

CONFORMAL SYMMETRY IN MOMENTUM SPACE AND ANOMALY ACTIONS IN GRAVITY

Matteo Maria Maglio

Heidelberg University - ITP

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UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386



STRUCTURES
CLUSTER OF
EXCELLENCE

- ▶ Conformal invariance
- ▶ Weyl invariance and conformal invariance
- ▶ Weyl anomaly and effective action
- ▶ Anomalous contributions to correlation functions involving stress energy tensors in momentum space

[C. Corianò, M. M. M., Nucl.Phys.B 937 (2018)]

[C. Corianò, M. M. M., E. Mottola, Nucl.Phys.B 942 (2019)]

[C. Corianò, M. M. M., D. Theofilopoulos, Eur.Phys.J.C 81 (2021)]

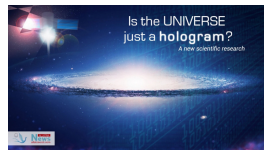
- ▶ Conformal invariance imposes strong constraints on correlation functions
- ▶ It determines 2- and 3-point functions of scalars, conserved vectors and the stress-energy tensor
- ▶ It determines the form of the higher functions up to functions of cross-ratios
- ▶ These results were obtained in position space and this is in contrast with general QFT where Feynman diagrams are typically computed in momentum space
- ▶ While position space methods are powerful, typically they can not be Fourier transformed easily in momentum space

Why study CFT directly in momentum space?

Momentum space results are needed in several recent applications

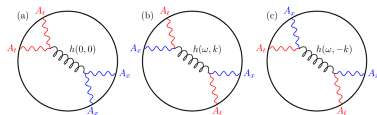
Holographic Cosmology

CMB primordial power spectra $\langle T_{\mu\nu}(\rho) T_{\rho\sigma}(-\rho) \rangle$ and
Non-Gaussianities $\langle T_{ij} T_{kl} T_{mn} \rangle$



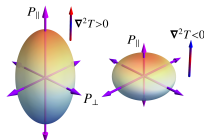
Studies of 3d critical phenomena

AdS/CFT correspondence for non-Fermi liquid metallic system. Chemical potential to a strongly-coupled CFT.



Anomalies

Breaking of conformal invariance by quantum effects.



Conformal invariance in Coordinate Space

Conformal transformations $x_\mu \rightarrow x'_\mu(x)$ preserve the infinitesimal length up to a local factor

$$dx_\mu dx^\mu \rightarrow dx'_\mu dx'^\mu = \Omega(x)^{-2} dx_\mu dx^\mu$$

and in the infinitesimal form $x'_\mu(x) = x_\mu + a_\mu + \omega_{\mu\nu} x^\nu + \lambda x_\mu + b_\mu x^2 - 2b \cdot x x_\mu$ with $\Omega(x) = 1 - \sigma(x) = 1 - \lambda + 2b \cdot x$.

Translation $\delta x^\mu = a_\mu,$

$$P_\mu = -i\partial_\mu,$$

Rotation $\delta x^\mu = \omega_{\mu\nu} x^\nu$

\implies

$$L_{\mu\nu} = -i(x_\nu \partial_\mu - x_\mu \partial_\nu),$$

Dilatations $\delta x^\mu = \lambda x^\mu,$

$$D = -ix^\mu \partial_\mu,$$

SCT $\delta x^\mu = b^\mu x^2 - 2x^\mu b \cdot x.$

$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu).$$

Defining the generators $J_{ab}, J_{ab} = -J_{ba}$ with $a, b = -1, 0, 1, \dots, d$ as

$$J_{\mu\nu} = L_{\mu\nu}, \quad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu), \quad J_{-1,0} = D, \quad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu)$$

$$[J_{ab}, J_{cd}] = -i(\eta_{ac} J_{bd} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad})$$

with $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)$. Then $\text{Conf}(\mathbb{R}^d) \cong \text{SO}(d+1, 1) \Rightarrow \frac{(d+2)(d+1)}{2}$ parameters.

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{C_{12} \delta_{\Delta_i \Delta_j}}{(x-y)^{\Delta_i + \Delta_j}}, \quad \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{\lambda_{123}}{|x_{12}|^{\Delta_{123}} |x_{13}|^{\Delta_{132}} |x_{23}|^{\Delta_{231}}}$$

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_4) \rangle = \frac{f(u, v)}{|x_{12}|^{2\Delta} |x_{34}|^{2\Delta}}, \quad u = \frac{x_{12}^2 x_{24}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

Dilatation

$$0 = \left[\sum_{j=1}^n \Delta_j + \sum_{j=1}^n x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right] \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

SCT

$$0 = \left[\sum_{j=1}^n \left(2\Delta_j x_j^\kappa + 2x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_j^\kappa} \right) \right] \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

$$2 \sum_{j=1}^n \sum_{h=1}^{r_j} \left[(x_j)_{\alpha_j h} \delta^{\kappa \mu_j h} - x_j^{\mu_j h} \delta_{\alpha_j h}^\kappa \right] \langle \mathcal{O}^{\mu_{11} \mu_{12} \dots \mu_{1r_1}}(x_1) \dots \mathcal{O}^{\mu_{j1} \mu_{j2} \dots \mu_{jr_j}}(x_j) \dots \mathcal{O}^{\mu_{n1} \mu_{n2} \dots \mu_{nr_n}}(x_n) \rangle$$

The position space expressions are only valid at separated points and do not possess a Fourier transform prior to renormalization. For instance

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle = \frac{C_{\mathcal{O}}}{|x|^{2\Delta}}$$

does not have a Fourier transform when

$$\Delta = \frac{d}{2} + k, \quad k = 0, 1, 2, \dots$$

because of short-distance singularities. In momentum space

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2) \rangle = (2\pi)^d \delta(p_1 + p_2) \frac{C_{\mathcal{O}} \pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} p_1^{2\Delta-d}, \quad \Delta \text{ generic}$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2) \rangle = (2\pi)^d \delta(p_1 + p_2) \left[\frac{C_{\mathcal{O}} \pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} \ln \frac{p_1^2}{\mu^2} + c_{\Delta'} \right] p_1^{2k}, \quad \Delta = \frac{d}{2} + k$$

with $c_{\Delta'}$ scheme dependent constant that can be absorbed by a redefinition of the scale μ .

[C. Coriano, L. Delle Rose, E. Mottola, and M. Serino(2012); A. Bzowski, P. McFadden and K. Skenderis (2016); C. Corianò and M.M.M. (2019)]

The infinitesimal transformation $x^\mu(x) \rightarrow x'^\mu(x) = x^\mu + \xi^\mu$ is classified as an isometry if it leaves the metric $g_{\mu\nu}(x)$ invariant in form $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$. This property leads to the **Killing equation**

$$\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0.$$

The condition $g'_{\mu\nu}(x') = \Omega^{-2}(x)g_{\mu\nu}(x')$ instead brings to the **conformal Killing equation**

$$\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 2\sigma g_{\mu\nu}$$

In the flat spacetime limit the conformal Killing equation has the general solution

$$\xi^\mu(x) = a_\mu + \omega_{\mu\nu}x^\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b \cdot x$$

Weyl transformations: $g_{\mu\nu} \rightarrow g'_{\mu\nu} = e^{2\sigma(x)}g_{\mu\nu} \implies \delta_\sigma^W g_{\mu\nu} = 2\sigma g_{\mu\nu}$

Diffeomorphism: $x_\mu \rightarrow x_\mu + \xi_\mu \implies \delta_\xi^E g_{\mu\nu} = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu$

and the conformal killing equations are given by the invariance of Weyl rescaling and diffeomorphism

$$\delta_\xi^E g_{\mu\nu} + \delta_\sigma^W g_{\mu\nu} = 0$$

Weyl invariance

The transformations of the conformal group cause specific rescaling of the metric in curved space, i.e. $g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu}$ for specific $\sigma(x)$, but Weyl transformations in curved space allow for a completely general $\sigma(x)$.

Weyl invariance implies conformal invariance when explicitly choosing a fixed “background” metric manifold.

Weyl invariance + Diffeomorphism \implies Conformal invariance in flat space

Weyl invariance + Diffeomorphism $\not\Leftarrow$ Conformal invariance in flat space

Example Weyl+Diff \implies CFT

Let Δ be a linear covariant differential operator built with $g_{\mu\nu}$ and ∇_μ of rank s ($\Delta \sim \partial^s + \dots$) such that

$$\Delta \rightarrow \Delta' = e^{-\frac{d+s}{2}\sigma} \Delta e^{\frac{d-s}{2}\sigma}.$$

Introducing a field Φ of dimensions $\frac{d-s}{2}$,

$$\Phi \rightarrow \Phi' = e^{-\frac{d-s}{2}\sigma} \Phi,$$

and $\bar{\Phi}$ conjugate with the same dimension, we construct a Weyl invariant action in an arbitrary dimension

$$S[\Phi] = - \int d^d x \sqrt{-g} \bar{\Phi} \Delta \Phi.$$

There exist interesting families of operators Δ defined above (Graham - Jenne - Mason - Sparling). They are rank $2n$ operators $n \in \mathbb{N}_0$ which act on real scalar fields ϕ of dimension $\delta_{2n} = \frac{d-2n}{2}$

$$\Delta_{2n} = \Delta_{2n,1} + \frac{d-2n}{2} Q_{2n},$$

where $\Delta_{2n,1}$ is purely derivative part and Q_{2n} the curvature part.

[A. Stergiou, G. P. Vacca, O. Zanusso (2022)]

The first member is the **Yamabe operator**

$$\Delta_2 = -\nabla^2 + \frac{d-2}{4(d-1)}R.$$

giving the action of a scalar conformally coupled

$$S[\phi, g] = -\frac{1}{2} \int d^d x \sqrt{-g} \phi \Delta_2 \phi \quad \xrightarrow{g \rightarrow \delta} \quad S[\phi, g] = \frac{1}{2} \int d^d x \phi \nabla^2 \phi$$

The **energy momentum tensor** in a general background can be defined as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

The variational EMT is traceless and conserved when evaluated on-shell using the equation of motion of ϕ

$$\nabla_\mu T^{\mu\nu} \Big|_{on-shell} = g_{\mu\nu} T^{\mu\nu} \Big|_{on-shell} = 0$$

Let us consider a four-dimensional system of massless fields in interaction with an external gravitational field whose action is invariant under conformal transformations of the metric tensor with appropriate rescalings of the matter fields.

In absence of self-interaction of the matter fields, the effective action \mathcal{W} is given exactly by its one-loop value and it is a non-local functional of the metric tensor and the fields. In dimensional regularization \mathcal{W} will be a function of the dimensions d and exhibits a simple pole at $d = 4$

$$\mathcal{W}(d) = \frac{1}{d-4} \int d^d x \sqrt{-g} A(d)$$

with

$$A(d) \sim R \square^{\frac{d-4}{2}} R, F_{\mu\nu}^a \square^{\frac{d-4}{2}} F^{a\mu\nu}, \dots$$

The infinities must be removed by a counterterm $\Delta\mathcal{W}$ which is d dependent and

$$\mathcal{W}_{reg} \equiv \mathcal{W}(d) + \delta\mathcal{W}(d)$$

is finite at $d = 4$.

Weyl anomaly

$\Delta\mathcal{W}$ must be a local functional and conformally invariant in $d = 4$. On dimensional grounds the only possibility is

$$\Delta\mathcal{W} = \frac{\mu^{d-4}}{d-4} \int d^d x \sqrt{-g} \left[bC^2 + b' E_4 + b'' e^2 F^2 \right]$$

$$C^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \quad E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

The regularized stress tensor will be finite at $d = 4$ and its trace is

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta\mathcal{W}_{reg}}{\delta g_{\mu\nu}},$$

$$T^\mu_\mu = \frac{2}{\sqrt{-g}} \left[g_{\mu\nu} \frac{\delta\mathcal{W}(d)}{\delta g_{\mu\nu}} + g_{\mu\nu} \frac{\delta\Delta\mathcal{W}(d)}{\delta g_{\mu\nu}} \right]_{d=4} = \frac{2}{\sqrt{-g}} \left[g_{\mu\nu} \frac{\delta\Delta\mathcal{W}(d)}{\delta g_{\mu\nu}} \right]_{d=4}$$

but

$$\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^d x \sqrt{-g} \left\{ C^2 ; E_4 ; F^2 \right\} = (d-4) \left\{ C^2 + \frac{2}{3} \square R ; E_4 ; F^2 \right\}$$

$$\langle T^\mu_\mu \rangle = b \left(C^2 + \frac{2}{3} \square R \right) + b' E_4 + b'' e^2 F^2$$

The effective action that results from renormalization can be written as

$$\mathcal{S} = \mathcal{S}_{inv}(g) + \mathcal{S}_{local}(g) + \mathcal{S}_{\mathcal{A}}(g)$$

with arbitrary Weyl invariant term $\mathcal{S}_{inv}(e^{2\sigma}g) = \mathcal{S}_{inv}(g)$ and $\mathcal{S}_{local}(g) \sim \int R^2$ term. The general form of the anomalous trace is

$$\langle T_{\mu}^{\mu}(x) \rangle_g = \mathcal{A} \equiv b C^2 + b' \left(E - \frac{2}{3} \square R \right) + \sum_i \beta_i \mathcal{L}_i,$$

and the Wess-Zumino consistency condition

$$\mathcal{S}_{\mathcal{A}}[e^{2\sigma} \bar{g}] = \mathcal{S}_{\mathcal{A}}[\bar{g}] + \Gamma_{WZ}[\bar{g}; \sigma],$$

$$\frac{\delta \Gamma_{WZ}[\bar{g}; \sigma]}{\delta \sigma(x)} = [\sqrt{-\bar{g}} \mathcal{A}]_{g=e^{2\sigma} \bar{g}}.$$

Anomaly effective action in $d = 4$

Integrating the anomaly along the orbit of the local conformal group, one finds the form of the **anomaly effective action in the local form** (depending on σ). There are two cases in $d = 4$ in which one can invert $\sigma = \Sigma(g)$ to obtain

$$\mathcal{S}_{\mathcal{A}}[g, \Sigma(g)] = -\frac{b'}{4} \int d^4x \sqrt{-g} \Sigma \Delta_4 \Sigma - \frac{1}{4} \int d^4x \sqrt{-g} \left[b C^2 + b' \left(E - \frac{2}{3} \square R + \sum_i \beta_i \mathcal{L} \right) \right] \Sigma$$

with the **two choices**

$$\Sigma_{FV}(g) = 2 \ln \left(1 + \frac{1}{6} G_{\square - R/6} \right), \quad \Sigma_R(g) = -\frac{1}{2} G_{\Delta_4} \left(E - \frac{2}{3} \square R \right),$$

where G_{Δ_4} and $G_{\square - R/6}$ are the Green's functions of the operators Δ_4 and $\square - R/6$ respectively. [E. Fradkin and G. A. Vilkovisky (1978); R. J. Riegert (1984)]

Momentum Space

No direct way to find the structure of the correlators.

One has to solve the CWI's

$$0 = \left[\sum_{j=1}^n \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right] \langle \mathcal{O}_1(p_1) \dots \mathcal{O}_n(\bar{p}_n) \rangle$$

$$0 = \left[\sum_{j=1}^{n-1} \left(2(\Delta_j - d) \frac{\partial}{\partial p_{j\kappa}} - 2p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_{j\kappa}} + p_j^\kappa \frac{\partial^2}{\partial p_{j\alpha} \partial p_j^\alpha} \right) \right] \langle \mathcal{O}_1(p_1) \dots \mathcal{O}_n(\bar{p}_n) \rangle$$

$$2 \sum_{j=1}^n \sum_{h=1}^{r_j} \left[\delta^{\kappa\mu_{jh}} \frac{\partial}{\partial p_j^{\alpha_{jh}}} - \delta_{\alpha_{jh}}^\kappa \frac{\partial}{\partial p_j^{\mu_{jh}}} \right] \langle \mathcal{O}^{\mu_{11}\mu_{12}\dots\mu_{1r_1}}(x_1) \dots \mathcal{O}^{\mu_{j1}\mu_{j2}\dots\mu_{jr_j}}(x_j) \dots \mathcal{O}^{\mu_{n1}\mu_{n2}\dots\mu_{nr_n}}(x_n) \rangle$$

Invariance under Weyl transformation \implies Trace Ward Identities

$$g_{\mu\nu} \langle T^{\mu\nu}(x) \rangle_g = 0$$

↓ Functional Differentiation, flat space limit and Fourier Transform

$$\delta_{\mu_1\nu_1} \langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_3) \rangle = -2 \langle T^{\mu_2\nu_2}(p_1 + p_2) T^{\mu_3\nu_3}(p_3) \rangle - 2 \langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_1 + p_3) \rangle$$

Invariance under diffeomorphisms of the background metric \implies Transverse Ward Identities

$$\nabla_{\mu}^{(g)} \langle T^{\mu\nu}(x) \rangle_g = 0$$

↓ Functional Differentiation, flat space limit and Fourier Transform

$$\begin{aligned} p_{1\nu_1} \langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_3) \rangle &= -p_2^{\mu_1} \langle T^{\mu_2\nu_2}(p_1 + p_2) T^{\mu_3\nu_3}(p_3) \rangle - p_3^{\mu_1} \langle T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_1 + p_3) \rangle \\ &+ p_{2\alpha} [\delta^{\mu_1\nu_2} \langle T^{\mu_2\alpha}(p_1 + p_2) T^{\mu_3\nu_3}(p_3) \rangle + \delta^{\mu_1\mu_2} \langle T^{\nu_2\alpha}(p_1 + p_2) T^{\mu_3\nu_3}(p_3) \rangle] \\ &+ p_{3\alpha} [\delta^{\mu_1\nu_3} \langle T^{\mu_3\alpha}(p_1 + p_3) T^{\mu_2\nu_2}(p_2) \rangle + \delta^{\mu_1\mu_3} \langle T^{\nu_3\alpha}(p_1 + p_3) T^{\mu_2\nu_2}(p_2) \rangle]. \end{aligned}$$

[Bzowski et al. , Implication of conformal invariance in momentum space, JHEP vol. 03, (2014), arXiv:1304.7760]

The separation of the stress energy tensor

$$T^{\mu\nu} = t^{\mu\nu} + t_{loc}^{\mu\nu}$$

where

$$t^{\mu\nu}(p) = \Pi_{\alpha\beta}^{\mu\nu}(p) T^{\alpha\beta}(p) \equiv \left[\pi_{\alpha}^{(\mu}(p) \pi_{\beta}^{\nu)}(p) - \frac{1}{d-1} \pi^{\mu\nu}(p) \pi_{\alpha\beta}(p) \right] T^{\alpha\beta}(p)$$

$$t_{loc}^{\mu\nu}(p) = \Sigma_{\alpha\beta}^{\mu\nu}(p) T^{\alpha\beta}(p) \equiv \left\{ \frac{p_{\beta}}{p^2} \left[2p^{(\mu} \delta_{\alpha}^{\nu)} - \frac{p_{\alpha}}{d-1} \left(\delta^{\mu\nu} + (d-2) \frac{p^{\mu} p^{\nu}}{p^2} \right) \right] + \frac{\pi^{\mu\nu}(p)}{d-1} \delta_{\alpha\beta} \right\} T^{\alpha\beta}(p)$$

$$\pi^{\mu\nu}(p) \equiv \delta^{\mu\nu} - \frac{p^{\mu} p^{\nu}}{p^2}$$

allows to decompose the correlator in **transverse-traceless**, **longitudinal-trace** parts

$$\begin{aligned} \langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle &= \langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t^{\mu_3\nu_3} \rangle + \langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} t_{loc}^{\mu_3\nu_3} \rangle + \langle T^{\mu_1\nu_1} t_{loc}^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle \\ &+ \langle t_{loc}^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle - \langle T^{\mu_1\nu_1} t_{loc}^{\mu_2\nu_2} t_{loc}^{\mu_3\nu_3} \rangle - \langle t_{loc}^{\mu_1\nu_1} t_{loc}^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle - \langle t_{loc}^{\mu_1\nu_1} T^{\mu_2\nu_2} t_{loc}^{\mu_3\nu_3} \rangle + \langle t_{loc}^{\mu_1\nu_1} t_{loc}^{\mu_2\nu_2} t_{loc}^{\mu_3\nu_3} \rangle \end{aligned}$$

In the decomposition

$$\begin{aligned} \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle = & \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle + \langle t_{loc}^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle + \langle T^{\mu_1 \nu_1} t_{loc}^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \\ & + \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} t_{loc}^{\mu_3 \nu_3} \rangle - \langle T^{\mu_1 \nu_1} t_{loc}^{\mu_2 \nu_2} t_{loc}^{\mu_3 \nu_3} \rangle - \langle t_{loc}^{\mu_1 \nu_1} t_{loc}^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle - \langle t_{loc}^{\mu_1 \nu_1} T^{\mu_2 \nu_2} t_{loc}^{\mu_3 \nu_3} \rangle + \langle t_{loc}^{\mu_1 \nu_1} t_{loc}^{\mu_2 \nu_2} t_{loc}^{\mu_3 \nu_3} \rangle \end{aligned}$$

the **longitudinal and trace parts** can be written in terms of **two point functions**

$$\begin{aligned} \langle t_{loc}^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle &= \Sigma_{\alpha_1 \beta_1}^{\mu_1 \nu_1}(\rho_1) \langle T^{\alpha_1 \beta_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \\ &= \left\{ \frac{\rho_1 \beta_1}{\rho_1^2} \left[2\rho_1^{\mu_1} \delta_{\alpha_1}^{\nu_1} - \frac{\rho_1 \alpha_1}{d-1} \left(\delta^{\mu_1 \nu_1} + (d-2) \frac{\rho_1^{\mu_1} \rho_1^{\nu_1}}{\rho_1^2} \right) \right] + \frac{\pi^{\mu_1 \nu_1}(\rho_1)}{d-1} \delta_{\alpha_1 \beta_1} \right\} \langle T^{\alpha_1 \beta_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \\ &= -\frac{2\pi^{\mu_1 \nu_1}(\rho_1)}{(d-1)} \left[\langle T^{\mu_2 \nu_2}(\rho_1 + \rho_2) T^{\mu_3 \nu_3}(\rho_3) \rangle + \langle T^{\mu_2 \nu_2}(\rho_2) T^{\mu_3 \nu_3}(\rho_1 + \rho_3) \rangle \right] \\ &+ \frac{1}{\rho_1^2} \left[2\rho_1^{\mu_1} \delta_{\alpha_1}^{\nu_1} - \frac{\rho_1 \alpha_1}{d-1} \left(\delta^{\mu_1 \nu_1} + (d-2) \frac{\rho_1^{\mu_1} \rho_1^{\nu_1}}{\rho_1^2} \right) \right] \left\{ -\rho_2^{\alpha_1} \langle T^{\mu_2 \nu_2}(\rho_1 + \rho_2) T^{\mu_3 \nu_3}(\rho_3) \rangle - \rho_3^{\alpha_1} \langle T^{\mu_2 \nu_2}(\rho_2) T^{\mu_3 \nu_3}(\rho_1 + \rho_3) \rangle \right. \\ &\quad \left. + \rho_2 \beta \left[\delta^{\alpha_1 \mu_2} \langle T^{\beta \mu_2}(\rho_1 + \rho_2) T^{\mu_3 \nu_3}(\rho_3) \rangle + \delta^{\alpha_1 \nu_2} \langle T^{\beta \mu_2}(\rho_1 + \rho_2) T^{\mu_3 \nu_3}(\rho_3) \rangle \right] \right. \\ &\quad \left. + \rho_3 \beta \left[\delta^{\alpha_1 \mu_3} \langle T^{\nu_2 \mu_2}(\rho_2) T^{\beta \nu_3}(\rho_1 + \rho_3) \rangle + \delta^{\alpha_1 \nu_3} \langle T^{\mu_2 \nu_2}(\rho_2) T^{\mu_3 \beta}(\rho_1 + \rho_3) \rangle \right] \right\} \end{aligned}$$

The transverse and traceless part can be decomposed in a minimal and manifestly symmetric form

$$\begin{aligned} \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle &= \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1}(p_1) \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2}(p_2) \Pi_{\alpha_3 \beta_3}^{\mu_3 \nu_3}(p_3) \left\{ A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3} \right. \\ &+ A_2 \delta^{\beta_1 \beta_2} p_2^{\alpha_1} p_3^{\alpha_2} p_1^{\alpha_3} p_1^{\beta_3} + A_2 (p_1 \leftrightarrow p_3) \delta^{\beta_2 \beta_3} p_3^{\alpha_2} p_1^{\alpha_3} p_2^{\alpha_1} p_2^{\beta_1} + A_2 (p_2 \leftrightarrow p_3) \delta^{\beta_3 \beta_1} p_1^{\alpha_3} p_2^{\alpha_1} p_3^{\alpha_2} p_3^{\beta_2} \\ &+ A_3 \delta^{\alpha_1 \alpha_2} \delta^{\beta_1 \beta_2} p_1^{\alpha_3} p_1^{\beta_3} + A_3 (p_1 \leftrightarrow p_3) \delta^{\alpha_2 \alpha_3} \delta^{\beta_2 \beta_3} p_2^{\alpha_1} p_2^{\beta_1} + A_3 (p_2 \leftrightarrow p_3) \delta^{\alpha_3 \alpha_1} \delta^{\beta_3 \beta_1} p_3^{\alpha_2} p_3^{\beta_2} \\ &\left. + A_4 \delta^{\alpha_1 \alpha_3} \delta^{\alpha_2 \beta_3} p_2^{\beta_1} p_3^{\beta_2} + A_4 (p_1 \leftrightarrow p_3) \delta^{\alpha_2 \alpha_1} \delta^{\alpha_3 \beta_1} p_3^{\beta_2} p_1^{\beta_3} + A_4 (p_2 \leftrightarrow p_3) \delta^{\alpha_3 \alpha_2} \delta^{\alpha_1 \beta_2} p_1^{\beta_3} p_2^{\beta_1} + A_5 \delta^{\alpha_1 \beta_2} \delta^{\alpha_2 \beta_3} \delta^{\alpha_3 \beta_1} \right\} \end{aligned}$$

in terms of **five independent form factors** depending on three variables p_i , the magnitude of the momenta $p_i = \sqrt{p_i^2}$ with $i = 1, 2, 3$. These form factors have also some symmetry conditions, in particular A_1 and A_5 are S_3 invariant, and the others are symmetric under the permutation $p_1 \leftrightarrow p_2$.

What is the procedure to find these form factors in a general CFT?

Condition on the form factors

Imposing the dilatation and special conformal Ward identities to correlator one can find some constraints on the form factors in particular

Dilatation Ward Identities \implies A_i are homogeneous functions of the momenta of specific degree
 $\text{deg}(A_i) = d - N_n$

Special Conformal Ward Identities \implies

- ▶ Second order differential equations on A_i
- ▶ First order differential equations on A_i

$$A_1(p_1, p_2, p_3) = \alpha_1 J_{6(000)} = \alpha_1 \int_0^\infty x^{\frac{d}{2}+5} p_1^{\frac{d}{2}} p_2^{\frac{d}{2}} p_3^{\frac{d}{2}} K_{\frac{d}{2}}(p_1 x) K_{\frac{d}{2}}(p_2 x) K_{\frac{d}{2}}(p_3 x)$$

...

The Special Conformal Ward Identities fix the 3-point function to depend on **two undetermined constants** α_1 and α_2 and 2-point function normalization c_T .

Perturbative Realization

The quantum actions for the scalar and fermion fields are respectively

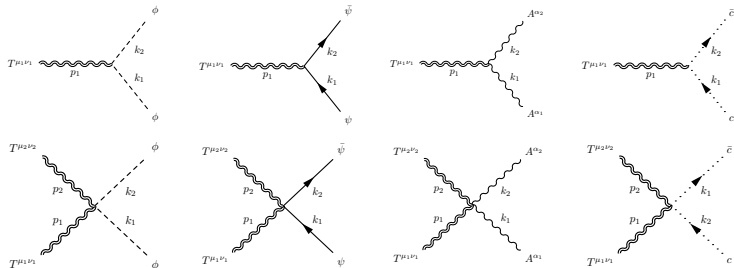
$$S_{\text{scalar}} = \frac{1}{2} \int d^d x \sqrt{-g} \left[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{(d-2)}{4(d-1)} R \phi^2 \right],$$

$$S_{\text{fermion}} = \frac{i}{2} \int d^d x e e_a^\mu \left[\bar{\psi} \gamma^a (D_\mu \psi) - (D_\mu \bar{\psi}) \gamma^a \psi \right], \quad D_\mu = \partial_\mu + \Gamma_\mu = \partial_\mu + \frac{1}{2} \Sigma^{ab} e_a^\sigma \nabla_\mu e_{b\sigma}.$$

The Σ^{ab} are the generators of the Lorentz group in the spin 1/2 representation. In $d = 4$ there is an additional conformal field theory

$$S_{\text{abelian}} = S_M + S_{gf} + S_{gh},$$

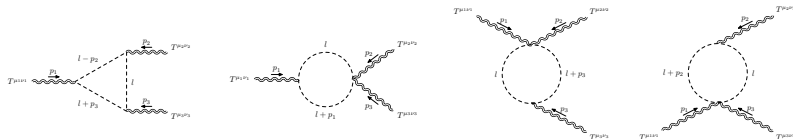
$$S_M = -\frac{1}{4} \int d^4 x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \quad S_{gf} = -\frac{1}{\xi} \int d^4 x \sqrt{-g} (\nabla_\mu A^\mu)^2, \quad S_{gh} = \int d^4 x \sqrt{-g} \partial^\mu \bar{c} \partial_\mu c.$$



The 3-point function we are interested in studying is

$$\begin{aligned} \langle T^{\mu_1 \nu_1}(x_1) T^{\mu_2 \nu_2}(x_2) T^{\mu_3 \nu_3}(x_3) \rangle &= 8 \left\{ - \left\langle \frac{\delta S}{\delta g_{\mu_1 \nu_1}(x_1)} \frac{\delta S}{\delta g_{\mu_2 \nu_2}(x_2)} \frac{\delta S}{\delta g_{\mu_3 \nu_3}(x_3)} \right\rangle \right. \\ &+ \left\langle \frac{\delta^2 S}{\delta g_{\mu_1 \nu_1}(x_1) \delta g_{\mu_2 \nu_2}(x_2)} \frac{\delta S}{\delta g_{\mu_3 \nu_3}(x_3)} \right\rangle + \left\langle \frac{\delta^2 S}{\delta g_{\mu_1 \nu_1}(x_1) \delta g_{\mu_3 \nu_3}(x_3)} \frac{\delta S}{\delta g_{\mu_2 \nu_2}(x_2)} \right\rangle \\ &\left. + \left\langle \frac{\delta^2 S}{\delta g_{\mu_2 \nu_2}(x_2) \delta g_{\mu_3 \nu_3}(x_3)} \frac{\delta S}{\delta g_{\mu_1 \nu_1}(x_1)} \right\rangle - \left\langle \frac{\delta^3 S}{\delta g_{\mu_1 \nu_1}(x_1) \delta g_{\mu_2 \nu_2}(x_2) \delta g_{\mu_3 \nu_3}(x_3)} \right\rangle \right\} \end{aligned}$$

where the angle brackets denote the vacuum expectation value. In **momentum space**



$$\begin{aligned} V_S^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(p_1, p_2, p_3) &= \int \frac{d^d \ell}{(2\pi)^d} \frac{V_{T\phi\phi}^{\mu_1 \nu_1}(\ell - p_2, \ell + p_3) V_{T\phi\phi}^{\mu_2 \nu_2}(\ell, \ell - p_2) V_{T\phi\phi}^{\mu_3 \nu_3}(\ell, \ell + p_3)}{\ell^2 (\ell - p_2)^2 (\ell + p_3)^2} \\ W_{S,1}^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(p_1, p_2, p_3) &= \frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \frac{V_{T\phi\phi}^{\mu_1 \nu_1}(\ell, \ell + p_1) V_{TT\phi\phi}^{\mu_2 \nu_2 \mu_3 \nu_3}(\ell, \ell + p_1)}{\ell^2 (\ell + p_1)^2} \end{aligned}$$

[C. Corianò and M.M.M., The general 3-graviton vertex $\langle TTT \rangle$ of conformal field theories in momentum space in $d = 4$, Nucl.Phys. B937 (2018) 56-134]

One calculates the diagrams contributions in dimensional regularization and constructs the transverse and traceless part for each sectors as

$$\langle t^{\mu_1 \nu_1}(p_1) t^{\mu_2 \nu_2}(p_2) t^{\mu_3 \nu_3}(p_3) \rangle_S = \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1}(p_1) \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2}(p_2) \Pi_{\alpha_3 \beta_3}^{\mu_3 \nu_3}(p_3) \left[-V_S^{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3}(p_1, p_2, p_3) + \sum_{j=1}^3 W_{S,j}^{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3}(p_1, p_2, p_3) \right].$$

Using the properties

$$\begin{aligned} p^\alpha \Pi_{\alpha\beta}^{\mu\nu}(p) &= 0, & g^{\alpha\beta} \Pi_{\alpha\beta}^{\mu\nu}(p) &= 0, & p_3^{\alpha_1} \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1}(p_1) &= -p_2^{\alpha_1} \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1}(p_1) \\ p_1^{\alpha_2} \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2}(p_2) &= -p_3^{\alpha_2} \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2}(p_2), & p_2^{\alpha_3} \Pi_{\alpha_3 \beta_3}^{\mu_3 \nu_3}(p_3) &= -p_1^{\alpha_3} \Pi_{\alpha_3 \beta_3}^{\mu_3 \nu_3}(p_3) \end{aligned}$$

we reorganize the transverse and traceless part as

$$\begin{aligned} \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle_S &= \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1}(p_1) \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2}(p_2) \Pi_{\alpha_3 \beta_3}^{\mu_3 \nu_3}(p_3) \left\{ A_1^S p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3} \right. \\ &+ A_2^S \delta^{\beta_1 \beta_2} p_2^{\alpha_1} p_3^{\alpha_2} p_1^{\alpha_3} p_1^{\beta_3} + A_2^S (p_1 \leftrightarrow p_3) \delta^{\beta_2 \beta_3} p_3^{\alpha_2} p_1^{\alpha_3} p_2^{\alpha_1} p_2^{\beta_1} + A_2^S (p_2 \leftrightarrow p_3) \delta^{\beta_3 \beta_1} p_1^{\alpha_3} p_2^{\alpha_1} p_3^{\alpha_2} p_3^{\beta_2} \\ &+ A_3^S \delta^{\alpha_1 \alpha_2} \delta^{\beta_1 \beta_2} p_1^{\alpha_3} p_1^{\beta_3} + A_3^S (p_1 \leftrightarrow p_3) \delta^{\alpha_2 \alpha_3} \delta^{\beta_2 \beta_3} p_2^{\alpha_1} p_2^{\beta_1} + A_3^S (p_2 \leftrightarrow p_3) \delta^{\alpha_3 \alpha_1} \delta^{\beta_3 \beta_1} p_3^{\alpha_2} p_3^{\beta_2} \\ &\left. + A_4^S \delta^{\alpha_1 \alpha_3} \delta^{\alpha_2 \beta_3} p_2^{\beta_1} p_3^{\beta_2} + A_4^S (p_1 \leftrightarrow p_3) \delta^{\alpha_2 \alpha_1} \delta^{\alpha_3 \beta_1} p_3^{\beta_2} p_1^{\beta_3} + A_4^S (p_2 \leftrightarrow p_3) \delta^{\alpha_3 \alpha_2} \delta^{\alpha_1 \beta_2} p_1^{\beta_3} p_2^{\beta_1} + A_5^S \delta^{\alpha_1 \beta_2} \delta^{\alpha_2 \beta_3} \delta^{\alpha_3 \beta_1} \right\} \end{aligned}$$

The form factors A_j^S are functions of the kinematical invariants and the space-time dimensions d . Furthermore they are written in terms of master integrals B_0 and C_0 .

Finally we construct the entire transverse and traceless part noticing that the number of fermion, scalar and gauge families is arbitrary and then the constants n_l , $l = F, S, G$ have to be considered.

$$\langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle = \sum_{l=F,S,G} n_l \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle_l$$

Case $d = 4$

Divergences!

$$A_2^{Div} = \frac{\pi^2}{45\varepsilon} [26n_G - 7n_F - 2n_S]$$

$$A_3^{Div} = \frac{\pi^2}{90\varepsilon} [3(p_1^2 + p_2^2)(6n_F + n_S + 12n_G) + p_3^2(11n_F + 62n_G + n_S)]$$

$$A_4^{Div} = \frac{\pi^2}{90\varepsilon} [(p_1^2 + p_2^2)(29n_F + 98n_G + 4n_S) + p_3^2(43n_F + 46n_G + 8n_S)]$$

$$A_5^{Div} = \frac{\pi^2}{180\varepsilon} \left\{ n_F(43p_1^4 - 14p_1^2(p_2^2 + p_3^2) + 43p_2^4 - 14p_2^2p_3^2 + 43p_3^4) \right. \\ \left. + 2[n_G(23p_3^2 + 26p_1^2(p_2^2 + p_3^2) + 23p_2^4 + 26p_2^2p_3^2 + 23p_3^4) + 2n_S(2p_1^4 - p_1^2(p_2^2 + p_3^2) + 2p_2^4 - p_2^2p_3^2 + 2p_3^4)] \right\}$$

In perturbation theory the one loop counterterm Lagrangian is

$$S_{\text{count}} = -\frac{1}{\varepsilon} \sum_{I=F,S,G} n_I \int d^d x \sqrt{-g} \left(\beta_a(I) C^2 + \beta_b(I) E \right)$$

where

$$V_{C^2}^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(p_1, p_2, p_3) = 8 \int d^d x_1 d^d x_2 d^d x_3 d^d x \left(\frac{\delta^3(\sqrt{-g} C^2)(x)}{\delta g_{\mu_1 \nu_1}(x_1) \delta g_{\mu_2 \nu_2}(x_2) \delta g_{\mu_3 \nu_3}(x_3)} \right)_{\text{flat}} e^{-i(p_1 x_1 + p_2 x_2 + p_3 x_3)} \\ \equiv 8[\sqrt{-g} C^2]^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(p_1, p_2, p_3)$$

$$V_E^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(p_1, p_2, p_3) = 8 \int d^d x_1 d^d x_2 d^d x_3 d^d x \left(\frac{\delta^3(\sqrt{-g} E)(x)}{\delta g_{\mu_1 \nu_1}(x_1) \delta g_{\mu_2 \nu_2}(x_2) \delta g_{\mu_3 \nu_3}(x_3)} \right)_{\text{flat}} e^{-i(p_1 x_1 + p_2 x_2 + p_3 x_3)} \\ \equiv 8[\sqrt{-g} E]^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3}(p_1, p_2, p_3).$$

Renormalized TTT

The renormalization of the transverse-traceless part of TTT ensures also the renormalization of the contributions coming from the two point function.

$$\begin{aligned}
 \langle t_{loc}^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle &= \left(\mathcal{I}_{\alpha_1}^{\mu_1 \nu_1}(p_1) p_{1\beta_1} + \frac{\pi^{\mu_1 \nu_1}(p_1)}{(d-1)} \delta_{\alpha_1 \beta_1} \right) \langle T^{\alpha_1 \beta_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \\
 &= -\frac{2\pi^{\mu_1 \nu_1}(p_1)}{(d-1)} \left[\langle T^{\mu_2 \nu_2}(p_1 + p_2) T^{\mu_3 \nu_3}(p_3) \rangle + \langle T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_1 + p_3) \rangle \right] \\
 &\quad + \mathcal{I}_{\alpha_1}^{\mu_1 \nu_1}(p_1) \left\{ -p_2^{\alpha_1} \langle T^{\mu_2 \nu_2}(p_1 + p_2) T^{\mu_3 \nu_3}(p_3) \rangle - p_3^{\alpha_1} \langle T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_1 + p_3) \rangle \right\} \\
 &\quad + p_{2\beta} \left[\delta^{\alpha_1 \mu_2} \langle T^{\beta \mu_2}(p_1 + p_2) T^{\mu_3 \nu_3}(p_3) \rangle + \delta^{\alpha_1 \nu_2} \langle T^{\beta \mu_2}(p_1 + p_2) T^{\mu_3 \nu_3}(p_3) \rangle \right] \\
 &\quad + p_{3\beta} \left[\delta^{\alpha_1 \mu_3} \langle T^{\nu_2 \mu_2}(p_2) T^{\beta \nu_3}(p_1 + p_3) \rangle + \delta^{\alpha_1 \nu_3} \langle T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \beta}(p_1 + p_3) \rangle \right] \}
 \end{aligned}$$

$$\begin{aligned}
 \langle t_{loc}^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle_{count} &= -\frac{1}{\varepsilon} \frac{(d-4)}{(d-1)} \pi^{\mu_1 \nu_1}(p_1) \left(4n_l \beta_b [E]^{\mu_2 \nu_2 \mu_3 \nu_3}(p_2, p_3) + 4n_l \beta_a [C^2]^{\mu_2 \nu_2 \mu_3 \nu_3}(p_2, p_3) \right) \\
 &\quad + \frac{1}{\varepsilon} \frac{2}{(d-1)} \pi^{\mu_1 \nu_1}(p_1) \left(4n_l \beta_a [C^2]^{\mu_2 \nu_2 \mu_3 \nu_3}(p_1 + p_2, p_3) + 4n_l \beta_a [C^2]^{\mu_2 \nu_2 \mu_3 \nu_3}(p_2, p_1 + p_3) \right) \\
 &\quad - \frac{1}{\varepsilon} \mathcal{I}_{\alpha_1}^{\mu_1 \nu_1}(p_1) \left\{ -4p_2^{\alpha_1} n_l \beta_a [C^2]^{\mu_2 \nu_2 \mu_3 \nu_3}(p_1 + p_2, p_3) - p_3^{\alpha_1} n_l \beta_a [C^2]^{\mu_2 \nu_2 \mu_3 \nu_3}(p_2, p_1 + p_3) \right. \\
 &\quad + 4p_{2\beta} \left[\delta^{\alpha_1 \mu_2} n_l \beta_a [C^2]^{\beta \nu_2 \mu_3 \nu_3}(p_1 + p_2, p_3) + \delta^{\alpha_1 \nu_2} n_l \beta_a [C^2]^{\mu_2 \beta \mu_3 \nu_3}(p_1 + p_2, p_3) \right] \\
 &\quad \left. + 4p_{3\beta} \left[\delta^{\alpha_1 \mu_3} n_l \beta_a [C^2]^{\mu_2 \nu_2 \beta \nu_3}(p_2, p_1 + p_3) + \delta^{\alpha_1 \nu_3} n_l \beta_a [C^2]^{\mu_2 \nu_2 \mu_3 \beta}(p_2, p_1 + p_3) \right] \right\}.
 \end{aligned}$$

The renormalized TTT is given by

$$\langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle_{Ren} = \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle_{Ren} + \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle_{Ren} + \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle_{extra}$$

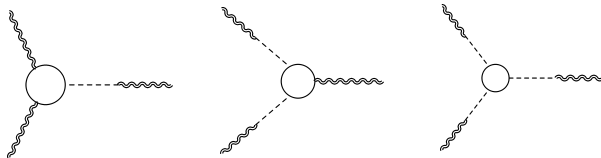
Anomalous Conformal Ward Identities \implies Explicit breaking of Conformal Invariance

[C. Corianò and M.M.M., The general 3-graviton vertex (TTT) of conformal field theories in momentum space in $d = 4$, Nucl.Phys. B937 (2018) 56-134]

The extra part of the TTT after the renormalization is the anomaly part

$$\begin{aligned} \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle_{\text{extra}}^{(4)} = & \left(\frac{\hat{\pi}^{\mu_1 \nu_1}(\rho_1)}{3\rho_1^2} \left(4n_l \beta_b [E]^{\mu_2 \nu_2 \mu_3 \nu_3}(\rho_2, \bar{\rho}_3) + 4n_l \beta_a [C^2]^{\mu_2 \nu_2 \mu_3 \nu_3}(\rho_2, \bar{\rho}_3) \right) + (\text{cycl. perm.}) \right) \\ & - \left(\frac{\hat{\pi}^{\mu_1 \nu_1}(\rho_1)}{3\rho_1^2} \frac{\hat{\pi}^{\mu_2 \nu_2}(\rho_2)}{3\rho_2^2} \delta_{\alpha_2 \beta_2} \left(4n_l \beta_b [E]^{\alpha_2 \beta_2 \mu_3 \nu_3}(\rho_2, \bar{\rho}_3) + 4n_l \beta_a [C^2]^{\alpha_2 \beta_2 \mu_3 \nu_3}(\rho_2, \bar{\rho}_3) \right) + (\text{cycl. perm.}) \right) \\ & + \frac{\hat{\pi}^{\mu_1 \nu_1}(\rho_1)}{3\rho_1^2} \frac{\hat{\pi}^{\mu_2 \nu_2}(\rho_2)}{3\rho_2^2} \frac{\hat{\pi}^{\mu_3 \nu_3}(\bar{\rho}_3)}{3\rho_3^2} \delta_{\alpha_2 \beta_2} \delta_{\alpha_3 \beta_3} \left(4n_l \beta_b [E]^{\alpha_2 \beta_2 \alpha_3 \beta_3}(\rho_2, \bar{\rho}_3) + 4n_l \beta_a [C^2]^{\alpha_2 \beta_2 \alpha_3 \beta_3}(\rho_2, \bar{\rho}_3) \right). \end{aligned}$$

We have anomaly interactions!



Anomaly interactions mediated by the exchange of one, two or three poles.

Consider then the **anomaly effective action** in its **non-local form** for the two choices Σ_{FV} and Σ_R . We have calculated their variations to get the anomaly contribution to the $\langle TTT \rangle$

$$\begin{aligned} \mathcal{S}_{\text{anom}}^{(3)}[\Sigma_{FV}] = \mathcal{S}_{\text{anom}}^{(3)}[\Sigma_R] = & \frac{b'}{9} \int dx \int dx' \int dx'' \left\{ \left(\partial_\mu R^{(1)} \right)_x \left(\frac{1}{\bar{\square}} \right)_{xx'} \left(R^{(1)\mu\nu} - \frac{1}{3} \eta^{\mu\nu} R^{(1)} \right)_{x'} \left(\frac{1}{\bar{\square}} \right)_{x'x''} \left(\partial_\nu R^{(1)} \right)_{x''} \right\} \\ & - \frac{1}{6} \int dx \int dx' \left(b' E^{(2)} + b [C^2]^{(2)} \right)_x \left(\frac{1}{\bar{\square}} \right)_{xx'} R_{x'}^{(1)} + \frac{b'}{18} \int dx R^{(1)} \left(2R^{(2)} + (\sqrt{-g})^{(1)} R^{(1)} \right) \end{aligned}$$

[C. Corianò, M.M.M. and E. Mottola, (2019)]

$$\begin{aligned} (A) \mathcal{S}_3^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} (p_1, p_2, p_3) = & \frac{8}{3} \left\{ \pi^{\mu_1 \nu_1} (p_1) \left[b' E^{(2)} + b [C^2]^{(2)} \right]^{\mu_2 \nu_2 \mu_3 \nu_3} (p_2, p_3) + (\text{cyclic}) \right\} \\ & - \frac{16b'}{9} \left\{ \pi^{\mu_1 \nu_1} (p_1) Q^{\mu_2 \nu_2} (p_1, p_2, p_3) \pi^{\mu_3 \nu_3} (p_3) + (\text{cyclic}) \right\} \\ & + \frac{16b'}{27} \pi^{\mu_1 \nu_1} (p_1) \pi^{\mu_2 \nu_2} (p_2) \pi^{\mu_3 \nu_3} (p_3) \left\{ p_3^2 p_1 \cdot p_2 + (\text{cyclic}) \right\} \end{aligned}$$

For a generic nT correlator, the only counterterm needed for its renormalization is

$$\mathcal{S}_{ct} = -\frac{\mu^\epsilon}{\epsilon} \int d^d x (b C^2 + b' E)$$

$$\mathcal{S}_{ct} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

The renormalized effective action \mathcal{S}_R is then defined by the sum of the two terms

$$\mathcal{S}(g)_R = \mathcal{S}(g) + \mathcal{S}(g)_{ct}.$$

The correlation functions extracted by the renormalized action can be expressed as the sum of a finite (f) correlator and of an anomaly term (\mathcal{A}) in the form

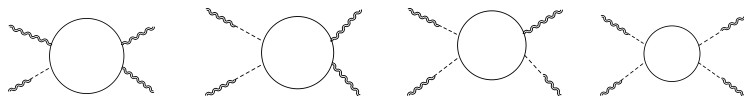
$$\langle T^{\mu_1 \nu_1} \dots T^{\mu_n \nu_n} \rangle_{Ren} = \langle T^{\mu_1 \nu_1} \dots T^{\mu_n \nu_n} \rangle_f^{(d=4)} + \langle T^{\mu_1 \nu_1} \dots T^{\mu_n \nu_n} \rangle_{\mathcal{A}}^{(d=4)}$$

4-point function

The anomaly part of the four point function

$$\langle T^{\mu_1 \nu_1} \dots T^{\mu_4 \nu_4} \rangle_{\mathcal{A}} = \langle T^{\mu_1 \nu_1} \dots T^{\mu_4 \nu_4} \rangle_{poles} + \langle T^{\mu_1 \nu_1} \dots T^{\mu_4 \nu_4} \rangle_{0-trace}$$

where the poles part can be summarised by the four types of diagrams



and the 0-trace part is essential to satisfy the conservation Ward identities, but not the anomalous trace relations, indeed

$$\delta_{\mu_i \nu_i} \langle T^{\mu_1 \nu_1} \dots T^{\mu_4 \nu_4} \rangle_{0-trace} = 0, \quad i = 1, \dots, 4$$

$$\delta_{\mu_1 \nu_1} \langle T^{\mu_1 \nu_1} \dots T^{\mu_4 \nu_4} \rangle_{poles} = \mathcal{A}^{\mu_2 \nu_2 \dots \mu_4 \nu_4}$$

$$\begin{aligned} & p_{i\mu_i} \langle T^{\mu_1 \nu_1} \dots T^{\mu_4 \nu_4} \rangle_{\mathcal{A}} = \\ & = p_{i\mu_i} \langle T^{\mu_1 \nu_1} \dots p_{i\mu_i} T^{\mu_4 \nu_4} \rangle_{poles} + p_{i\mu_i} \langle T^{\mu_1 \nu_1} \dots T^{\mu_4 \nu_4} \rangle_{0-trace} \propto \langle TTT \rangle_{\mathcal{A}} \end{aligned}$$

[C. Corianò, M. M. M. and D. Theofilopoulos, Eur. Phys. J. C 81 (2021) 8, 740.]

$$\begin{aligned}
 & \langle T^{\mu_1 \nu_1}(p_1) T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_3) T^{\mu_4 \nu_4}(\bar{p}_4) \rangle_{0\text{-residue}} = \\
 & = \mathcal{I}_{\alpha_1}^{\mu_1 \nu_1}(p_1) p_{1\beta_1} \langle T^{\alpha_1 \beta_1}(p_1) T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_3) T^{\mu_4 \nu_4}(\bar{p}_4) \rangle_{\text{anomaly}} + (\text{perm.}) \\
 & + (\mathcal{II} - \text{terms}) + (\mathcal{I}\pi - \text{terms}) + (\mathcal{II}\pi - \text{terms}) + (\mathcal{I}\pi\pi - \text{terms}) \\
 & + (\mathcal{II}\pi\pi - \text{terms}) + (\mathcal{I}\pi\pi\pi - \text{terms}).
 \end{aligned}$$

$$\begin{aligned}
 & \langle T^{\mu_1 \nu_1}(p_1) T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_3) T^{\mu_4 \nu_4}(\bar{p}_4) \rangle_{\text{poles}} = \\
 & = \frac{\pi^{\mu_1 \nu_1}(p_1)}{3} \langle T(p_1) T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_3) T^{\mu_4 \nu_4}(\bar{p}_4) \rangle_{\text{anomaly}} + (\text{perm.}) \\
 & - \frac{\pi^{\mu_1 \nu_1}(p_1)}{3} \frac{\pi^{\mu_2 \nu_2}(p_2)}{3} \langle T(p_1) T(p_2) T^{\mu_3 \nu_3}(p_3) T^{\mu_4 \nu_4}(\bar{p}_4) \rangle_{\text{anomaly}} + (\text{perm.}) \\
 & + \frac{\pi^{\mu_1 \nu_1}(p_1)}{3} \frac{\pi^{\mu_2 \nu_2}(p_2)}{3} \frac{\pi^{\mu_3 \nu_3}(p_3)}{3} \langle T(p_1) T(p_2) T(p_3) T^{\mu_4 \nu_4}(\bar{p}_4) \rangle_{\text{anomaly}} + (\text{perm.}) \\
 & - \frac{\pi^{\mu_1 \nu_1}(p_1)}{3} \frac{\pi^{\mu_2 \nu_2}(p_2)}{3} \frac{\pi^{\mu_3 \nu_3}(p_3)}{3} \frac{\pi^{\mu_4 \nu_4}(p_4)}{3} \langle T(p_1) T(p_2) T(p_3) T(\bar{p}_4) \rangle_{\text{anomaly}}.
 \end{aligned}$$

- ▶ Implications of conformal invariance in momentum space
- ▶ Matching of the exact solutions with the perturbative ones for the TTT
- ▶ Structure of the anomalous part for the TTT
- ▶ Structure of the anomalous part for the $TTTT$

Next:

- ▶ What is the anomalous structure of higher point functions?
- ▶ Does the anomaly effective action in $d = 4$ predicts the anomaly part of correlators involving T at all orders?

