Some recent advances in the theory of moment models

Teddy Pichard

CMAP, école polytechnique

Context

1D kinetic PDE

$$\partial_t f + v\partial_x f = Q(f)$$

satisfying certain properties

- Well-posed with $f \geq 0$ (together with IC and BC)
- Hyperbolic (at fixed v)
- Entropy decay

$$\partial_t \mathcal{H}(f) + \partial_x \mathcal{G}(f) = \mathcal{D}(f) \leq 0,$$

$$\mathcal{H}(f) = \int_v \eta(f), \quad \mathcal{G}(f) = \int_v v\eta(f), \quad \mathcal{D}(f) = \int_v \eta'(f)Q(f)$$

with $\mathcal{D}(f) = 0 \quad \Leftrightarrow \quad f = M$

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Objective: Discretize w.r.t. v such that

- These properties are preserved
- Capture exactly physical regimes (equilibrium, purely anisotropic)

Context

Other (toy) models :

- Radiative transfer $\mu \in [-1, 1]$

$$\frac{1}{|v|}\partial_t f + \mu \partial_x f = L(f),$$

- Spray modelling $S \in \mathbb{R}^+$

$$\partial_t f + v\partial_x f + \partial_S(Kf) = 0,$$

$\hookrightarrow$ see A. Loison et al

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State of the art and new alternatives

Alternatives (non-exhaustive):

- Brute force: numerical cost, no equilibrium
  - Monte-Carlo
  - Discrete velocities
- Moments methods:
  - Euler equations $\rightarrow$ restricted to low order
  - Grad's methods $\rightarrow$ non-hyperbolic, non-positive approximation

  $\hookrightarrow$ regularizations (non-conservative)

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Novelties around

- Quadrature methods: HyQMOM
- Entropy method: φ-divergence
- Realizability method: Projection technique

Principle

Principle: $\partial_t f + v\partial_x f = Q(f)$

1. Choose basis of weight functions

$$\boldsymbol{w}(v) = \boldsymbol{w}_N(v) = (1, v, v^2, ..., v^N)^T$$

2. Integrate the equation against $\boldsymbol{w}(v)$ over v

$$\partial_t \boldsymbol{f} + \partial_x \boldsymbol{F} = \boldsymbol{Q},$$

$$\boldsymbol{f} = \int \boldsymbol{w}(v) f(v) dv, \quad \boldsymbol{F} = \int v \boldsymbol{w}(v) f(v) dv, \quad \boldsymbol{Q} = \int \boldsymbol{w}(v) Q(f)(v) dv$$

$\hookrightarrow$ Work with $\boldsymbol{f}$ instead of f

3. Express $\boldsymbol{F}(\boldsymbol{f})$ and $\boldsymbol{Q}(\boldsymbol{f})$ (**closure**) based on $\boldsymbol{f}$

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3. Express $\boldsymbol{F}(\boldsymbol{f})$ and $\boldsymbol{Q}(\boldsymbol{f})$ (**closure**) based on $\boldsymbol{f}$

Difficulty: Choose a **closure** that

- Preserves property
- Captures regimes

Construction and properties of the closure

Seek

$$\boldsymbol{F} = \int v\boldsymbol{w}(v)f(v)dv = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_N, \boldsymbol{f}_{N+1}) \qquad \text{knowing} \qquad \boldsymbol{f} = \int \boldsymbol{w}(v)f(v)dv = (\boldsymbol{f}_0, \dots, \boldsymbol{f}_N)$$

Common idea:

- Solve the "problem of moments"

$$\text{from} \quad \boldsymbol{f} \in \mathbb{R}^{N+1}, \quad \text{find} \quad f_R \quad \text{s.t.} \quad \boldsymbol{f} = \int \boldsymbol{w} f_R \qquad (1)$$

- Closure: replace f by f_R in $\boldsymbol{F}$

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Problems:

- **Existence of a solution to** (1)?
  $\hookrightarrow$ Under condition $\Rightarrow$ When? $\Rightarrow$ Realizability
- **Uniqueness?**
  $\hookrightarrow$ Very rarely $\Rightarrow$ How to choose f_R? $\Rightarrow$ Closure

Definition and properties

Definition

Realizability domain

$$\mathcal{R}_{\boldsymbol{w}_N} := \left\{ \boldsymbol{f} \in \mathbb{R}^{N+1} \quad \text{s.t.} \quad \exists f_R \in L^1_{\boldsymbol{w}_{N+1}}(\mathbb{R})^+ \quad \boldsymbol{f} = \int \boldsymbol{w}_N f_R \right\}$$

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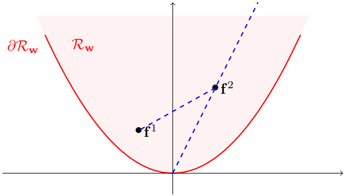
$$= \left\{ \int \boldsymbol{w}_N f, \quad f \in L^1_{\boldsymbol{w}_{N+1}}(\mathbb{R})^+ \right\}$$

Remark:

- $\mathcal{R}_{\boldsymbol{w}_N}$ is a convex cone,
  i.e. stable by positive combinations

  If $\boldsymbol{f}^i \in \mathcal{R}_{\boldsymbol{w}_N}$ and $\alpha_i \geq 0$, then

  $$\sum_i \alpha_i \boldsymbol{f}^i \in \mathcal{R}_{\boldsymbol{w}_N}$$



- $\mathcal{R}_{\boldsymbol{w}_N}$ is open and $\partial\mathcal{R}_{\boldsymbol{w}_N} \not\subset \mathcal{R}_{\boldsymbol{w}_N}$

Hamburger moment problem

Definition: Riesz functional

$$R_{\boldsymbol{f}}\left(\sum_i \alpha_i X^i\right) = \sum_i \alpha_i \boldsymbol{f}_i$$

Proposition ([1])

Even case :	$\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N}}$	$\Leftrightarrow$	$H_N(\boldsymbol{f}) = (f_{i+j})_{0\le i,j\le N}$	*is SPD*
Odd case :	$\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N+1}}$	$\Leftrightarrow$	$H_N(\boldsymbol{f}) = (f_{i+j})_{0\le i,j\le N}$	*is SPD*

Example:

$$H_N(\boldsymbol{f}) = R_{\boldsymbol{f}}(\boldsymbol{w}_N^T \boldsymbol{w}_N) = \begin{pmatrix} \boldsymbol{f}_0 & \boldsymbol{f}_1 & \dots & \boldsymbol{f}_N \\ \boldsymbol{f}_1 & \boldsymbol{f}_2 & \dots & \boldsymbol{f}_{N+1} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{f}_N & \boldsymbol{f}_{N+1} & \dots & \boldsymbol{f}_{2N} \end{pmatrix}$$

$\hookrightarrow \boldsymbol{f}_{2N+1}$ is free

[1] Hamburger (1920), Akhiezer, Krein

Hamburger moment problem

Proposition ([2])

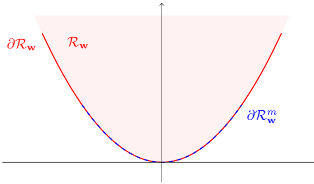
Suppose that $J := rank(H_N) < N$ *and* $H_J(\boldsymbol{f})$ *is SPD*
Then $\exists!$ *representing measure*

$$\gamma = \sum_{i=1}^{J} \alpha_i \delta_{s_i}, \qquad \alpha_i > 0,$$

- Available only on part of $\partial\mathcal{R}_{\boldsymbol{w}_N}$ (call it $\partial\mathcal{R}^m_{\boldsymbol{w}_N}$)
  Example: $\boldsymbol{f} = (0,0,1)^T \in \partial\mathcal{R}_{\boldsymbol{w}_2}$ since

$$H_2(\boldsymbol{f}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{is SPSD},$$

but $J = Rank(H_2(\boldsymbol{f})) = 1$
and $H_1(\boldsymbol{f}) = (0)$ is not SPD



[2] Curto & Fialkow (1991-)

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- Notion of "recursiveness"
  - $\hookrightarrow$ Moments $\boldsymbol{f}_{0,\dots,2J}$ in $H_J(\mathrm{f})$ and $\boldsymbol{f}_{2J+1}$ are independent,
  - $\hookrightarrow$ Other moments $\boldsymbol{f}_{2J+2,\dots,N}$ are functions of $\boldsymbol{f}_{0,\dots,2J+1}$

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Corollary

$\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N+1}}$ *iff* $\exists \boldsymbol{g}_{2N+2}$ *s.t.* $(\boldsymbol{f},\ \boldsymbol{g}_{2N+2}) \in \partial\mathcal{R}^m_{\boldsymbol{w}_{2N+2}}$
or equivalently

$$\begin{pmatrix} \boldsymbol{f}_0 & \cdots & \boldsymbol{f}_N & \boldsymbol{f}_{N+1} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{f}_N & \cdots & \boldsymbol{f}_{2N} & \boldsymbol{f}_{2N+1} \\ \boldsymbol{f}_{N+1} & \cdots & \boldsymbol{f}_{2N+1} & \boldsymbol{g}_{2N+2} \end{pmatrix} \quad \textit{symmetric positive singular}$$

$\hookrightarrow$ extends to multi-variate problems !

Corollary ([3])

$$\mathcal{R}_{\boldsymbol{w}_{2N+1}} = \left\{ \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i), \qquad \alpha_i \in \mathbb{R}^+, \quad v_i \in \mathbb{R} \right\}$$

[3] Curto-Fialkow (1998-)

Other sets of integration

Proposition (Stieltjes)

$$\boldsymbol{f} = \int_{\mathbb{R}^+} \boldsymbol{w}_N f \ \textit{iff}$$

- Even case $N = 2M$: $H_M(\boldsymbol{f})$ *and* $(\boldsymbol{f}_{i+j+1})_{i,j=0,\dots,M-1}$ *are SPD*
- Odd case $N = 2M+1$: $H_{M-1}(\boldsymbol{f})$ *and* $(\boldsymbol{f}_{i+j+1})_{i,j=0,\dots,M}$ *are SPD*

Proposition (Hausdorff)

$$\boldsymbol{f} = \int_{-1}^{+1} \boldsymbol{w}_N f \ \textit{iff}$$

- Even case $N = 2M$: $H_M(\boldsymbol{f})$ *and* $(\boldsymbol{f}_{i+j+2} - \boldsymbol{f}_{i+j})_{i,j=0,\dots,M-1}$ *are SPD*
- Odd case $N = 2M+1$: $(\boldsymbol{f}_{i+j} \pm \boldsymbol{f}_{i+j+1})_{i,j=0,\dots,M}$ *are SPD*

$\hookrightarrow$ **Property:** $\partial\mathcal{R}_{\boldsymbol{w}} = \partial\mathcal{R}^m_{\boldsymbol{w}} = \left\{ \sum_{i=1}^{M-1} \alpha_i \boldsymbol{w}(v_i) \right\}$

QMOM[4]: Principle

Considering $\boldsymbol{f} = (\boldsymbol{f}_0, \dots, \boldsymbol{f}_{2N-1}) \in \mathcal{R}_{\boldsymbol{w}_{2N-1}} \subset \mathbb{R}^{2N}$, then

$$f_R \equiv \sum_{i=1}^{N} \alpha_i \delta_{v_i} \quad \Leftrightarrow \quad \boldsymbol{f} = \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i)$$

How to compute α_i and v_i?

$$\boldsymbol{f} = \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i)$$

[4] McGraw (1997), R. Fox, F. Laurent, D. Marchisio, M. Massot, C. Chalon

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How to compute α_i and v_i?

1 Construct **orthogonal** polynomials w.r.t. $\boldsymbol{f}$ (any representing measure), i.e.

$$R_{\boldsymbol{f}}(P_i P_j) = 0 \quad \forall i \neq j$$

Recursively: $P_{-1} = 0$, $P_0 = 1$ and

$$P_{k+1} = (X - a_k(\boldsymbol{f}))\, P_k + b_k(\boldsymbol{f}) P_{k-1},$$

$$a_k(\boldsymbol{f}) = \frac{R_{\boldsymbol{f}}(X P_k^2)}{R_{\boldsymbol{f}}(P_k^2)}, \qquad b_k(\boldsymbol{f}) = \frac{R_{\boldsymbol{f}}(P_k^2)}{R_{\boldsymbol{f}}(P_{k-1}^2)}$$

for $k \leq N-1$

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$$P_{k+1} = (X - a_k(\boldsymbol{f}))\, P_k + b_k(\boldsymbol{f}) P_{k-1},$$

2. Seek $\boldsymbol{f}_{2N}$ s.t. $(\boldsymbol{f}, \boldsymbol{f}_{2N}) \in \partial \mathcal{R}^m_{\boldsymbol{w}_{2N}} \quad \Leftrightarrow \quad R_{(\boldsymbol{f}, \boldsymbol{f}_{2N})}(P_N^2) = 0$

$$\{v_i, \quad i = 1, \ldots, N\} = Z(P_N)$$

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3. Masses α_i solve a Van der Monde system

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Properties

QMOM closure

$$f_R \equiv \sum_{i=1}^{N} \alpha_i \delta_{v_i}, \qquad \boldsymbol{f} = \sum_{i=1}^{N} \alpha_i \boldsymbol{w}(v_i), \qquad \boldsymbol{F} = \sum_{i=1}^{N} \alpha_i v_i \boldsymbol{w}(v_i)$$

Properties:

- **Positivity**: if $\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}}$ then $\alpha_i > 0$
- Capture Diracs, no equilibrium
- **Weak hyperbolicity**:

$$J_{\boldsymbol{f}} \boldsymbol{F} = J_{(\boldsymbol{\alpha},\boldsymbol{v})} \boldsymbol{F} (J_{(\boldsymbol{\alpha},\boldsymbol{v})} \boldsymbol{f})^{-1} = P Diag(J_1, \ldots, J_N) P^{-1}, \quad J_i = \begin{pmatrix} v_i & \alpha_i \\ 0 & v_i \end{pmatrix}$$

- **Entropy**? Not related to kinetic

Appearance of δ-shocks

$$\partial_t u + \partial_x u = 0, \qquad \partial_t v + \partial_x v + \partial_x u = 0$$

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Extensions

- Algorithm: DQMOM
- Multi-variate $v \in \mathbb{R}^3$: CQMOM
- Multi-Gaussian a.k.a. EQMOM
- Strongly hyperbolic: HyQMOM (and multi-variate CHyQMOM)

HyQMOM[5]

Considering $\boldsymbol{f} = (\boldsymbol{f}_0, \ldots, \boldsymbol{f}_{2N-1}) \in \mathcal{R}_{\mathrm{w}_{2N-1}} \subset \mathbb{R}^{2N}$, then

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- Bijection $(a_0, b_0, \ldots, a_{N-1}, b_{N-1}) \leftrightarrow \boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N-1}}$

QMOM : $\chi_{J_f \boldsymbol{F}} = P_N^2$,

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- Construct a last one

$$\tilde{P}_{N+1} = (X - \tilde{a}_N)P_k + \tilde{b}_N P_{N-1}, \quad \tilde{a}_N = \frac{1}{N}\sum_{k=0}^{N-1} a_k, \qquad \tilde{b}_N = \frac{2N+1}{N} b_N$$

- Bijection $(a_0, b_0, \dots, a_{N-1}, b_{N-1}, \tilde{a}_N, \tilde{b}_N) \leftrightarrow \boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_{2N}}$

QMOM : $\chi_{J_{\boldsymbol{f}}\boldsymbol{F}} = P_N^2$, HyQMOM : $\chi_{J_{\boldsymbol{f}}\boldsymbol{F}} = P_N \tilde{P}_{N+1}$

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Minimum entropy

Consider $\partial_t f + v\partial_x f = Q(f)$ such that

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$$\mathcal{H}(f) = \int_v \eta(f), \quad \mathcal{G}(f) = \int_v v\eta(f), \quad \mathcal{D}(f) = \int_v \eta'(f)Q(f)$$

Proposition ([6])

Choice: $f_R = \underset{\int \boldsymbol{w} f = \boldsymbol{f}}{argmin}\ \mathcal{H}(f)$,

Boltzmann/Shanon entropy: $\eta(f) = f \log f$, then $(\eta^*)' = \exp$

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- $\partial_t \left(\int \boldsymbol{w} f_R\right) + \partial_x \left(\int v\boldsymbol{w} f_R\right)$ *is symmetric hyperbolic*[7]
  $A\partial_t \boldsymbol{\lambda} + B\partial_x \boldsymbol{\lambda}$

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- $\partial_t \left(\int \boldsymbol{w} f_R\right) + \partial_x \left(\int v \boldsymbol{w} f_R\right)$ *is symmetric hyperbolic*[7]
- $\int \boldsymbol{\lambda}^T \boldsymbol{w} Q(f_R) = \int \eta'(f_R) Q(f_R) = \mathcal{D}(f_R) \leq 0$

Boltzmann/Shanon entropy: $\eta(f) = f \log f$, then $(\eta^*)' = \exp$

[6] Levermore (1996), Junk, Schneider, Borwein & Lewis, Mead & Papanicolaou

[7] Lax & Friedrichs (1971), S. Kawashima, Y. Shizuta, W.-A. Yong

Issues

Two difficulties:

- Computational costs
- Undefined for some $\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}}$:

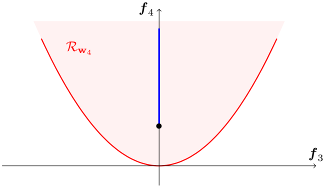
Proposition ([8])

$$\exists \boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}} \quad s.t \quad \nexists \boldsymbol{\lambda} \text{ satisfying } \int_{\mathbb{R}} \boldsymbol{w}(v)(\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}(v))dv = \boldsymbol{f}.$$

$\hookrightarrow$ lack of integrability

$$\exp\left(\sum_{i=0}^{3} \boldsymbol{\lambda}_i v^i + \lambda_4 v^4\right)$$

when $\lambda_4 \to 0^-$



[8] Junk (1998), Schneider, McDonald, Groth

Alternative: φ-divergence[9]

Idea (via minimization techniques): Case $\eta(f) = f \log f$

- Use relative entropy[10]

$$D(f, M) = \int M\eta\left(\frac{f}{M}\right) = \int f \log\left(\frac{f}{M}\right)$$

- Replace η by

$$\eta_N(f) = f\left(\frac{N^2}{N+1} f^{1/N} - N\right) + \frac{N}{N+1} \approx f \log f$$

s.t. $D_N(f, M) = \int M\eta_N(\frac{f}{M})$ and obtain

$$f_R^N = \underset{\int \boldsymbol{w} f = \boldsymbol{f}}{\text{argmin}}\ D_N(f, M) = M\left(1 + \frac{\boldsymbol{\lambda}^T \boldsymbol{w}}{N}\right)_+^N \approx \exp(\boldsymbol{\lambda}^T \boldsymbol{w})$$

[9] M. Abdelmalik (2016-), H. Van Brummelen
[10] Kullback & Leibler (1951)

Alternative: φ-divergence[9]

Entropy inequality

$$\partial_t \int \eta(f) + \partial_x \int v\eta(f) = \int \eta'(f)Q(f) \leq 0$$

Reconstruction

$$f_R^N = M\left(1 + \frac{\boldsymbol{\lambda}^T \boldsymbol{w}}{N}\right)_+^{N+1}$$

Properties:

- $\partial_t \left(\int \boldsymbol{w} f_R\right) + \partial_x \left(\int v \boldsymbol{w} f_R\right)$ is symmetric hyperbolic
- Construction of Q_N s.t.

$$\int \boldsymbol{\lambda}^T \boldsymbol{w} Q_N(f_R^N) = \int \eta_N'(f_R^N) Q_N(f_R^N) \leq 0$$

- Still need to compute $\boldsymbol{\lambda}$

[9] M. Abdelmalik (2016-), H. Van Brummelen

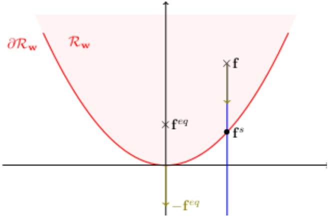
General idea[11]

Receipe:

1. Choose an equilibrium $f^{eq} \in L^1_{\boldsymbol{w}_{N+1}}(\mathbb{R})^+$
2. Project $\boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}_N}$ toward $\partial\mathcal{R}_{\boldsymbol{w}_N}$ along $-\boldsymbol{f}^{eq}$

$$\boldsymbol{f} = \alpha_0 \boldsymbol{f}^{eq} + \boldsymbol{f}^s, \qquad \boldsymbol{f}^s \in \partial\mathcal{R}_{\boldsymbol{w}_N}$$

3. Reconstruct f_R from f^{eq} and $\boldsymbol{f}^s$



$\hookrightarrow$ also for high order schemes (see K. Ait-Ameur et al)

[11] T.P. (2020-)

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Problem: Choose $f^{eq}(\boldsymbol{f})$ such that

- Maxwellians $f^{eq} \equiv M(\rho, u, T) \equiv (\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w})$
- $\boldsymbol{f}^s \in \partial\mathcal{R}^m_{\boldsymbol{w}_N} \subset \partial\mathcal{R}_{\boldsymbol{w}_N}$ to reconstruct $\boldsymbol{f}^s = \sum_i \alpha_i \boldsymbol{w}(v_i)$

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Remark: In radiative transfer,

- $f^{eq} \equiv 1$ (independent of $\boldsymbol{f}$)
- and $\partial\mathcal{R}^m_{\boldsymbol{w}_N} = \partial\mathcal{R}_{\boldsymbol{w}_N}$

Precomputed $f^{eq}(t, x)$

- f^{eq} independent of $\boldsymbol{f}$
- but $f^s \notin \partial\mathcal{R}^m_{\boldsymbol{w}_N}$ a priori

[11] T.P. (2020-)

A parametrization of $\mathcal{R}_{\boldsymbol{w}}$

Suppose that $f^{eq}(\boldsymbol{\lambda}) \equiv (\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}_J(v))$ and $2M = N - (J+1)$, seek

$$\boldsymbol{f} = \boldsymbol{f}^{eq}(\boldsymbol{\lambda}) + \sum_{i=1}^{M} \alpha_i \boldsymbol{w}(v_i)$$

Is $\left\{ \begin{array}{rll} & \Lambda \times (\mathbb{R}^+ \times \mathbb{R})^M & \to \mathcal{R}_{\boldsymbol{w}} \\ \boldsymbol{v} = & (\boldsymbol{\lambda}, \alpha_1, v_1, \dots, \alpha_K, v_K) & \mapsto \boldsymbol{f} \end{array} \right.$ a bijection ?

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No: *det* $J_{\boldsymbol{v}} \boldsymbol{f} = 0$ when $\alpha_K = 0$ (and $\boldsymbol{f}(\alpha_K = 0) \in \mathcal{R}_{\boldsymbol{w}}$)

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$\hookrightarrow$ **Open questions**:

- Work in $\boldsymbol{f}\left(\Lambda \times (\mathbb{R}^+ \times \mathbb{R})^M\right) \subsetneq \mathcal{R}_{\boldsymbol{w}}$
  $\hookrightarrow$ $\alpha_K < 0$ attained?
- Complete with $\alpha_K < 0$
  $\hookrightarrow$ Existence $\forall \boldsymbol{f} \in \mathcal{R}_{\boldsymbol{w}}$, but lose uniqueness

Hyperbolicity

Suppose that $f^{eq}(\boldsymbol{\lambda}) \equiv (\eta^*)'(\boldsymbol{\lambda}^T \boldsymbol{w}_J(v))$ and $2M = N - (J+1)$, seek

$$\boldsymbol{f} = \boldsymbol{f}^{eq}(\boldsymbol{\lambda}) + \sum_{i=1}^{M} \alpha_i \boldsymbol{w}(v_i)$$

Proposition

Weak hyperbolicity with wave speeds v_i (multiplicity 2) and $Sp(J)$ with

$$J = \left(\int v \prod_i (v - v_i)^2 \boldsymbol{w}_J \boldsymbol{w}_J^T (\eta^*)'' \left(\boldsymbol{\lambda}^T \boldsymbol{w}_J \right) \right) \left(\int \prod_i (v - v_i)^2 \boldsymbol{w}_J \boldsymbol{w}_J^T (\eta^*)'' \left(\boldsymbol{\lambda}^T \boldsymbol{w}_J \right) \right)^{-1}$$

Properties and on-going work

Properties:

- Positivity

$$\boldsymbol{f} = \int \boldsymbol{w}(v)\left(\alpha_0 f^{eq}(v)dv + \sum_i \alpha_i \delta_{v_i}\right)$$

- Captures f^{eq} and δ_v
- Weak hyperbolicity
- Entropy decay: on-going work

On-going work:

- Choice of closure $\boldsymbol{Q}(\boldsymbol{f}) \to$ Entropy decay
- Computation (numerical) of $\boldsymbol{\lambda}, (\alpha_i, v_i)_{i=1,\dots,J}$

In a nutshell

- **Quadrature**:
  - Weakly hyperbolic $\rightarrow$ strongly
  - Positive reconstruction
  - Capture Diracs, not equilibria
  - Entropy $\rightarrow$ not kinetic
  - **Algorithm !**
- **Entropy**:
  - **Symmetric hyperbolic and dissipative**
  - Choice for a positive reconstruction
  - Capture equilibria and Diracs (on the boundary)
- **Projection**:
  - Weakly hyperbolic
  - Positive reconstruction
  - **Capture equilibria and Diracs**