Topological Corretions and Conformal Backreaction
in Einstein-Gauss-Bonnet-Weyl Theory of Gravity at d=4

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We investigate the gravitational backreaction, generated by coupling a general conformal sector to external, classical gravity, as described by a conformal anomaly effective action. We address the issues raised by the regularization of the topological Gauss–Bonnet and Weyl terms in these actions and the use of dimensional regularization (DR). We discuss both their local and nonlocal expressions, as possible IR and UV descriptions of conformal theories, below and above the conformal breaking scale.

Einstein Gauss-Bonnet theories as ordinary, Wess-Zumino conformal anomaly actions #2

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Topological Corrections and Conformal Backreaction in the Einstein Gauss-Bonnet/Weyl Theories of Gravity at D=4

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and works in preparation (**M. Creti', M. Maglio, R. Tommasi, S. Lionetti)**

Implications of these effective field theory descriptions in topological materials

The search for corrections to general relativity (GR) and to its Einstein-Hilbert (EH) action by higher derivative terms, is characterized by a large number of both older and of more recent proposals. Their goal is to address unsolved issues, such as the nature of dark energy and the mechanism of inflation of the early universe, in a more satisfactory way.

An important open question concerns the quantum consistency of these extensions, since the presence of higher order derivatives in the action leads, in general, to equations of motion of higher order.

particular attention is paid to the stability and the consistency of such theories, by showing, for instance, the absence of tachyonic solutions as well as of ghosts and, eventually, addressing their renormalizability in a perturbative context.

Among these proposals, of particular interest are those extensions that lead to second-order equations of motion, even though they are generated by higher derivatives Lagrangians.

Such Lagrangians may be introduced at classical level, or, alternatively, they may originate from the inclusion of quantum corrections, in models where gravity is still treated classically.

Their structure depends on the specific type of matter sector that is integrated out of the quantum partition function.

INDUCED GRAVITY (A. Sakharov)

GRAVITY MATTER

Gravity is entirely generated by integrating out of the functional integral a matter sector (nicely reviewed by M. Visser)

These induce an effective action of the form

$$\mathcal{S} = \int d^4x\sqrt{g}\left(c_1\frac{M_P^2}{2}R + c_2R^2 + \ldots\right),$$

If the matter sector that is integrated out is conformally coupled to gravity, then this formulation adds to a tree-level Lagrangian, which may be conformal or not, the effect of conformal backreaction on the metric.

The ellipsis refer to extra contributions built out of higher order geometrical invariants.

In this approach, the entire gravitational theory can be viewed as the result of the quantum backreaction on a freely fluctuating metric, induced by the path integration over the matter sector.

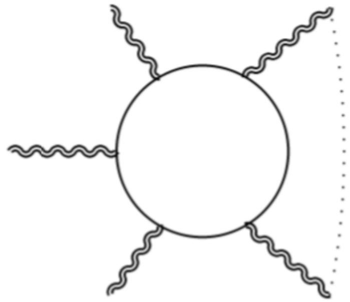
The backreaction of a conformal sector on the gravitational metric can be discussed via the partition function

$$\mathcal{Z}_B(g) = \mathcal{N} \int D\chi e^{-S_0(g,\chi)},$$

$$e^{-\mathcal{S}_B(g)} = \mathcal{Z}_B(g) \leftrightarrow \mathcal{S}_B(g) = -\log \mathcal{Z}_B(g).$$

$$S_0(g,\chi) = \frac{1}{2} \int d^dx \sqrt{-g} \left[g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi - c_0 R \chi^2 \right],$$

$$\mathcal{S}(g) = \sum_n$$



(n-point)

$$\mathcal{S}(g)_B \equiv \mathcal{S}(\bar{g})_B + \sum_{n=1}^{\infty} \frac{1}{2^n n!} \int d^d x_1 \ldots d^d x_n \sqrt{g_1} \ldots \sqrt{g_n}\, \langle T^{\mu_1 \nu_1} \ldots\, T^{\mu_n \nu_n} \rangle_{\bar{g}B} \delta g_{\mu_1 \nu_1}(x_1) \ldots \delta g_{\mu_n \nu_n}(x_n),$$

$$\langle T^{\mu_1 \nu_1}(x_1) \ldots T^{\mu_n \nu_n}(x_n) \rangle_B \equiv \frac{2}{\sqrt{g_1}} \cdots \frac{2}{\sqrt{g_n}} \frac{\delta^n \mathcal{S}_B(g)}{\delta g_{\mu_1 \nu_1}(x_1) \delta g_{\mu_2 \nu_2}(x_2) \ldots \delta g_{\mu_n \nu_n}(x_n)},$$

In DR the divergences appear as single poles if we couple a conformal sector to gravity, and their renormalization is performed by expanding the counterterms around d=4.

$$\mathcal{S}_B(g,d) = -\log\left(\int D\Phi e^{-S(\Phi,g)}\right) + \log\mathcal{N},$$

The two counterterms to be included are VE, VC
that will be discussed next, giving a regularized effective action
of the form

$$\mathcal{S}_R(g,d) = \mathcal{S}_B(g,d) + b'\frac{1}{\epsilon}V_E(g,d) + b\frac{1}{\epsilon}V_{C^2}(g,d).$$

VE and VC are related to the Euler density E and to the Weyl
tensor squared C^2 respectively, defined by the expressions

$$V_{C^2}(g,d) \equiv \mu^{\varepsilon}\int d^dx\,\sqrt{-g}\,C^2,$$

$$V_E(g,d) \equiv \mu^{\varepsilon}\int d^dx\,\sqrt{-g}\,E,$$

Advantages of the functional formalism. Invariance of the action under Killing/Conformal Killing vectors can immediately generate all the Conformal Ward identities

$$(ds')^2 = e^{2\sigma(x)}(ds)^2 \qquad \leftrightarrow \qquad \nabla_\mu \varepsilon_\nu^{(K)} + \nabla_\nu \varepsilon_\mu^{(K)} = 2\sigma\delta_{\mu\nu} \qquad \sigma = \frac{1}{4}\nabla\cdot\varepsilon^{(K)}.$$

$$\nabla\cdot\langle J_c\rangle = \frac{1}{4}\nabla\cdot\varepsilon^{(K)}\langle T^\mu_\mu\rangle + \varepsilon_\nu^{(K)}\nabla_\mu\langle T^{\mu\nu}\rangle.$$

$$\langle J^\mu\rangle = \varepsilon_\nu^{(K)}\langle T^{\mu\nu}\rangle,$$

$$K_\mu^{(C)\,\kappa} = 2x^\kappa x_\mu - x^2\delta_\mu^\kappa$$

$$\partial\cdot K^{(C)\,\kappa} = 2d\,x^\kappa$$

$$0 = \int d^dx\,\sqrt{-g}\,\nabla_\mu\,\langle J^\mu_{(K)}(x)\,T^{\mu_1\nu_1}(x_1)\rangle\,.$$

$$\delta_{KV}\mathscr{S} = 0.$$

$$0 = \int d^dx\left[\left(2x^\kappa x_\nu - x^2\delta_\nu^\kappa\right)\partial_\mu\,\langle T^{\mu\nu}(x)\,T^{\mu_1\nu_1}(x_1)\rangle + 2x^\kappa\,\langle T(x)\,T^{\mu_1\nu_1}(x_1)\rangle\right],$$

(Maglio, Theofilopoulos, C.C. 2021)

$$\left(2d\,x_1^\kappa + 2x_1^\kappa x_1^\mu\frac{\partial}{\partial x_1^\mu} + x_1^2\frac{\partial}{\partial x_{1\kappa}}\right)\langle T^{\mu_1\nu_1}(x_1)\rangle$$
$$+2\left(x_{1\lambda}\,\delta^{\mu_1\kappa} - x_1^{\mu_1}\delta_\lambda^\kappa\right)\langle T^{\lambda\nu_1}(x_1)\rangle + 2\left(x_{1\lambda}\,\delta^{\nu_1\kappa} - x_1^{\nu_1}\delta_\lambda^\kappa\right)\langle T^{\mu_1\lambda}(x_1)\rangle = 0$$

If we integrate out a conformal matter sector at quantum level, the gravitational action is modified only by contributions up to second order in the Riemann tensor

the integration of a conformal sector induces a renormalized effective action

$$\delta_\sigma \mathcal{S} = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left(c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \Box R \right) ,$$

constrained by the Wess-Zumino consistency condition

$$[\delta_{\sigma_1}, \delta_{\sigma_2}]\, \mathcal{S}_R = 0,$$

$$c_1 + c_2 + 3c_3 = 0,$$

$$\delta_\sigma \mathcal{S}_R = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \delta\sigma(x) \left(aE + bC^2 + c\Box R \right) .$$

Gauss-Bonnet

If the Riemann tensor is expanded around flat space up to quadratic order

$$R_{\mu\nu\rho\sigma} = R^{(1)}_{\mu\nu\rho\sigma} + R^{(2)}_{\mu\nu\rho\sigma} + \dots,$$

$$\mathcal{S}_2 = \int d^dx \left((R_{\mu\nu\rho\sigma})^2 + a(R_{\mu\nu})^2 + bR^2 \right),$$

the quadratic fluctuations in the action

$$\mathcal{S}_2^{(2)} = \frac{1}{4} \int d^dx \sqrt{g} \left((a+4) h^{\mu\nu} \Box^2 h_{\mu\nu} + (b-1) h \Box^2 h \right),$$

are affected by a single pole if we take the combinations for a,b anc c corresponding to the GB term

$$E = R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}.$$

$$V_E^{\mu\nu} = 4R_{\mu\alpha\beta\sigma} R_\nu{}^{\alpha\beta\sigma} - 8R_{\mu\alpha\nu\beta} R^{\alpha\beta} - 8R_{\mu\alpha} R_\nu{}^\alpha + 4RR_{\mu\nu} - g_{\mu\nu} E,$$

the equation so motion are fine as far as we stay away from d=4.
The basic issue is if we can "push" down to d=4 this contribution in a notrivial way.

The no go theorem is due to Lovelock

Lovelock's theorem identifies the EH action - with the inclusion of a cosmological constant - as the unique, purely gravitational action, yielding second order equations of motion at d=4

$$E_d = \frac{1}{2^{d/2}} \delta^{\nu_1 \cdots \nu_d}_{\mu_1 \cdots \mu_d} R^{\mu_1 \mu_2}{}_{\nu_1 \nu_2} \cdots R^{\mu_{d-1} \mu_d}{}_{\nu_{d-1} \nu_d} \,,$$

generalized topological contributions enter in higher dimensions

$$\mathcal{L} = -2\Lambda + \sum_{m=1}^{\overline{m}} \frac{1}{2^m} \frac{\alpha_m}{m} \delta^{\mu_1 \nu_1 \mu_2 \nu_2 \ldots \mu_m \nu_m}_{\lambda_1 \sigma_1 \lambda_2 \sigma_2 \ldots \lambda_m \sigma_m} \times R_{\mu_1 \nu_1}{}^{\lambda_1 \sigma_1} R_{\mu_2 \nu_2}{}^{\lambda_2 \sigma_2} \ldots R_{\mu_m \nu_m}{}^{\lambda_m \sigma_m} ,$$

where

$$\delta^{\mu_1 \mu_2 \ldots \mu_p}_{\nu_1 \nu_2 \ldots \nu_p} = \det \begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \cdots & \delta^{\mu_1}_{\nu_p} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \cdots & \delta^{\mu_2}_{\nu_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{\mu_p}_{\nu_1} & \delta^{\mu_p}_{\nu_2} & \cdots & \delta^{\mu_p}_{\nu_p} \end{pmatrix}$$

The d=2 case

In d=2 the Einstein Hilbert action is topological

TOPOLOGICAL TERMS IN THE EARLY UNIVERSE FROM THE CONFORMAL ANOMALY

$$E_d = \frac{1}{2^{d/2}} \delta^{\nu_1 \cdots \nu_d}_{\mu_1 \cdots \mu_d} R^{\mu_1 \mu_2}{}_{\nu_1 \nu_2} \cdots R^{\mu_{d-1} \mu_d}{}_{\nu_{d-1} \nu_d} \ ,$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} + \frac{\alpha}{d-4} \mathcal{H}_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

$$\begin{aligned}
\mathcal{L}_0 &= 1 \\
\mathcal{L}_1 &= R \\
\mathcal{L}_2 &= E_4 \\
\mathcal{L}_3 &= R^3 - 12 R R_{\mu\nu} R^{\mu\nu} + 16 R_{\mu\nu} R^{\mu}{}_{\rho} R^{\nu\rho} + 24 R_{\mu\nu} R_{\rho\sigma} R^{\mu\rho\nu\sigma} + 3 R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
&\quad - 24 R_{\mu\nu} R^{\mu}{}_{\rho\sigma\kappa} R^{\nu\rho\sigma\kappa} + 4 R_{\mu\nu\rho\sigma} R^{\mu\nu\eta\zeta} R^{\rho\sigma}{}_{\eta\zeta} - 8 R_{\mu\rho\nu\sigma} R^{\mu}{}_{\eta}{}^{\nu}{}_{\zeta} R^{\rho\eta\sigma\zeta} \, .
\end{aligned}$$

Pushing down the topological terms (E_2 to d=4)

$$g_{\mu\nu} = \bar{g}_{\mu\nu} e^{2\phi}.$$

In d=2

$$\begin{aligned} S_D^{\mathrm{EH}} &= \hat{\kappa}\left(\int d^D x \sqrt{-\tilde{g}}\tilde{R} - \int d^D x \sqrt{-g} R\right) \\ &= \hat{\kappa}\int d^D x \left[e^{\epsilon\psi/2}\left[(R - (\epsilon+1)\Box\psi) - \frac{1}{4}\epsilon(\epsilon+1)(\partial\psi)^2\right] - R\right], \end{aligned}$$

$$S_2^{\mathrm{EH}} \equiv \lim_{\epsilon\to 0} S_D^{\mathrm{EH}} = \kappa \int d^2 x \sqrt{-g}\left[\psi R - \psi\Box\psi - \frac{1}{2}(\partial\psi)^2\right] = \kappa \int d^2 x \sqrt{-g}\left[\psi R + \frac{1}{2}(\partial\psi)^2\right],$$

The idea is to perform an infinite renormalization of the coupling at d=2, with a prefactor 1/(d-2) combined with a Weyl transformation on the metric The result is a form of diolaton gravity, where we are left with a metric and a dilaton

Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime #1

Dražen Glavan (Louvain U., CP3), Chunshan Lin (Warsaw U.) (May 9, 2019)

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$$S[g_{\mu\nu}] = \int d^D x \sqrt{-g}\left[\frac{M_P^2}{2} R - \Lambda_0 + \frac{\alpha}{D-4}\mathcal{G}\right], \qquad \alpha \to \alpha/(D-4),$$

$$\frac{g_{\mu\nu}}{\sqrt{-g}}\frac{\delta S_{\rm GB}}{\delta g_{\mu\nu}} = (D-4) \times \frac{\alpha}{2}\mathcal{G},$$

$$\frac{g_{\nu\rho}}{\sqrt{-g}}\frac{\delta S_{\rm GB}}{\delta g_{\mu\rho}} = 15\alpha\delta^{\mu}_{[\nu} R^{\rho\sigma}_{\rho\sigma} R^{\alpha\beta}_{\alpha\beta]} = -2R^{\mu\alpha}_{\rho\sigma}R^{\rho\sigma}_{\nu\alpha} + 4R^{\mu\alpha}_{\nu\beta}R^{\beta}_{\alpha} + 4R^{\mu}_{\alpha}R^{\alpha}_{\nu} - 2RR^{\mu}_{\nu} + \frac{1}{2}\mathcal{G}\delta^{\mu}_{\nu},$$

The GB term is topological in d=4, but it can be part of a sequenc eof Lagrangians in higher dimensions.

The idea to bring down the GB term via an infinite renormalization of the coupling was introduced by Glavan and Lin, assuming that the limit was indeed finite. In other word, the vanishing of this term would be removed via a 0/0 redefinition (using the new coupling)

$$\lim_{D\to 4}\left[\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda_0 g_{\mu\nu}\right)+\frac{\alpha}{D-4}\mathcal{H}_{\mu\nu}\right]=0,$$

$$\begin{aligned}\mathcal{H}_{\mu\nu} = 2\Big[& RR_{\mu\nu}-2R_{\mu\alpha\nu\beta}R^{\alpha\beta}+R_{\mu\alpha\beta\sigma}R_{\nu}{}^{\alpha\beta\sigma}-2R_{\mu\alpha}R^{\alpha}_{\nu} \\ & -\tfrac{1}{4}g_{\mu\nu}\left(R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}-4R_{\alpha\beta}R^{\alpha\beta}+R^2\right)\Big]. \quad (1)\end{aligned}$$

$$\frac{\mathcal{H}_{\mu\nu}}{D-4}=2\frac{\mathcal{L}_{\mu\nu}}{D-4}+\frac{2\,(D-3)}{(D-1)\,(D-2)}S_{\mu\nu},$$

However it was noticed that this smooth limit does not exits, one violates conservation of the energy momentum tensor in Einsetein.s equation

In a series of papers, several people realized that in order to regulate the 0/0 term one had to be careful and, as in the case of d=2, one needed to discuss a version of dilaton gravity

(Mann et al)

The result is that Lovelock's theorem is not violated, because an infinite renormalization generates, if treated this way, a new Horndesky theory, rather than a pure gravitational theory.

They also noticed a surprising constraint between the equations of motion of the dilaton and the gravitational equations for the fiducial metric

$$2g_{\mu\nu}\frac{\delta\mathcal{S}_E^{WZ}}{\delta g_{\mu\nu}} - \frac{\delta\mathcal{S}_E^{WZ}}{\delta\phi} = -\sqrt{\bar{g}}\bar{E}.$$

We wanted to understand more on this EGB theory, given that it is basically a subset of an ordinary conformal anomaly action.

So I will go back to the original action S.

$$Z_R(g) = \mathcal{N}\int D\Phi e^{-S_0(g,\Phi)+b'\frac{1}{\varepsilon}V_E(g,d)+b\frac{1}{\varepsilon}V_{C^2}(g,d)},$$

The real issue is: When we regulate a theory with a topological term, how do we perform the regularization? In order to answer this question we need to discuss the structure of the VE and VC counterterms very carefully
(**Maglio, Theofilopoulos, C.C)**

In particular, we can perform a finite renormalization of the theory theory in order to render the theory nonlocal, by removing the dilaton from the spectrum.

So we will turn back to the usual quastion: are anomaly actions local or nonlocal?

The answer, at the end, boils down to the question if the dilaton gravity action is quadratic or quartic.

It can be both. It depends on what type of action we are interested in.

Quadratic dilaton actions describe a certain theory around the largest scale where this theory is almost scaleless (except for the anomaly). We are in the far UV (close to the Planck scale)

Quartic dilaton actions are also possible, but contain a scale (f) and are valid only around that scale.

Consider the loop contributions and go to d=4
We rewrite this part of the effective action in terms of a finite piece plus the poles, whose structure is known

$$V_{C^2}(g,d) \equiv \mu^{\varepsilon} \int d^d x \sqrt{-g}\, C^2,$$
$$V_E(g,d) \equiv \mu^{\varepsilon} \int d^d x \sqrt{-g}\, E,$$

$$\mathcal{S}_B(d) = \mathcal{S}_f(d) - \frac{b}{\epsilon} V_{C^2}(4) - \frac{b'}{\epsilon} V_E(4),$$

We define the renormalised action SR by the addition of the two VE aand VC counterterms, expanding everything around d=4 as

$$\begin{aligned} \mathcal{S}_R(d) &= \left(\mathcal{S}_f(d) - \frac{b}{\epsilon} V_{C^2}(4) - \frac{b'}{\epsilon} V_E(4)\right) + \frac{1}{\varepsilon}\left(b' V_E(4) + \varepsilon b' V_E'(4) + O(\varepsilon^2)\right) \\ &\quad + \frac{1}{\varepsilon}\left(b V_{C^2}(4) + \varepsilon b V_{C^2}'(4) + O(\varepsilon^2)\right), \end{aligned}$$

These expansions are all well defined, though formal. We need to extend the metric from 4 to d spacetime dimensions, and by doing this we are forced to introduce extra degrees of freedom, in the form of a dilaton field.

$$\mathcal{S}_R \equiv \mathcal{S}_R(4) = \mathcal{S}_f(4) + V'_E(4) + V'_{C^2}(4).$$

Rescalings extracting the dilaton

$$\sqrt{g}E = \sqrt{\bar{g}}e^{(d-4)\phi}\left\{\bar{E} + (d-3)\bar{\nabla}_\mu \bar{J}^\mu(\bar{g},\phi) + (d-3)(d-4)\bar{K}(\bar{g},\phi)\right\},$$

where we have defined

$$\bar{J}^\mu(\bar{g},\phi) = 8\bar{R}^{\mu\nu}\bar{\nabla}_\nu\phi - 4\bar{R}\bar{\nabla}^\mu\phi + 4(d-2)(\bar{\nabla}^\mu\phi\bar{\Box}\phi - \bar{\nabla}^\mu\bar{\nabla}^\nu\phi\bar{\nabla}_\nu\phi + \bar{\nabla}^\mu\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi),$$

$$\bar{K}(\bar{g},\phi) = 4\bar{R}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 2\bar{R}\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + 4(d-2)\bar{\Box}\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + (d-1)(d-2)(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2.$$

$$\frac{\delta}{\delta\phi}\int d^dy\sqrt{-g}E(y) = \epsilon\sqrt{g}E(x)$$

$$\begin{aligned}\int d^dx\sqrt{\bar{g}}\frac{\delta}{\delta\phi}\left(\bar{E} + \bar{\nabla}_M\bar{J}^M\right) &= 0,\\ \int d^dx\sqrt{\bar{g}}\left(\phi\frac{\delta}{\delta\phi}\left(\bar{E} + \bar{\nabla}_M\bar{J}^M\right) + \frac{\delta}{\delta\phi}K(\bar{g},\phi)\right) &= O(\epsilon^2).\end{aligned}$$

The Wess-Zumino action versus DR

$$V_E' \equiv \frac{1}{\epsilon}\left(V_E(g,d) - V_E(\bar{g},4)\right) = \frac{\partial}{\partial d} V_E(\bar{g},d)\,|_{d=4} + \int d^dx\sqrt{\bar{g}}\phi\left(\bar{E} + \bar{\nabla}_M \bar{J}^M\right) + \int d^dx\sqrt{\bar{g}}K,$$

$$\mathcal{S}_E^{WZ} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(V_E(\bar{g}e^{2\phi},d) - V_E(\bar{g},d)\Big),$$

$$\mathcal{S}_{C^2}^{WZ} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(V_{C^2}(\bar{g}e^{2\phi},d) - V_{C^2}(\bar{g},d)\Big),$$

Anomaly constraints

$$2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} = \frac{\delta}{\delta\phi},$$

$$\begin{aligned}\mathcal{S}_R \;&=\; \mathcal{S}_f + b'\int d^4x\sqrt{g}\Big[\phi_4 E - (4G^{\mu\nu}(\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi) + 2(\nabla_\lambda\phi\nabla^\lambda\phi)^2 + 4\,\Box\phi\nabla_\lambda\phi\nabla^\lambda\phi)\Big] \\ &\quad + b\int d^dx\sqrt{\bar{g}}\phi\bar{C}^2 + \log(L\mu)\int d^4x\sqrt{\bar{g}}\left(b'\bar{E} + b\bar{C}^2\right),\end{aligned}$$

Quartic local action

Rescalings at d=4 and away (by a finite renormalization)

$$\Delta_4 = \nabla^2 + 2\, R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\, R\Box + \frac{1}{3}\,(\nabla^\mu R)\nabla_\mu\,.$$

$$\sqrt{g}\left(E - \frac{2}{3}\,\Box R\right) \;=\; \sqrt{\bar g}\left(\bar E - \frac{2}{3}\,\bar\Box \bar R + 4\bar\Delta_4\phi\right),$$

Only valid at d=4

$$\sqrt{-g}\,\Delta_4\chi = \quad \bar J(x) \equiv \sqrt{\bar g}\Big(\quad J(x) = \bar J(x) + 4\sqrt{g}\Delta_4\phi(x), \quad E - \frac{2}{3}\Box R\Big)$$

In this case one can remove the dilaton from the spectrum

$$\bar J(x) \equiv \sqrt{\bar g}\left(\bar E - \frac{2}{3}\bar\Box \bar R\right), \qquad J(x) \equiv \sqrt{g}\left(E - \frac{2}{3}\Box R\right)$$

$$J(x) = \bar J(x) + 4\sqrt{g}\Delta_4\phi(x),$$

Indentify the Green function

$$\phi(x) = \frac{1}{4}\int d^4y\, D_4(x,y)(J(y) - \bar{J}(y)).$$

$$\mathcal{S}_{WZ} = \mathcal{S}_{anom}(g) - \mathcal{S}_{anom}(\bar{g}),$$

$$\mathcal{S}_{anom}(g) = \frac{1}{8}\int d^4xd^4yJ(x)D_4(x,y)J(y),$$

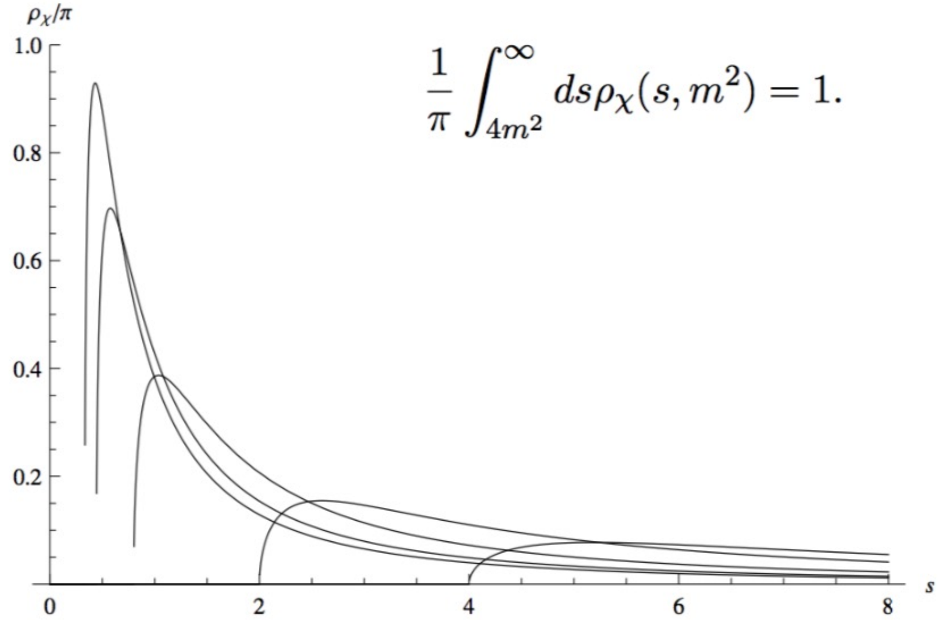
Riegert

$$\mathcal{S}_{\rm anom}(g) = \frac{1}{8}\int d^4x\sqrt{-g_x}\,\left(E - \frac{2}{3}\Box R\right)_x \int d^4x'\sqrt{-g_{x'}}\,D_4(x,x')\left[\frac{b'}{2}\,\left(E - \frac{2}{3}\Box R\right) + b\,C^2\right]_{x'}.$$

$$\mathcal{S}_A \sim \int d^4x\, d^4y R^{(1)}(x) \left(\frac{1}{\square}\right)(x,y) \left(b'\, E_4^{(2)}(y) + b\,(C^2)^{(2)}(y)\right),$$

This behavior is generic for conformal and chiral anomalies

$$\mathcal{S}_A \sim \beta(e) \int d^4x\, d^4y\, R^{(1)}(x) \left(\frac{1}{\square}\right)(x,y) F^{\mu\nu} F_{\mu\nu}(y),$$



$$\frac{1}{\pi}\int_{4m^2}^{\infty} ds\,\rho_\chi(s,m^2) = 1.$$

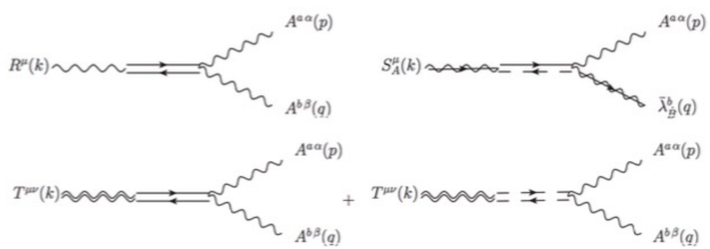
EXACT SUM RULE

The form factor that
carries the chiral
And conformal anomaly
away from the critical point c
shows a branch cut.

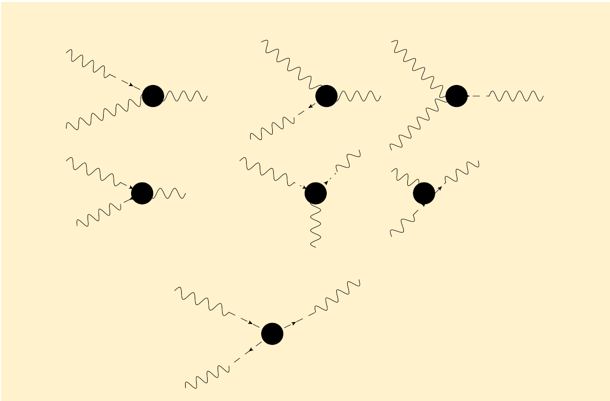
The spectral dentity exhibits a pole
As m→0

$$\lim_{m\to 0}\rho_\chi(s,m^2) = \lim_{m\to 0}\frac{2\pi m^2}{s^2}\log\left(\frac{1+\sqrt{\tau(s,m^2)}}{1-\sqrt{\tau(s,m^2)}}\right)\theta(s-4m^2) = \pi\delta(s)$$

Delle Rose, CC



THESE ANALYSIS
ARE PURELY
PERTURBATIVE.

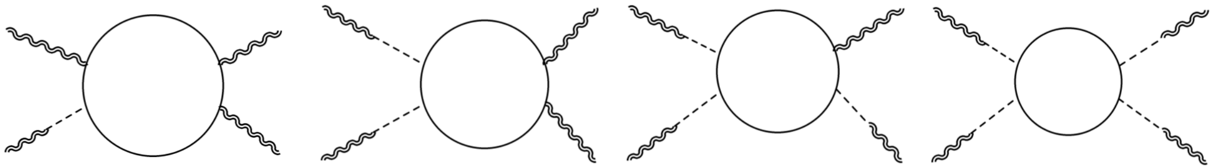


(a)

the anomaly part

$$
\begin{aligned}
\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} &= \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)}{3\,p_1^2}\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} \\
&+ \frac{\hat{\pi}^{\mu_2\nu_2}(p_2)}{3\,p_2^2}\langle T^{\mu_1\nu_1}(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} + \frac{\hat{\pi}^{\mu_3\nu_3}(p_3)}{3\,p_3^2}\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T(p_3)\rangle_{anomaly} \\
&- \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_2\nu_2}(p_2)}{9\,p_1^2p_2^2}\langle T(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)\rangle_{anomaly} - \frac{\hat{\pi}^{\mu_2\nu_2}(p_2)\hat{\pi}^{\mu_3\nu_3}(p_2)}{9p_2^2p_3^2}\langle T^{\mu_1\nu_1}(p_1)T(p_2)T(p_3)\rangle_{anomaly} \\
&- \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_3\nu_3}(\bar{p}_3)}{9p_1^2p_3^2}\langle T(p_1)T^{\mu_2\nu_2}(p_2)T(p_3)\rangle_{anomaly} + \frac{\hat{\pi}^{\mu_1\nu_1}(p_1)\hat{\pi}^{\mu_2\nu_2}(p_2)\hat{\pi}^{\mu_3\nu_3}(\bar{p}_3)}{27p_1^2p_2^2p_3^2}\langle T(p_1)T(p_2)T(\bar{p}_3)\rangle_{anomaly}.
\end{aligned}
$$

The 4rth order anomaly action: organization



$$\mathcal{S}_A = \int d^4x_1 d^4x_2 \langle T\cdot h(x_1) T\cdot h(x_2)\rangle + \int d^4x_1 d^4x_2 d^4x_3 \langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3)\rangle_{pole}$$
$$+ \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left(\langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3) T\cdot h(x_4)\rangle_{pole} + \right.$$
$$\left. + \langle T\cdot h(x_1) T\cdot h(x_2) T\cdot h(x_3) T\cdot h(x_4)\rangle_{0T}\right),$$

Moving away from d=4 in the rescaling: finite renormalization and a new EGB

$$\tilde{V}_E = \int d^d x \sqrt{g} \left(E_4 + \epsilon \frac{R^2}{2(d-1)^2} \right).$$

Perform a finite renormalization on the topological density

$$\delta_\phi(\sqrt{g} E_{ext}) = \delta\phi \epsilon \left(\sqrt{g} E_{ext} - \frac{2}{d-1} \sqrt{g} \Box R \right),$$

$$\delta_\phi \int d^d x \sqrt{g} E_{ext} = \epsilon \sqrt{g} (E_{ext} - \frac{2}{d-1} \Box R).$$

$$
\begin{aligned}
\frac{\delta \mathcal{S}^{(WZ)}_{GB}}{\delta \phi} &= \alpha \sqrt{g}\left(E - \frac{2}{3}\,\Box R\right) \\
&= \alpha \sqrt{\bar{g}}\left(\bar{E} - \frac{2}{3}\,\bar{\Box}\bar{R} + 4\bar{\Delta}_4 \phi\right),
\end{aligned}
$$

$$
\mathcal{S}^{(WZ)}_{GB} = \alpha \int d^4x\,\sqrt{-\bar{g}}\,\left\{\left(\overline{E} - \frac{2}{3}\,\bar{\Box}\,\overline{R}\right)\phi + 2\,\phi\bar{\Delta}_4\phi\right\},
$$

$$
\begin{aligned}
\mathcal{S}^{(WZ)}_{GB} &= \frac{\alpha}{8}\int d^4x\,\sqrt{-g}\,\int d^4x'\,\sqrt{-g'}\,\left(E_4 - \frac{2}{3}\,\Box R\right)_x \\
&\quad \times D_4(x,x')\left(E - \frac{2}{3}\,\Box R\right)_{x'},
\end{aligned}
$$

Localization and order by order expansion

Mazur Mottola

$$\mathcal{S}_{\rm anom}(g,\phi) \equiv -\frac{1}{2}\int d^4x\,\sqrt{-g}\left[(\Box\phi)^2 - 2\big(R^{\mu\nu} - \tfrac{1}{3}Rg^{\mu\nu}\big)(\nabla_\mu\phi)(\nabla_\nu\phi)\right] + \frac{1}{2}\int d^4x\,\sqrt{-g}\,\left[\big(E - \tfrac{2}{3}\Box R\big)\right]\phi,$$

Test with the general TTT:
Maglio, Mottola, CC

that can be varied with respect to ϕ, giving

$$\sqrt{-g}\,\Delta_4\,\phi = \sqrt{-g}\left[\frac{E}{2} - \frac{\Box R}{3}\right].$$

The metric can be expanded perturbatively in the form

$$\begin{aligned} g_{\mu\nu} &= g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} + \cdots \equiv \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \ldots \\ \phi &= \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \ldots \end{aligned}$$

$$\overline{\square}^2\phi^{(0)} = 0$$

$$(\sqrt{-g}\Delta_4)^{(1)}\phi^{(0)} + \overline{\square}^2\phi^{(1)} = \left[\sqrt{-g}\left(\frac{E}{2} - \frac{\square R}{3}\right)\right]^{(1)} = -\frac{1}{3}\,\overline{\square}\,R^{(1)}$$

Expansion around flat space

$$(\sqrt{-g}\Delta_4)^{(2)}\phi^{(0)} + (\sqrt{-g}\Delta_4)^{(1)}\phi^{(1)} + \overline{\square}^2\phi^{(2)} = \left[\sqrt{-g}\left(\frac{E}{2} - \frac{\square R}{3}\right)\right]^{(2)}$$

$$= \frac{1}{2}E^{(2)} - \frac{1}{3}\,[\sqrt{-g}\,\square R]^{(2)},$$

$$\phi^{(1)} = -\frac{1}{3\overline{\square}}\,R^{(1)}$$

The dilaton is removed recursively

$$\phi^{(2)} = \frac{1}{\overline{\square}^2}\left\{(\sqrt{-g}\Delta_4)^{(1)}\frac{1}{3\overline{\square}}\,R^{(1)} + \frac{1}{2}E^{(2)} - \frac{1}{3}\,[\sqrt{-g}\,\square R]^{(2)}\right\}.$$

$$\mathcal{S}^{(3)}_{\rm anom} = \frac{1}{9}\int d^4x\int d^4x'\int d^4x''\left\{(\partial_\mu R^{(1)})_x\left(\frac{1}{\overline{\square}}\right)_{xx'}\left(R^{(1)\mu\nu} - \frac{1}{3}\eta^{\mu\nu}R^{(1)}\right)_{x'}\left(\frac{1}{\overline{\square}}\right)_{x'x''}(\partial_\nu R^{(1)})_{x''}\right\}$$

$$-\frac{1}{6}\int d^4x\int d^4x'\left(E^{(2)}\right)_x\left(\frac{1}{\overline{\square}}\right)_{xx'}R^{(1)}_{x'} + \frac{1}{18}\int d^4x\,R^{(1)}\left(2\,R^{(2)} + (\sqrt{-g})^{(1)}R^{(1)}\right), \qquad (12.36)$$

Concerning the asymptotic structure of $\tilde{\mathcal{S}}_{EGBW_1}$, it is convenient to organize the terms appearing in it as an expansion in the two scales $1/f$ and $1/(f^n M_P^2)$, obtaining

$$
\begin{aligned}
\tilde{\mathcal{S}}_{EGBW_1} \;=\; & \frac{M_P^2}{2}\int d^4x\sqrt{g}\Bigg(R-\frac{\tilde{\phi}}{\bar{f}}R+\frac{1}{2\bar{f}^2}(\partial_\mu\tilde{\phi})^2+2\frac{\tilde{\phi}}{\bar{f}}\Lambda-2\Lambda+O(1/\bar{f}^2) \\
& -\frac{\tilde{\phi}}{\bar{f}M_P^2}(\alpha E+\alpha' C^2)+O(1/(\bar{f}^2M_P^2)\Bigg)\,,
\end{aligned}
\tag{9.7}
$$

where we have redefined $\tilde{\phi}\to\tilde{\phi}/\sqrt{3}$ and $\bar{f}=\sqrt{3}f$.
At large $\bar{f}$, with $f\ll M_P$, the dilaton field can be expressed in terms of the fiducial metric using the nonlocal relation

$$
\tilde{\phi}\sim\frac{1}{\Box}\left(-\bar{f}(R+\Lambda)-\frac{\bar{f}}{M_P^2}(b'E+bC^2)\right),
\tag{9.8}
$$

$$
\begin{aligned}
\tilde{\mathcal{S}}_{EGBW_1} = & \frac{M_P^2}{2}\int d^4x\sqrt{g}\left(R-\frac{1}{f}\tilde{\phi}R+\frac{3}{2}\frac{1}{(1-\frac{\tilde{\phi}}{f})f^2}\partial_\lambda\tilde{\phi}\partial^\lambda\tilde{\phi}-2(1-\frac{\tilde{\phi}}{f})\Lambda\right)+\mathcal{S}_f(4) \\
& +\int d^4x\sqrt{g}\Bigg[\frac{1}{2}\log(1-\frac{\tilde{\phi}}{f})(b'E+bC^2)-b'\Bigg(G^{\mu\nu}\frac{1}{(1-\frac{\tilde{\phi}}{f})^2f^2}(\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi})+\frac{1}{8(1-\frac{\tilde{\phi}}{f})^4f^4}(\partial_\lambda\tilde{\phi}\partial^\lambda\tilde{\phi})^2 \\
& -\frac{1}{2(1-\frac{\tilde{\phi}}{f})^3f^3}\Box_0\tilde{\phi}\partial_\lambda\tilde{\phi}\partial^\lambda\tilde{\phi}-\frac{1}{2(1-\frac{\tilde{\phi}}{f})^4f^4}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}\partial_\nu\tilde{\phi}\partial^\nu\tilde{\phi}+\frac{1}{2(1-\frac{\tilde{\phi}}{f})^2f^3}\Gamma^\lambda\partial_\lambda\tilde{\phi}\partial_\sigma\tilde{\phi}\partial^\sigma\tilde{\phi}\Bigg)\Bigg],
\end{aligned}
$$

$$\tilde{S}_{EGBW_1} = \frac{1}{16\pi G}\int d^4x\sqrt{g}\; e^{-2\phi}\left([R + 6\nabla_\lambda\phi\nabla^\lambda\phi] - 2e^{-2\phi}\Lambda\right) + \mathcal{S}_f(4)$$

$$+\int d^4x\sqrt{g}\Big[-\phi(b'E + bC^2) - b'\left(4G^{\mu\nu}(\nabla_\mu\phi\nabla_\nu\phi) + 2(\nabla_\lambda\phi\nabla^\lambda\phi)^2 - 4\bar{\Box}\phi\nabla_\lambda\phi\nabla^\lambda\phi\right)\Big].$$

$$e^{-2\phi} = 1 - \frac{\tilde{\phi}}{f} \qquad \phi = -\frac{1}{2}\log(1 - \frac{\tilde{\phi}}{f})$$

$$\tilde{S}_{EGBW_1} = \frac{M_P^2}{2}\int d^4x\sqrt{g}\left(R - \frac{1}{f}\tilde{\phi}R + \frac{3}{2}\frac{1}{(1-\frac{\tilde{\phi}}{f})f^2}\partial_\lambda\tilde{\phi}\partial^\lambda\tilde{\phi} - 2(1 - \frac{\tilde{\phi}}{f})\Lambda\right) + \mathcal{S}_f(4)$$

$$+\int d^4x\sqrt{g}\Bigg[\frac{1}{2}\log(1 - \frac{\tilde{\phi}}{f})(b'E + bC^2) - b'\Bigg(G^{\mu\nu}\frac{1}{(1-\frac{\tilde{\phi}}{f})^2 f^2}(\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}) + \frac{1}{8(1-\frac{\tilde{\phi}}{f})^4 f^4}(\partial_\lambda\tilde{\phi}\partial^\lambda\tilde{\phi})^2$$

$$-\frac{1}{2(1-\frac{\tilde{\phi}}{f})^3 f^3}\Box_0\tilde{\phi}\partial_\lambda\tilde{\phi}\partial^\lambda\tilde{\phi} - \frac{1}{2(1-\frac{\tilde{\phi}}{f})^4 f^4}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}\partial_\nu\tilde{\phi}\partial^\nu\tilde{\phi} + \frac{1}{2(1-\frac{\tilde{\phi}}{f})^2 f^3}\Gamma^\lambda\partial_\lambda\tilde{\phi}\partial_\sigma\tilde{\phi}\partial^\sigma\tilde{\phi}\Bigg)\Bigg],$$

in an expansion in ϕ/f. The presence of a bilinear mixing in the EH part of the effective action $(\sim M_P^2/f)\tilde{\phi}R$ is indicating that we are describing a spontaneously broken phase. A solution of the equations of motion can be obtained by setting ϕ constant, and taking a flat fiducial metric $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$,

Concerning the asymptotic structure of $\tilde{S}_{EGBW_1}$, it is convenient to organize the terms appearing in it as an expansion in the two scales $1/f$ and $1/(f^n M_P^2)$, obtaining

$$
\begin{aligned}
\tilde{S}_{EGBW_1} \quad = \quad & \frac{M_P^2}{2}\int d^4x\sqrt{g}\left(R - \frac{\tilde{\phi}}{\bar{f}}R + \frac{1}{2\bar{f}^2}(\partial_\mu\tilde{\phi})^2 + 2\frac{\tilde{\phi}}{\bar{f}}\Lambda - 2\Lambda + O(1/\bar{f}^2)\right. \\
& \left. - \frac{\tilde{\phi}}{\bar{f}M_P^2}(\alpha E + \alpha' C^2) + O(1/(\bar{f}^2 M_P^2)\right),
\end{aligned}
$$

where we have redefined $\tilde{\phi} \to \tilde{\phi}/\sqrt{3}$ and $\bar{f} = \sqrt{3}f$.

At large $\bar{f}$, with $f \ll M_P$, the dilaton field can be expressed in terms of the fiducial metric using the nonlocal relation

$$
\tilde{\phi} \sim \frac{1}{\Box}\left(-\bar{f}(R+\Lambda) - \frac{\bar{f}}{M_P^2}(b'E + bC^2)\right),
$$

The description in terms of effective field theory of topological matter, in the case of toipological materials, establishes a profound link with anomalies (chiral and conformal) about which much still can be said.

Connection to gravity emerges in various ways.

A DISCUSSION IS CONTAINED IN THE RECENT REVIEW
"**Thermal transport, geometry, and anomalies**"
Maxim N. Chernodub, Yago Ferreiros, Adolfo G. Grushin, Karl Landsteiner, María A. H. Vozmediano

The gradient that drives a system out of equilibrium can be compensated by a non-uniform gravitational potential (at linear order) (LUTTINGER FORMULA)

$$\frac{1}{T}\nabla T = -\frac{1}{c^2}\nabla\Phi\,,$$

$$g_{00} = 1 + \frac{2\Phi}{c^2}\,,$$

Open questions:

What is the physical meaning of such massless excitations ?

When does the dilaton should be part of the spectrum?

(work in preparation with Maglio, Creti, Tommasi)

This is particularly relevant in the case in which we are in a Weyl-flat case, and the dilaton is part of the spectrum.

We expect a more complex structure of the CWI's studied so far in the flat spacetime limit