigma}$ did indeed obey the identity with

$$\Pi_{\mu\nu\rho\sigma} = \Pi(2)S(2)_{\mu\nu\rho\sigma} + \Pi(0)S(0)_{\mu\nu\rho\sigma}$$

where

$$S(2)_{\mu\nu\rho\sigma} = \frac{1}{2}(X_{\mu\rho}X_{\nu\sigma} + X_{\nu\rho}X_{\mu\sigma}) - \frac{1}{3}X_{\mu\nu}X_{\rho\sigma}$$

$$S(0)_{\mu\nu\rho\sigma} = \frac{1}{3}X_{\mu\nu}X_{\rho\sigma}$$

$$X_{\mu\nu} = \eta_{\mu\nu}p^2 - p_\mu p_\nu$$

Diffeomophism Ward identity

- If we denote by $D - 4 = \epsilon$ and expand about $\epsilon = 0$

$$\Pi_{\mu\nu\rho\sigma} = \frac{1}{\epsilon}\Pi_{\mu\nu\rho\sigma}(pole) + \Pi_{\mu\nu\rho\sigma}(finite)$$

$$\Pi(2)(pole) = \bar{c}/(4\pi)^2 \qquad \Pi(0)(pole) = 0$$

where $\bar{c}$ is a constant, in which case the infinity can then be removed by a generally covariant counterterm

$$\Delta L = \frac{1}{\epsilon}\frac{\bar{c}}{(4\pi)^2}\sqrt{-g}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} \tag{1}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. We found $\bar{c} = 18$ for $s = 1/2$ and $\bar{c} = 72$ for spin 1. A later calculation by Capper gave $\bar{c} = 6$ for conformally coupled $s = 0$.

Conformal Weyl identity

- We were also aware that since the actions for massless particles of spins $s = 0, 1/2, 1$ are invariant under the Weyl transformations

$$\Pi^{\mu}{}_{\mu\rho\sigma}(p) = 0 \qquad \rightarrow \Pi(0) = 0$$

but whereas

$$\Pi(2)(pole) \neq 0 \qquad \Pi(0)(pole) = 0$$

we found

$$\Pi(2)(finite) \neq 0 \qquad \Pi(0)(finite) \neq 0$$

The Weyl invariance displayed by classical massless field systems in interaction with gravity, first proposed by Hermann Weyl in 1918 no longer survives in the quantum theory! We rushed off a paper to *Nuovo Cimento* (How times have changed!).

Timeline 1974

- I first announced the existence of gravitational Weyl anomalies at The First Oxford Quantum Gravity Conference, organised by Isham, Penrose,Sciama, and held at the Rutherford Laboratory in February 1974
- Unfortunately, the announcement was somewhat overshadowed because Hawking chose the same conference to reveal to an unsuspecting world his result that black holes evaporate!
- Ironically, Christensen,Fulling were subsequently to link the Hawking effect and the trace anomaly.

Timeline 1976

- Non-local effective lagrangian for trace anomaly Deser, Duff and Isham By general covariance and dimensional analysis, it must take the following form:
- For D=2,

$$g^{\alpha\beta} < T_{\alpha\beta} >= aR$$

- For D=4,

$$g^{\alpha\beta} < T_{\alpha\beta} >= \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \delta \Box R + c F_{\mu\nu}{}^{a} F^{\mu\nu a}$$

where $a, \alpha, \beta, \gamma, \delta$ and c are constants.

- For $D = 6$,

$$g^{\alpha\beta} < T_{\alpha\beta} >= (curvature)^3$$

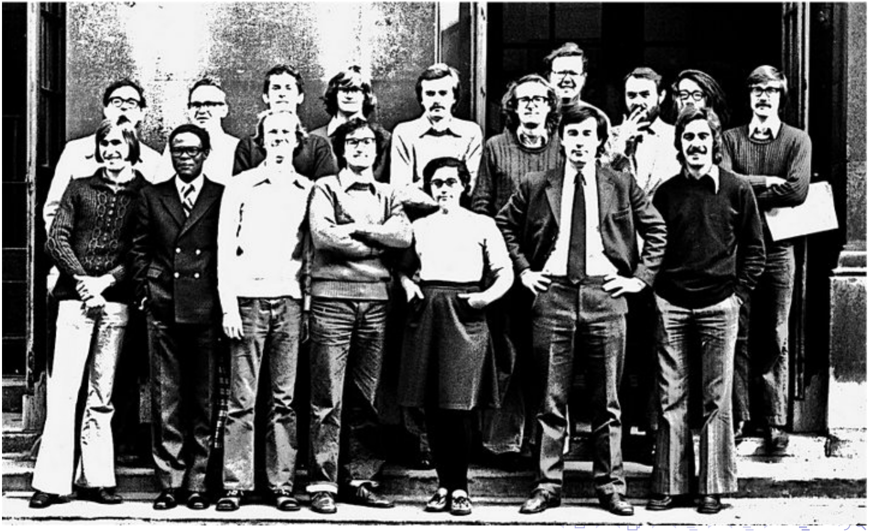
and so on.

- At one-loop, and ignoring boundary terms, there is no anomaly for D odd.

Non-local action

- The title of the paper *Non-local Conformal Anomalies* was chosen to emphasize that although the trace of the stress tensor and infinite counterterms, for example $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, are local, the finite effective action, for example $C_{\mu\nu\rho\sigma} \ln \Box C^{\mu\nu\rho\sigma}$, is not.

King's College London 1976



The heat kernel

- Zeta functions, heat kernels and anomalies
  Christensen
  Dowker
  Hawking
  Barvinsky and Vilkoviski

The heat kernel

- Classical action

$$S_0 = \int d^D x \frac{1}{2}(\Phi, \Delta\Phi)$$

where Δ is a conformally invariant d-dimensional operator.

- The one-loop effective action is given by

$$S_1 = ln[det\Delta]^{-1/2}$$

- Its kernel $F(x, y, \rho)$ obeys the heat equation

$$\frac{\partial}{\partial\rho}F(x, y, \rho) + \Delta F(x, y, \rho) = 0$$

with the initial conditions

$$F(x, y, 0) = \delta(x, y)$$

The heat kernel

- One can express F as

$$F(x, y, \rho) = \sum_n e^{-\rho\Delta} \phi_n(x)\phi_n(y)$$

$$= \sum_n e^{-\rho\lambda_n} \phi_n(x)\phi_n(y)$$

where ϕ_n are the eigenfunctions of Δ with eigenvalues λ_n:

$$\Delta\phi_n = \lambda_n\phi_n$$

normalized according to

$$\int d^D x \sqrt{g}(x)\phi_n(x)\phi_m(x) = \delta_{mn}$$

b_4 coefficients

- The one-loop action may thus be written as

$$S_1 = \int d\rho d^D x \rho^{-1} \sqrt{g}(x) A(x,\rho)$$

where $A(x,\rho) = F(x,x,\rho)$. $A(x,\rho)$ obeys an asymptotic expansion, valid for small ρ,

$$A(x,\rho) \sim \sum_n B_{2n}(x) \rho^{n-\frac{D}{2}}$$

where

$$B_{2n} = \int d^D x \sqrt{g} b_{2n}(x)$$

Zeta functions

- The Schwinger-DeWitt coefficients b_{2n} are scalar polynomials, which are of order $2n$ in derivatives of the metric. In $D = 4$, for example, when Δ is the conformally invariant Laplacian acting on scalars:

$$\Delta = -\Box + \frac{1}{6}R \quad \textit{Penrose}$$

$$g^{\mu\nu} < T_{\mu\nu} >= b_4 = \frac{1}{2880\pi^2}[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + 30\Box R]$$

- Furthermore,

$$B_4 = n_0 + \zeta(0)$$

where n_0 is the number of zero modes and

$$\zeta(s) = \Sigma_n \ \lambda_n^{-s}$$

is defined only over the non-zero eigenvalues of Δ.

Timeline 1976

- Asymptotic safety
  Weinberg
- Point splitting regularization
  Christensen, Duncan
- More anomaly coefficients
  Dowker and Critchley, Duncan
- Vacuum energy in two dimensions
  Davies and Fulling
- Particle creation
  Wald
- Robertson-Walker and applications to cosmology
  Birrell, Bunch, Christensen,Davies, Fulling
  Hartle et al
- Black holes
  Davies, Fulling,Unruh

Timeline 1977

- Trace of stress tensor $T^{\mu}{}_{\mu}$
  Divergence of axial current $\partial_{\mu} J^{5\mu}$
  Gamma trace of spinor current $\gamma^{\mu} S_{\mu}$
  form a supermultiplet
- and so, therefore, do the anomalies!
  Ferrara,Zumino

Timeline 1977

- CFTs and the *a* and *c* coefficients
  Duff
- Trace anomalies and the Hawking effect
  Christensen and Fulling

Conformal Field Theories (CFT)

- Weyl anomalies appear in their most pristine form when CFTs are coupled to an external gravitational field. In this case

$$\mathcal{A} = g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{1}{(4\pi)^2}(cF - aG)$$

where F is the square of the Weyl tensor:

$$F = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$

G is proportional to the Euler density:

$$G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

- Note no R^2 term.
- We ignore $\Box R$ terms whose coefficient can be adjusted to any value by adding the finite counterterm

$$\int d^4x\sqrt{g}R^2.$$

Central charges c and a

- In the CFT a and c are the central charges given in terms of the field content by

$$\bar{a} \equiv 720a = 2N_0 + 11N_{1/2} + 124N_1$$

$$\bar{c} \equiv 720c = 6N_0 + 18N_{1/2} + 72N_1$$

where N_s are the number of fields of spin s.

- In the notation of Duff 1977

$$(4\pi)^2 b = c \quad (4\pi)^2 b' = -a$$

Central charges c and a

- The story of the Weyl anomaly for CFTs is thus the story of the central charges c and a. The ratio is given by

$$\frac{a}{c} = \frac{\left(2N_0 + 11N_{1/2} + 124N_1\right)}{\left(6N_0 + 18N_{1/2} + 72N_1\right)}$$

and by inspection we can read off the inequalities

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}$$

where the upper and lower bounds are saturated by a single vector and a single scalar respectively.

Euler number

- When $F - G$ vanishes, anomaly reduces to

$$\mathcal{A} = A\frac{1}{32\pi^2} R^{*\mu\nu\rho\sigma} R^{*}{}_{\mu\nu\rho\sigma}$$

where

$$360A = \bar{c} - \bar{a} = 4N_0 + 7N_{1/2} - 52N_1$$

so that in Euclidean signature

$$\int d^4x \sqrt{g} g^{\mu\nu} T_{\mu\nu} = A\chi(M^4)$$

where $\chi(M^4)$ is the Euler number of spacetime.

Non-conformal theories

- Since the anomaly arises because the operations of regularizing and tracing do not commute, the anomaly $\mathcal{A}$ in a theory which is not classically Weyl invariant may be defined as:

$$\mathcal{A} = tr\ reg\ T - reg\ tr\ T$$

- Of course, the second term happens to vanish when the classical invariance is present.
- We still have

$$\mathcal{A} = b_4$$

but may now involve R^2.

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Timeline 1978

- Conformal (and axial) anomalies for arbitrary spin
  Christensen,Duff
- Conformal anomalies for interacting theories: QED, ϕ^4 etc
  Drummond
  Shore
  Hathrell

Arbitrary spin

- Calculate b_4 for arbitrary (n, m) reps of Lorentz group, then physical anomaly given by

$$A = A(n, m) + A(n-1, m-1) - 2A(n-1/2, m-1/2)$$

so in total

$$A_{total} = 4N_0 + 7N_{1/2} - 52N_1 - 233N_{3/2} + 848N_2$$

where N_s are the number of fields of spin s.

- The b_4 coefficient for chiral reps (1/2,0) (1,0) etc also involve R*R unless we add (0,1/2) (0,1) etc
- Pseudoscalar contribution to Weyl anomaly?
  Nakayama
  Bonora et al
  Bastianelli et al
  Frob and Zein

Timeline 1980

- Anomaly-driven inflation
  Starobinsky (NB not $R + R^2$)
  Mukhanov et al
  Grishchuk,Zeldovich
  Vilenkin
- *p*-forms and inequivalent anomalies
  Duff, van Nieuwenhuizen
  Grisaru et al
  Siegel
- The path-integral approach to anomalies
  Fujikawa
  Bastianelli, van Nieuwenhuizin
  Nicolai,Townsend

p-forms and inequivalent anomalies

- Particularly interesting examples are provided by p-form gauge fields whose laplacians are not conformally invariant but for which

$$\int \mathcal{A}(\phi_2) - \int \mathcal{A}(\phi_0) = \chi$$

$$\int \mathcal{A}(\phi_3) = -2\chi$$

Such quantum inequivalence of p-forms and their duals has been called into question on the grounds that their total stress tensors are the same. Nevertheless, their partition functions differ by Euler number factors.
Donelly, Michel Wall

Central charges c and a

- In the supersymmetric case we have the values and bounds given below. Remarkably, these bounds continue to hold true when the CFT is interacting Maldacena

Fields	a	c	Bounds
$\mathcal{N} = 0$ spin 0	$1/360$	$1/120$	$31/18 \geq a/c \geq 1/3$
$\mathcal{N} = 0$ spin $1/2$	$11/720$	$1/40$	
$\mathcal{N} = 0$ spin 1	$31/180$	$1/10$	
$\mathcal{N} = 1$ chiral multiplet	$1/48$	$1/24$	$3/2 \geq a/c \geq 1/2$
$\mathcal{N} = 1$ vector multiplet	$3/16$	$1/8$	
$\mathcal{N} = 2$ hyper multiplet	$1/24$	$1/12$	$5/4 \geq a/c \geq 1/2$
$\mathcal{N} = 2$ vector multiplet	$5/24$	$1/6$	
$\mathcal{N} = 4$ vector multiplet	$1/4$	$1/4$	$a/c = 1$

Table: The central charges a and c for supersymmetric CFTs

Timeline 1981

- When we allow for a cosmological constant Λ the anomaly is

$$A\chi + BV$$

where V is the volume. We find

$$B = 6N_0 + 18N_{1/2} + 72N_1 - 822N_{3/2} + 3132N_2 \qquad (2)$$

Moreover in gauged supergravity

$$e^2 = G\Lambda$$

and B also determines the Yang-Mills beta-function.
- This yields vanishing β-function in gauged $N > 4$ supergravity Christensen,Duff,Gibbons,Rocek
- Spin sum rules

$$\sum_{\lambda} (-1)^{2\lambda} \lambda^k = 0$$

for $N > k$ Curtwright Christensen,Duff

Timeline 1981

- Critical dimensions for bosonic and super strings
  Polyakov

Bosonic string

- In the first quantized theory of the bosonic string, one starts with a Euclidean functional integral

$$e^{-\Gamma} = \int \frac{D\gamma\ DX}{Vol(Diff)}\ e^{-S[\gamma,X]}$$

where

$$S[\gamma, X] = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$$

- As shown by Polyakov, the Weyl anomaly in the worldsheet stress tensor is given by

$$\gamma^{ij} < T_{ij} >= \frac{1}{24\pi}(D-26)R(\gamma)$$

D is the contribution of the scalars while the -26 arises from the diffeomorphism ghosts that must be introduced into the functional integral.

Fermionic string

- In the case of the fermionic string, the result is

$$\gamma^{ij} < T_{ij} >= \frac{1}{16\pi}(D - 10)R(\gamma)$$

- Thus the critical dimensions $D = 26$ and $D = 10$ correspond to the preservation of the two dimensional Weyl invariance $\gamma_{ij} \to \Omega^2(\xi)\gamma_{ij}$.

Spacetime Einstein equations from worldsheet anomaly

- $$S[\gamma, X] = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$$
  $$\beta(g)_{\mu\nu} = R_{\mu\nu} + ..$$
  vanishing anomaly implies Einstein equations!
  Callan, Friedan,Perry

Timeline 1983

- Conformal anomaly and W-Z consistency (no R^2)
  Bonora et al
- Anomaly in conformal supergravity $N = 1, 2, 3, 4$

$$S = \int d^4x \sqrt{-g} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \dots$$

$N = 4$ conformal supergravity coupled to 4 $N = 4$ Maxwell multiplets (or $SU(2) \times U(1)$ SYM) is anomaly free/finite Fradkin and Tseytlin This theory has no chiral anomalies as well as found later in 85 Romer and van Nieuwenhuizen

Timeline 1984

- Local version of effective action
  Fradkin and Tseytlin
  Riegert

Local action

- Conformal operators

$$\sqrt{g}\Delta_d = \sqrt{g'}\Delta'_d$$

$$\Delta_2 = \Box$$

$$\Delta_4 \equiv \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}(\nabla^\mu R)\nabla_\mu - \frac{2}{3}R\Box$$

  Fradkin and Tseytlin
  Riegert
  Paneitz
- Subsequent work by
  Osborn et al Antoniadis, Mazur and Mottola Barvinsky et al Nicolai et al
- Local action

$$S_{anom} = \frac{b}{2}\int d^4x\sqrt{g}F\phi - \frac{b'}{2}\int d^4x\sqrt{g}[\phi\Delta_4\phi - (G - \frac{2}{3}\Box R)\phi]$$

Timeline 1985

- Conformal invariants
  Fefferman,Graham

Timeline 1986

- Recall in D=2

$$g^{\alpha\beta} < T_{\alpha\beta} >= cR \quad c = constant$$

- The *c*-theorem : in 2D CFT with coupling constants g_i and energy scale μ there exists a positive real function, $c(g_i, \mu)$ which decreases monotonically under the renormalization group flow and at fixed points where $g_i = g_i^*$, the function

$$c(g_i^*, \mu) = c$$

- Theories at high energies have more degrees of freedom than theories at low energies and that information is lost as we flow from the former to the latter.
  Zamolodchikov

Timeline 1988

- *c*-theorem and/or *a*-theorem in four dimensions?
  Cardy
  Osborn
  Capelli et al
  Shore
  Shapere
  Antoniadis et al

Timeline 1993

- Geometric classification of conformal anomalies in arbitrary dimensions
  Deser,Schwimmer

Timeline 1998

- The holographic Weyl anomaly
  Henningson,Skenderis
  Imbimbo
  Graham
  Bastianelli
  Manvelyan
  Fukuma
- Einstein manifolds and the *a* and *c* coefficients
  Gubser,Martelli

Holography

- A distinguished coordinate system, boundary at $\rho = 0$

$$G_{MN} dx^M dx^N = \frac{{L_{d+1}}^2}{4} \rho^{-2} d\rho d\rho + \rho^{-1} g_{\mu\nu} dx^\mu dx^\nu$$

- The effective action may be written

$$S_B = \int d\rho d^d x \rho^{-1} \sqrt{g}(x) B(x, \rho)$$

where the specific form of $B(x, \rho)$ depends on initial action.

$$B(x, \rho) \sim \sum_n b_{2n}(x) \rho^{n - \frac{d}{2}}$$

- Formal similarity with Schwinger-DeWitt coefficients, indeed $\mathcal{A} \sim b_4$ same for N=4 Yang-Mills but not in general.

Timeline 2000

- Anomaly-driven inflation revived
  Hawking et al
  Hamada
  Nojiri
  Shapiro
  de Paula Netto, Pelinson, Shapiro, Starobinsky
- *a* and *c* and corrections to Newton's law
  Duff and Liu
- Anomalies and entropy bounds
  Nojiri et al

Corrections to Newton's law

- In my 1972 PhD thesis, at the suggestion of Abdus Salam, I calculated one-loop CFT corrections to Newton's law (Schwarzschild solution)

$$V(r) = \frac{G_4 M}{r}\left(1 + \frac{8cG_4}{3\pi r^2}\right),$$

where G_4 is the four-dimensional Newton's constant and c is a purely numerical coefficient. In fact it turned out to be the c coefficient in the Weyl anomaly

N=4 Yang-Mills

- A particularly important example of a CFT is provided by $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$, for which

$$(N_1, N_{1/2}, N_0) = (N^2, 4N^2, 6N^2)$$

Then

$$a = c = \frac{N^2}{4}$$

and hence

$$\mathcal{A} = \frac{c}{(4\pi)^2}\left(2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2\right) = \frac{N^2}{32\pi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$$

- The contribution of a single $\mathcal{N} = 4$ $U(N)$ Yang-Mills CFT is

$$V(r) = \frac{G_4 M}{r}\left(1 + \frac{2N^2 G_4}{3\pi r^2}\right).$$

Randall-Sundrum

- Now fast-forward to 1999 when Randall and Sundrum proposed that our four-dimensional world is a 3-brane embedded in an infinite five-dimensional universe. They showed that there is an r^{-3} correction coming from the massive Kaluza-Klein modes

$$V(r) = \frac{G_4 M}{r}\left(1 + \frac{2{L_5}^2}{3r^2}\right).$$

where L_5 is the radius of AdS_5.

- Superficially, our 4D quantum correction seems unrelated to their 5D classical one.
- But through the miracle of AdS/CFT

$$N^2 = \frac{\pi L_5^3}{2G_5} \qquad G_4 = \frac{2G_5}{L_5}$$

the two are in fact equivalent. Duff and Liu

Timeline 2001

- *a* and *c* and the graviton mass
  Dilkes et al
  Aharony
- Weyl cohomology revisited
  Mazur and Mottola

Timeline 2005

- Anomalies as an infra-red diagnostic; IR free or interacting?
  Intriligator

Timeline 2006

- Macroscopic effects of the quantum trace anomaly
  Mottola et al
  Gianotti et al

Timeline 2007

- Anomalies and the hierarchy problem
  Meissner

Timeline 2008

- Viscosity bounds
  Buchel et al
- Conformal collider physics
  Hofman and Maldacena
- Weyl invariance and mass
  Waldron et al
- Entanglement Entropy, Trace Anomalies and Holography
  Schwimmer and Theisen

Timeline 2009

- Entanglement Entropy
  Nishioka
- Log corrections to black hole entropy
  Cai
  Solodukhin
  Sen et al

Timeline 2010

- Holographic c-theorems in arbitrary dimensions
  Myers et al
- Generalized mirror symmetry and trace anomalies
  Duff et al
- Vanish without a trace
  Duff et al

M-theory on X^7

- We consider compactification of ($\mathcal{N} = 1, D = 11$) supergravity on a 7-manifold X^7 with betti numbers $(b_0, b_1, b_2, b_3, b_3, b_2, b_1, b_0)$ and define a generalized mirror symmetry

$$(b_0, b_1, b_2, b_3) \rightarrow (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2)$$

  under which

$$\rho(X^7) \equiv 7b_0 - 5b_1 + 3b_2 - b_3$$

  changes sign

$$\rho \rightarrow -\rho$$

- The massless sectors of these compactifications have

$$f = 4(b_0 + b_1 + b_2 + b_3)$$

  degrees of freedom.
- Generalized self-mirror theories are defined to be those for which $\rho = 0$

M-theory on X^7

- In backgrounds for which $F - G$ vanishes, the Weyl anomaly reduces to

$$T = A\frac{1}{32\pi^2}R^{*\mu\nu\rho\sigma}R^{*}{}_{\mu\nu\rho\sigma} \tag{3}$$

where

$$A = 2(c - a) \tag{4}$$

so that in Euclidean signature

$$\int d^4x\sqrt{g}T = A\chi(M^4) \tag{5}$$

where $\chi(M^4)$ is the Euler number of spacetime.

Anomalies

	Field	f	ΔA	$360A$	X^7
g_{MN}	$g_{\mu\nu}$	2	-3	848	b_0
	$\mathcal{A}_\mu$	2	0	-52	b_1
	$\mathcal{A}$	1	0	4	$-b_1 + b_3$
ψ_M	ψ_μ	2	1	-233	$b_0 + b_1$
	χ	2	0	7	$b_2 + b_3$
A_{MNP}	$A_{\mu\nu\rho}$	0	2	-720	b_0
	$A_{\mu\nu}$	1	-1	364	b_1
	A_μ	2	0	-52	b_2
	A	1	0	4	b_3
total A					$-\rho/24$

Table: X^7 compactification of D=11 supergravity

Vanish without a trace!

- Remarkably, we find that the anomaly depends on ρ

$$A = -\frac{1}{24}\rho(X^7)$$

It flips sign under generalized mirror symmetry and vanishes for generalized self-mirror theories. For $X^{(8-\mathcal{N})} \times T^{(\mathcal{N}-1)}$ with $\mathcal{N} \geq 3$ the anomaly vanishes identically. Duff and Ferrara

Odd-dimensional analogue of Euler

- More generally, Euler $\chi = \Sigma(-1)^p b_p$ obeys a Kunneth formula $\chi(X \times Y) = \chi(X)\chi(Y)$; $\rho = \Sigma(-1)^p p b_p$ is an odd-dimensional analogue obeying $\rho(X \times Y) = \chi(X)\rho(Y)$. Borsten Duff and Nagy
- For example, the 4-dimensional Weyl anomaly for M-theory on $X_4 \times Y_7$ is given by $\chi(X_4)\rho(X^7) = \rho(X_4 \times Y_7)$ and hence vanishes when Y_7 is self-mirror. Since, in particular, $\rho(Y \times S^1) = \chi(Y)$, this is consistent with the corresponding anomaly for Type IIA on $X_4 \times Y_6$ given by $\chi(X_4)\chi(Y_6) = \chi(X_4 \times Y_6)$, which vanishes when Y_6 is self-mirror.

Four curious supergravities

- Of particular interest are the four cases

$$(b_0, b_1, b_2, b_3) = (1, \mathcal{N} - 1, 3\mathcal{N} - 3, 4\mathcal{N} + 3)$$

with $\mathcal{N} = 1, 2, 4, 8$, namely the four "curious" supergravities, discussed in Duff and Ferrara which enjoy some remarkable properties.
$\mathcal{N} = 1$, 7 WZ multiplets, $f = 32$,
$\mathcal{N} = 2$, 3 vector multiplets, 4 hypermultiplets, $f = 64$,
$\mathcal{N} = 4$, 6 vector mutiplets, $f = 128$,
$\mathcal{N} = 8$, $f = 256$.

O, H, C, R theories

Field	$360A$	**O**	**H**	**C**	**R**
$g_{\mu\nu}$	848	1	1	1	1
B_μ	-52	7	6	0	0
S	4	28	16	10	7
ψ_μ	-233	8	4	2	1
χ	7	56	28	14	7
$A_{\mu\nu\rho}$	-720	1	1	1	1
$A_{\mu\nu}$	364	7	3	1	0
A_μ	-52	21	6	4	0
A	4	35	19	11	7
		$A = 0$	$A = 0$	$A = 0$	$A = 0$

Table: Vanishing anomaly in **O**, **H**, **C**, **R** theories.

Timeline 2011

- Models for particle physics
  't Hooft
- Renormalization group and Weyl anomalies
  Percacci
- Proof of the four-dimensional a-theorem
  Komargodski et al
  Luty et al
  Elvang et al

Timeline 2012

- Gravitational anomalies and thermal Hall effect in topological insulators
  Stone
- A one-loop test of quantum gravity
  Bhattacharyya et al

Timeline 2013

- The a-theorem and entanglement entropy Solodukhin et al

Timeline 2015

- Holographic c-theorems in arbitrary dimensions
  Stone
- A one-loop test of quantum supergravity
  Bhattacharyya et al
- Anomalies and conformal manifolds
  Gomis
- More on boundary terms in the anomaly
  Fursaev
  Solodukhin
- Evanescent Effects Can Alter Ultraviolet Divergences
  Bern et al

Timeline 2016

- C_T for non-unitary CFTs in higher dimensions
  Osborn et al

Timeline 2017

- On the Flow of $\Box R$ Weyl-Anomaly
  Prochazka et al
- Higher spins
  Tseytlin
- More Weyl cosmology
  Dabholkar
  Nicolai
- The semi-classical stress-energy tensor in a Schwarzschild background
  Bardeen

Timeline 2017

- Conformal anomalies
  (1) gravitational wave
  (2) Einstein equations
  (3) Off-shell extensions Nicolai et al

Timeline 2021

- Applications of anomalies in condensed matter Chernodub et al
- Conformal Symmetry in Momentum Space and Anomaly Actions in Gravity Coriano et al Maglio et al
- Cancelling the Weyl anomaly in the standard model with dimension-zero scalar fields Boyle and Turok
- Boundary conformal invariants and the conformal anomaly in five dimensions Solodukhin et al

Timeline 2022

- Weyl anomalies of four dimensional conformal boundaries and defects Chalabi Herzog O'Bannon Brandon Robinson Sisti
- Anomaly-free scale symmetry and gravity Shaposhnikov and Tokareva
- The Effective Theory of Gravity and Dynamical Vacuum Energy Mottola

Summary

- Spacetime Weyl anomalies have found application in quantum corrections to the Schwarzschild solution and Newton's law, particle creation, the Hawking effect,. inflationary cosmology, asymptotic safety, wormholes, holography, viscosity bounds, condensed matter physics, hydrodynamics, the graviton mass in the braneworld, conformal collider physics, quantum entanglement, log corrections to black hole entropy, generalized mirror symmetry and double-copy theories.
- The cancellation of worldsheet Weyl anomalies not only determines the critical dimensions D=26 for strings and D=10 for superstrings, but also provides the derivation of the spacetime Einstein equations.
- The next 50 years?

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