

Some remarks on thermal CFTs and massless Feynman graphs

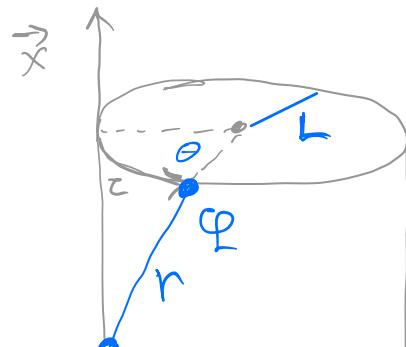
Tours
31/5 - 2/6
2022

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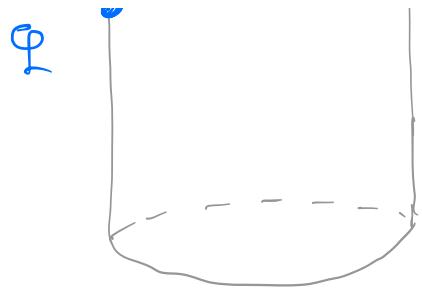


OVERVIEW

Consider thermal / finite-size correlators



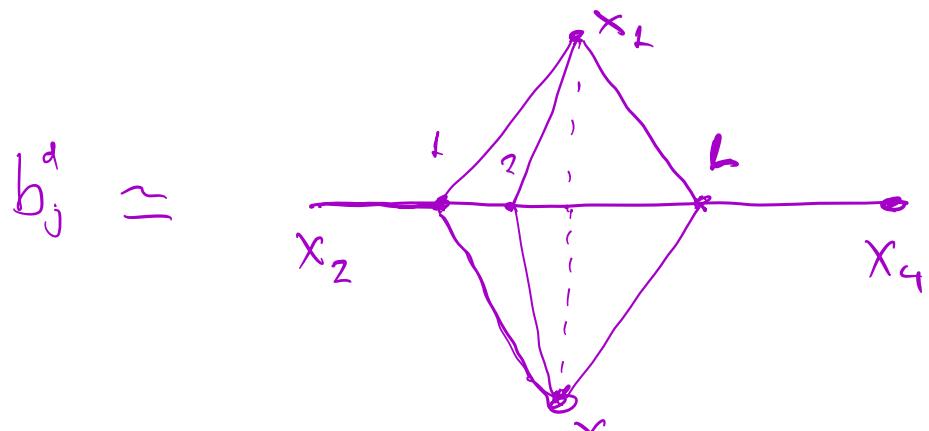
$$\sim \langle \phi(r, \theta) \phi(0, 0) \rangle =$$



$$= \frac{1}{r^{d-2}} + [\text{shadows}] + \frac{1}{L^{d-2}} b_\varphi^d + \frac{r \cos \theta}{L^{d-2}} b_j^d + \dots$$

Thermal Lpt
functions

// EQUAL
Massless Feynman
graphs



$$L = \frac{d-1}{2}$$

$$b_{\varphi^2}^d \approx$$

A Feynman diagram showing a central vertex connected to four external lines. The top line is labeled x_1 , the left line x_2 , the right line x_4 , and the bottom line x_3 . The vertex is enclosed in a dashed circle containing a question mark.

$\xrightarrow{\text{~~~~~}}$ The quantum harmonic oscillator as generator
of massless Feynman graphs.

$\xrightarrow{\text{~~~~~}}$ Keywords (... namedropping...)
modular invariance, single-valued polylogarithms,
partition functions / amplitude correspondence



THE HARMONIC OSCILLATOR AT $T \neq 0$

$$\hat{H} = \frac{\hat{P}^2}{2} + \frac{1}{2} \omega^2 \hat{x}^2$$

$$[\hat{P}, \hat{x}] = -i\hbar$$

$$Z_L = \text{Tr } e^{-\beta \hat{H}} = e^{-1/2 \beta \hbar \omega} \frac{1}{1 - e^{-\beta \hbar \omega}} = e^{-\beta F_L}$$

$$\rightarrow F_L = \frac{1}{\beta} C_L = \frac{1}{\beta} \left[\frac{1}{2} \beta \hbar \omega + k_B (1 - e^{-\beta \hbar \omega}) \right]$$

\downarrow

$$[\hbar = k = 1]$$

Consider $\langle \hat{x}(c=0) \hat{x}(0) \rangle \equiv \langle \hat{x}^2 \rangle$

$$= -2 \frac{2}{\beta} \ln Z_L = - \left[F_L + \frac{e^{-\omega}}{1 - e^{-\beta \hbar \omega}} \right] \Leftrightarrow$$

$$\partial \omega^2 \quad \hookrightarrow \quad 2\omega L \quad 1 - e^{-\omega} \quad \boxed{\boxed{}}$$

$$\Leftrightarrow \boxed{\frac{d}{d\omega} C_L(\omega) = \omega^2 \langle \hat{x}^2 \rangle}$$

$$\Leftrightarrow \boxed{C_L(\omega) = \int_0^\omega \omega' \langle \hat{x}^2(\omega') \rangle d\omega' + C_L(0)}$$

- Thermal partition function of massive free scalar

$$\boxed{Z_d(b; m) = \frac{1}{Z(0; 0)} \int \mathcal{D}\varphi e^{-\int_0^b \int d^dx \frac{1}{2} \varphi (-\partial_\mu^2 - \vec{\partial}^2 + m^2) \varphi}}$$

$$= e^{-b \cdot \frac{V_{d-1}}{b^d} C_d(bm)}$$

$$C_d(0) = \frac{\Gamma(d/2)}{\pi^{d/2}} J(d)$$

$$C_d(Bm) = -\frac{1}{2} K_d(-1)^d (Bm)^d - \frac{S_{d-1}}{(2\pi)^{d-1}} \int_0^{-Bm} \frac{dw}{w} \ln w [w^2 - Bm^2]^{\frac{d-3}{2}} \operatorname{er}(t-w)$$

$$K_d = \frac{\pi S_d}{d(2\pi)^d} \frac{1}{\sin(\frac{\pi d}{2})}, \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Recursive relations

$$\boxed{\frac{d}{dm} C_d(Bm) = -\frac{B^2}{2\pi} m C_{d-2}(Bm)}$$

$$d = 3, 5, 7, \dots, 2k+1 \quad (k=1, 2, \dots)$$

Combine that with:

$$\stackrel{(d=1)}{w \rightarrow m :} \frac{d}{dm} C_+(bm) \equiv C_{-+}(bm) = m \langle \hat{x}^2 \rangle$$

... we get the idea of a resummation ...

$$\begin{aligned} Z &= Z_1 \cdot Z_3 \cdot Z_5 \cdots Z_{2r+1} \cdots && [G=L] \\ &= e^{-C_L - \frac{\gamma_2}{G^2} C_3 - \frac{\gamma_4}{G^4} C_5 \cdots} \\ &= e^{-C(m)} \end{aligned}$$

$$C(m) = C_L + \lambda C_3 + \lambda^2 C_5 + \dots$$

$$\lambda = \frac{\gamma_2}{G^2}$$

It satisfies:

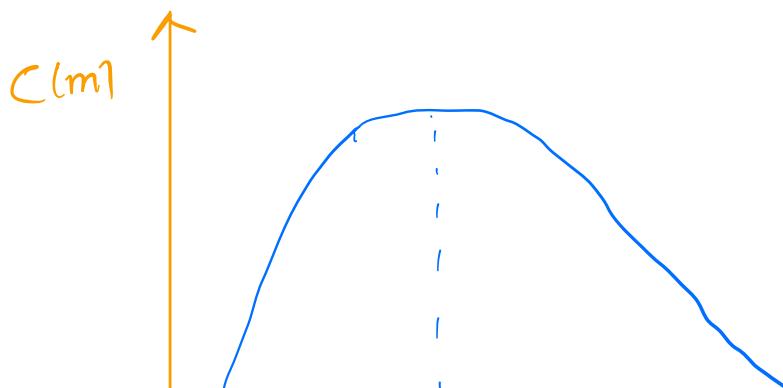
$$\boxed{\frac{d}{dm} C(m) + 2 \frac{6^2}{2\pi} m C(u) = C_{-L}(m)} \quad \left(2 \frac{6^2}{2\pi} = \frac{V_2}{2\pi} = 9 \right)$$

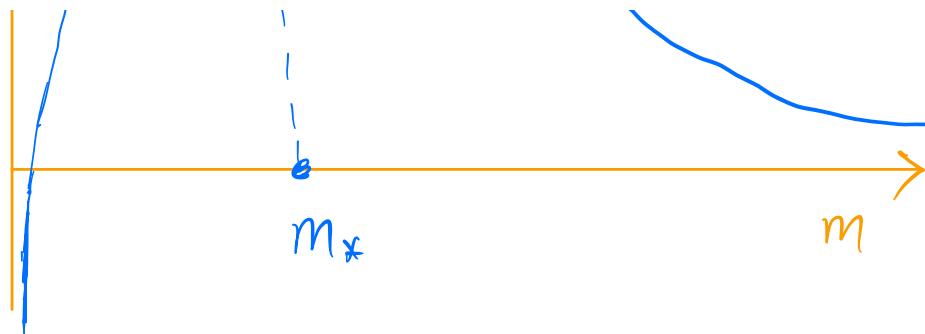
.... it can be solved ...

$$\boxed{C(m) = e^{-\frac{1}{2} g m^2} \int_0^m e^{\frac{1}{2} g m'^2} m' \langle \hat{x}^2 \rangle_{m'} dm'}$$

Remarkably.

$$\exists m_x \Rightarrow \frac{d}{dm} C(u) \Big|_{u_x} = 0$$





- Moral:

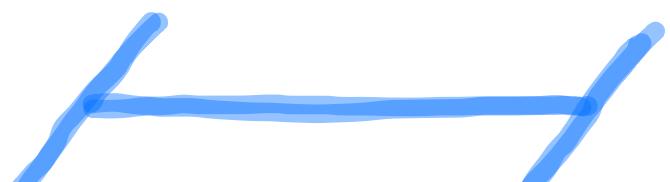
$$C(m) = \sum_{n=0}^{\infty} g^n G_n(m)$$

\$G_n(m)\$

generating functional
for d-dimensional
thermal free energies

Given st terms of
iterated integrals \longleftrightarrow Dyson series

$$G_n \sim \int_{m_{n-1}}^{m_n} \int_{m_{n-2}}^{m_n} \dots \int_{m_0}^{m_*} m_* \langle \hat{x} \rangle dm_0 \dots dm_{n-1}$$



IMAGINARY CHEMICAL POTENTIAL

Canonical vs grand canonical ensembles.

$$Z_c(\beta, Q) = \text{Tr} [\delta(\hat{Q} - Q) e^{-\beta \hat{H}}]$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta Q} \text{Tr} [e^{-\beta(\hat{H} - i\mu \hat{Q})}]$$

$$\mu = \frac{\theta}{\beta}$$

$$Z_{gc}(\beta, i\mu) \rightarrow Z_{gc}(\beta, \mu)$$

- $\hat{Q} \sim \text{global U(1)}$
- If $Q = \text{integer}$ (e.g. QCD)

$$\Rightarrow \theta \rightarrow \theta + \frac{2\pi}{N} \text{ period.} \Rightarrow \mathbb{Z}_N \text{ vacua}$$

• If $\theta \not\rightarrow \theta + \frac{2\pi}{N} \Rightarrow \mathbb{X}_N \sim$ deconfining phase.

\Rightarrow Main example: massive free scalar with imaginary chemical potential μ in the presence of A_0 -component of real Euclidean gauge field.

$$S_E(b; m, \mu) = \int_0^b d\zeta \int d^d \vec{x} [(\partial_\zeta - i\mu)^2 \varphi]^2 + |\vec{\partial} \varphi|^2 + m^2 \varphi^2$$

$$Z_{g.c.} = \frac{1}{Z(0; 0, 0)} \int D\bar{\varphi} D\varphi e^{-S_E} = e^{-b} \frac{\sqrt{d-1}}{b^d} C_d(b\mu, b\mu)$$

$$z = e^{-\ell u - i \ell \eta}, \quad \bar{z} = e^{-\ell u + i \ell \eta}$$

\Rightarrow

$C_d(z, \bar{z})$

$$C_d(z, \bar{z}) = -k_d \ell \epsilon^d |z| - \frac{\sum_{l=1}^{d-1}}{(2\pi)^{d-1}} [i_d(z, \bar{z}) + \bar{i}_d(z, \bar{z})]$$

$$i_d(z, \bar{z}) = \int_0^z \frac{dw}{w} \left(\ell u w - \frac{1}{2} \ell u \frac{z}{\bar{z}} \right) \left[\left(\ell u w - \frac{1}{2} \ell u \frac{z}{\bar{z}} \right)^2 - \ell u |z|^2 \right]^{\frac{d-3}{2}} \ell \epsilon (1-w)$$

$$\bar{i}_d(z, \bar{z}) = i_d(\bar{z}, z)$$

- For $d=1$: $i_1(z) = -\ell \epsilon (1-z), \quad \bar{i}_1(\bar{z}) = -\ell \epsilon (1-\bar{z})$

→ Result for $d = 2k+1$, $k=1, 2, \dots$

$$L_d(z, \bar{z}) + \bar{L}_d(z, \bar{z}) = - \frac{\Gamma(\frac{d+1}{2})}{d-1} I_{d+2}(z, \bar{z})$$

$$I_d(z, \bar{z}) = \sum_{n=0}^{\frac{d-3}{2}} \frac{(1)^n (d-3-n)! z^n}{(\frac{d-3}{2}-n)!} \frac{|z|^n}{n!} [L_{d-2-n}(z) + L_{d-2-n}(\bar{z})]$$

$$L_{i_n}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

→ Thermal 1pt-functions: consider the operators

$$\Phi_{\text{ex}}^2 \equiv |\Phi_{\text{ex}}|^2_{\text{ex}} \quad Q_{\text{ex}} \equiv i \bar{\Phi}_{\text{ex}} \overset{\leftrightarrow}{\partial}_c \Phi_{\text{ex}}$$

with uniform 1pt-functions:

$$\int \langle A(x) \rangle = \ell^d \langle A \rangle_d$$

\Rightarrow thermal Lpt functions are moments of C_d .

$$\frac{\partial}{\partial \mu^2} = \frac{\ell^2}{2\mu_1\mu_1} (z\partial_z + \bar{z}\partial_{\bar{z}}) = \ell^2 \hat{D}$$

$$\frac{\partial}{\partial \mu} = -i\ell (z\partial_z - \bar{z}\partial_{\bar{z}}) = -i\ell \hat{L}$$

\rightarrow we find:

$$\langle \varphi^2 \rangle_d = \frac{1}{\ell^{d-2}} \hat{D} \cdot C_d(z, \bar{z})$$



$$\langle Q \rangle_d = \frac{1}{6^{d-1}} \hat{L} \circ C_d(z, \bar{z})$$

 Harmonic oscillators with imaginary ch. pot.

$$\hat{H} = \sum_{i=1}^2 \frac{\hat{p}_i^2}{2} + \frac{1}{2} m^2 \dot{x}_i^2$$

$$\hat{Q}_i = \hat{a}_i^+ a_i \quad \rightarrow \quad \hat{Q} = \hat{Q}_1 - \hat{Q}_2$$

$$Z_1 = \text{Tr } e^{-\beta(\hat{H}_0 + m^2 \dot{x}^2 - i\mu \hat{Q})}$$

$$\hat{H}_0 = \frac{1}{2} (\hat{p}_1^2 + \hat{p}_2^2)$$

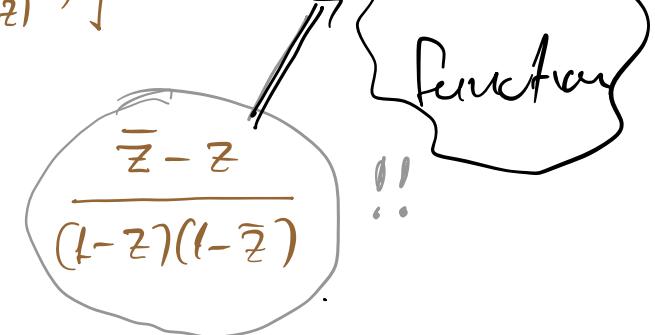
deformations of 

$$\langle \dot{x}^2 \rangle = - \frac{\beta}{[1 + |\beta|^2]} \left(\frac{1}{2} - \frac{1}{\pi} \right)$$

 Kallere

$$z \quad z \bar{z} | z |^2 \quad (z(1-\bar{z}) \quad \bar{z}(1-z)') \downarrow$$

$$\langle \hat{Q} \rangle_L = |z|^2 \left(\frac{1}{z(1-\bar{z})} - \frac{1}{\bar{z}(1-z)} \right) = \frac{\bar{z}-z}{(1-z)(1-\bar{z})}$$

!! { function } 



RESULTS I

→ Recursive relations:

$$|\stackrel{1}{D} \circ C_d(z, \bar{z}) = -\frac{1}{4\pi} C_{d-2}| : d=1, 3, 5 \dots$$

$$C_1 = -\frac{4\pi}{\ell} \langle \vec{x} \rangle_1$$

→ Thermal free energy and fpt functions:

$$\langle \varphi^2 \rangle_{d+2} = - \frac{1}{4\pi} \frac{1}{G^d} C_d(z, \bar{z})$$

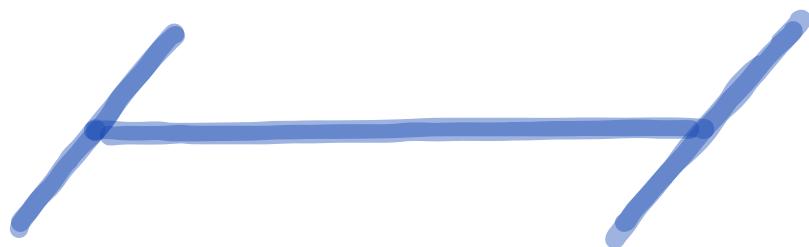
dimension d

- in $d+2$ -dimensions.



$$\langle Q \rangle_d = -4\pi G^2 \hat{D} \cdot \langle Q \rangle_{d+2}$$

recursive relation.



THERMAL FPT FUNCTIONS FROM THE OPE

→ General form:

$$\langle \bar{\varphi}(x) \varphi(0) \rangle = g(r, \cos\theta) = \sum_{Q_s} a_{Q_s} r^{\Delta_{Q_s}} \frac{1}{r^{d-2}} C_s(\cos\theta)$$

($r = d_2 - L$)

$$a_{Q_s} = \frac{s!}{2^s (v)_s} \frac{g_{\varphi\varphi Q_s} b_{Q_s}}{c_{Q_s}}$$

$$L = \ell$$

$$\langle Q_s(x) \rangle = b_{Q_s} (\hat{e}_{t_1} \dots \hat{e}_{t_s} - \text{traces}) \quad \hat{e} \cdot \hat{e} = 1.$$

→ The OPE:

$$g(r, \cos\theta) = \frac{1}{r^{d-2}} + [\text{shadows}] + \frac{g_{\varphi\varphi\varphi}^2}{C_\varphi^2} \langle \varphi^2 \rangle + r \cos\theta \frac{g_{\varphi\varphi}}{C_J} \langle Q \rangle + \dots$$

- When $m=0$ the spectrum consists of higher-spin operators $\Delta_s = d-2+s$; $s=0, 2, 4, \dots$ along with their higher-twist generalizations.
- When $m \neq 0$ there appear "shadow operators," with $\Delta_k = 2k$; $k=1, 2, \dots : G^k$.
- Requiring the vanishing of the r -independent term yields the gap-equation

$$\left| \frac{g_{\varphi\varphi\varphi^2}}{c_{\varphi^2}} \langle \varphi^2 \rangle_2 = \frac{\Gamma(\frac{d-1}{2})}{\Gamma(d-2)} \left(\frac{\Gamma(1-\frac{d}{2})}{2\pi} m^{d-2} + I_d(z, \bar{z}) \right) = 0 \right|$$

- The lot-function of the UCH charge:

$$\frac{g_{QQ3}}{C_3} \langle Q \rangle_d = -\frac{1}{2} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(d-2)} H_d(z, \bar{z})$$

$$H_d(z, \bar{z}) = \sum_{n=0}^{\frac{d-1}{2}} \frac{(-1)^n (d-l-n)! 2^n}{(\frac{d-1}{2}-n)!} \frac{L_l(z)}{n!} [L_{d-l-n}(z) - L_{d-l-n}(\bar{z})]$$

RESULT II

Single-valued polylogarithms.

A 2^{nd} order differential equation:

$$\partial_z \partial_{\bar{z}} C_d(z, \bar{z}) = -\frac{(d-1)}{8\pi} \frac{1}{|z|^2} C_{d-2}(z, \bar{z})$$

→ Setting: $|2L = d-1| \quad n=1, 2, 3, \dots$

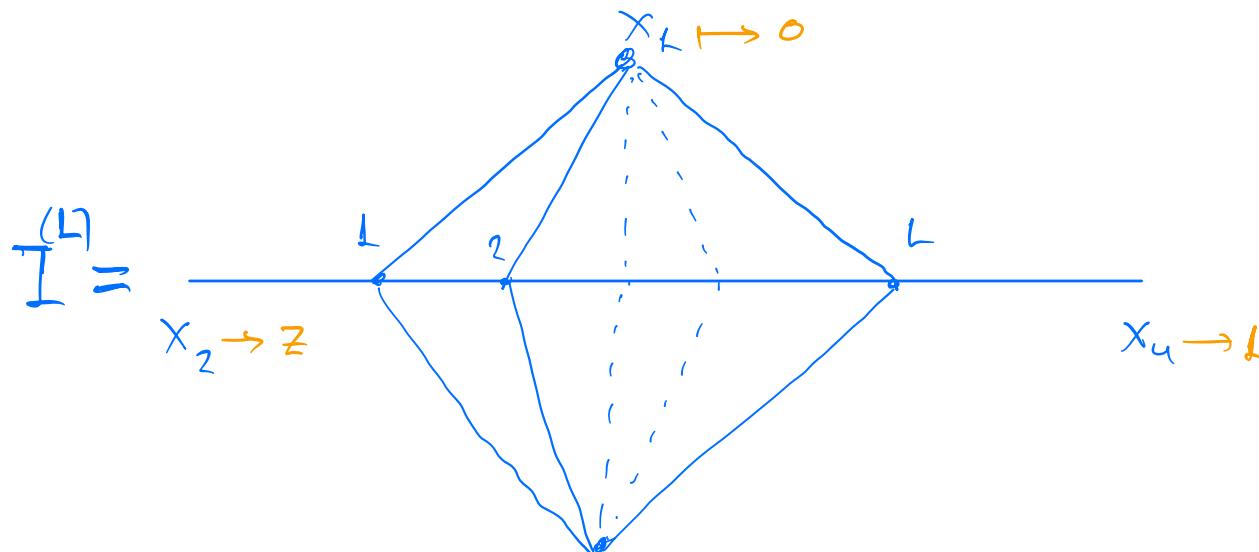
$\partial_z \partial_{\bar{z}} \langle Q \rangle_2 = \frac{1}{4\pi} \frac{z - \bar{z}}{(1-z)(1-\bar{z})} \frac{1}{|z|^2}$

$$\Rightarrow \partial_z \partial_{\bar{z}} \langle Q \rangle_n = - \frac{n}{4\pi} \frac{1}{z^2} \langle Q \rangle_{n+1}$$

$$(n=2, 3, \dots)$$

?? Where did we
see these equations?

! L-loop ladder integrals in $d=4$!



$$\left[\frac{1}{x_1 x_2} = \frac{1}{x_{12}^2} \right]$$

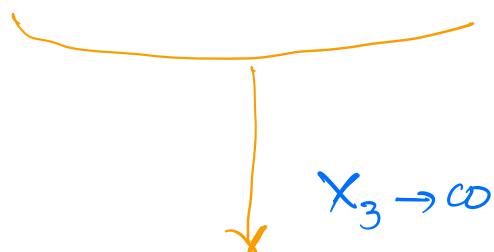
$$x_3 \rightarrow \infty$$

"Magic identities",

$$\prod_{12} \frac{1}{x_{12}^2} = -4\pi^2 \delta(x_{12})$$

→ conformal invariance:

$$x_{3u}^2 I^{(L)}(x_1, x_2, x_3, x_u) = \frac{1}{x_{12}^2 x_{3u}^2} g^{(L)}(v, u)$$



$$v = \frac{x_{12}^2 x_{3u}^2}{x_{1u}^2 x_{23}^2}$$

$$u = \frac{x_{12}^2 x_{3u}^2}{x_{13}^2 x_{2u}^2}$$

$$\boxed{I^{(L)}(x_1, x_2, x_u) = \frac{1}{x_{12}^2} g^{(L)}(v, u)}$$

$$\therefore v = \frac{x_{12}^2}{x_{1u}^2}, u = \frac{x_{12}^2}{x_{13}^2}$$

$$\square_2 I^{(L)}(x_1, x_2, x_3) = -\frac{4}{x_{12}^2} I^{(L-1)}(x_1, x_2, x_4)$$

Comparing this with the action of the Laplace operator

acting on $uv \rightarrow$

$$\rightarrow uv \Delta_2 g^{(L)}(u,v) = -g^{(L-1)}(u,v)$$

$$\Delta_2 = 2(\partial_u + \partial_v) + u\partial_u^2 + v\partial_v^2 - (1-u-v)\partial_u\partial_v$$

\rightarrow change variables as:

$$u = \frac{|z|^2}{(1-z)(1-\bar{z})}, \quad v = \frac{1}{(1-z)(1-\bar{z})}$$

$$\Rightarrow \partial_z \partial_{\bar{z}} \left[\frac{z - \bar{z}}{(1-z)(1-\bar{z})} g^{(L)}(z, \bar{z}) \right] = - \frac{1}{|z|^2} \frac{z - \bar{z}}{(1-z)(1-\bar{z})} g^{(L+1)}(z, \bar{z})$$

up to an overall factor, this is the equation satisfied by $\langle Q \rangle_d$

The solutions of the above equations are the single-valued polylogarithms of F. Brown that satisfy a shuffle algebra.....

(Partial) RESULT (II)

$$I(x_1, \dots, x_4) = \frac{1}{\pi^2} \int \frac{d^4 x}{(x_1 - x)^2 (x_2 - x)^2 (x_3 - x)^2 (x_4 - x)^2} = \frac{1}{x_{14}^2 x_{23}^2} \frac{4i}{z - \bar{z}} D(z)$$

$$D(z) = \text{Im}[L_{12}(z)] + 4|z| \arg(z) \quad \text{Bloch-Wigner.}$$

$$\langle Q \rangle_3 = -\frac{i}{\pi \epsilon^2} D(z)$$

Question: Does $\langle Q_3 \rangle \sim D(z)$ describe a "charge difference"?

Answer: Perhaps...

(Old result):

$$\frac{4i}{z - \bar{z}} D(z) = \lim_{\epsilon \rightarrow 0} T(\epsilon) \left[C_{2-\epsilon}^{(4)}(u, v) - \frac{C_{(4, 2-\epsilon)} v^\epsilon}{\epsilon} C_{2+\epsilon}^{(4)}(u, v) \right]$$

In dimensional analysis

7-dimensional scalar
conformal blocks.

Question: What about higher-loops ??

(Partial) RESULT \textcircled{IV}

$$\stackrel{(4)}{I}(x_1, \dots, x_u) = \frac{1}{\pi^2} \int \frac{d^4x}{(x-x_1)^2 \dots (x-x_u)^2} =$$

$$= \frac{1}{\pi^2} \frac{x_{23}^2}{x_{14}^2}$$

$$= \frac{1}{x_{14}^2} \Phi(v, u)$$

$$\Rightarrow \frac{1}{\pi^2} \int \frac{d^4x}{x^2 (x-z)^2 (x-\hat{e})^2} = \Phi(z^2, (z-e)^2)$$

$$\vee \rightarrow z\bar{z} = r^2$$

$$\frac{v}{u} \rightarrow (z-\hat{e})^2 = (1-z)(1-\bar{z})$$

$$\Rightarrow \frac{1}{\pi^2} \int_0^\infty \frac{x^3 dx d\Omega_3}{x^2 (x^2 + r^2 - 2rx \cos \varphi_1) (1+x^2 - 2x \cos \varphi_2)} \quad (*)$$

Gegenbauer:

$$\frac{1}{1+x^2 - 2x \cos \theta} = \sum_{n=0}^{\infty} x^n C_n^1(\cos \theta)$$

$$(*) \rightarrow \frac{1}{\pi^2} \int_0^\infty x dx \int d\Omega_3 \frac{1}{r^2} \left(\sum_{n=0}^{\infty} x^n C_n^1(\cos \varphi_1) \right) \left(\sum_{m=0}^{\infty} \left(\frac{x}{r}\right)^m C_m^1(\cos \varphi_2) \right)$$

$$= \frac{1}{\pi^2 r^2} \sum_{n,m} \int_0^\infty x^{n+m+1} dx \frac{1}{r^m} \underbrace{\int d\Omega_3 C_n^L(\omega \varphi_1) C_m^L(\omega \varphi_2)}$$

$$\frac{2\pi^2}{n+1} \delta_{nm} \underbrace{C_n^L(\omega \theta)}_{\downarrow}$$

$$C_n^L\left(\frac{z+\bar{z}}{2r}\right) = \frac{\sin(n+1)\theta}{\sin\theta}$$

$$= \frac{1}{r^n} \left(\frac{z^{n+1} - \bar{z}^{n+1}}{z - \bar{z}} \right)$$

$$\sum_{n=0}^{\infty} \frac{1}{r^{n+2}} \int_0^r x^{2n+1} dx \underset{n+1}{=} \frac{1}{r^n} \frac{z^{n+1} - \bar{z}^{n+1}}{z - \bar{z}}$$

$$1 \Gamma \dots 1 \bar{\Gamma} \bar{\gamma}$$

$$= \dots = \frac{1}{z - \bar{z}} [Li_2(z) - Li_2(\bar{z})]$$

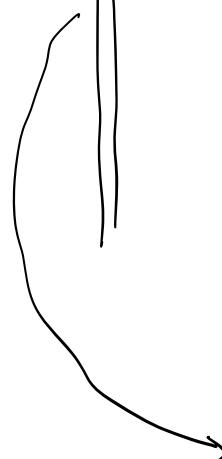


* To get $Li_u(z) + Li_u(\bar{z})$ we need

the Chebyshev polynomials.

$$\frac{1 - t \cos \theta}{1 + t^2 - 2t \cos \theta} = \sum_{n=0}^{\infty} T_n(\cos \theta) t^n$$

$$T_n(\cos \theta) = \cos(n\theta) = \frac{1}{2} (z^n + \bar{z}^n)$$



Another set of Feynman graphs

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