Approximation theory of tree tensor networks

Anthony Nouy

Centrale Nantes, Nantes Université,
Laboratoire de Mathématiques Jean Leray

Tensor networks

Tensor networks are prominent tools for the representation or approximation of multivariate functions or multidimensional arrays.

- A long history in quantum physics. A zoo of tools exploiting separation of variables (MPS, PEPS, MERA...)
- Tree tensor networks (Hierarchical Tucker tensors) appeared independently in numerical analysis and numerical linear algebra, as an extension of low-rank decompositions to high-order tensors [Hackbusch and Kuhn, Grasedyck, Oseledets and Tyrtyshnikov].
- Growing use in statistics, data science and probabilistic modelling.

Outline

1. Approximation tools based on tree tensor networks
2. Universality, Proximinality and Expressivity
3. Choice of tensor format
4. Approximation classes of tree tensor networks

Tensor product of functions

Let $V_\nu \subset \mathbb{R}^{\mathcal{X}_\nu}$ be a space of functions defined on $\mathcal{X}_\nu$.

$\mathcal{X}_\nu$ can be (a subset of) $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, or a set of vectors, sequences, graphs, images...

The tensor product of functions $v^{(\nu)} \in V_\nu$, denoted

$$v = v^{(1)} \otimes \ldots \otimes v^{(d)},$$

is a multivariate function defined on $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_d$ and such that

$$v(x_1, \ldots, x_d) = v^{(1)}(x_1) \ldots v^{(d)}(x_d)$$

Tensor product of functions

The algebraic tensor product of spaces V_ν is defined as

$$V_1 \otimes \ldots \otimes V_d = \text{span}\{v^{(1)} \otimes \ldots \otimes v^{(d)} : v^{(\nu)} \in V_\nu, 1 \leq \nu \leq d\}$$

which is the space of multivariate functions v which can be written as a finite linear combination of elementary (separated functions), i.e.

$$v(x_1, \ldots, x_d) = \sum_{k=1}^{r} v_k^{(1)}(x_1) \ldots v_k^{(d)}(x_d).$$

Rank of multivariate functions and canonical format

The canonical rank of a multivariate function $f(x_1, \dots, x_d)$ is the minimal integer such that f has a representation

$$f(x) = \sum_{k=1}^{r} v_k^1(x_1) \dots v_k^d(x_d)$$

Given a finite-dimensional tensor space $V = V_1 \otimes \dots \otimes V_d$ of multivariate functions we define a canonical tensor format in V as a set of functions

$$\mathcal{R}_r(V) = \{f \in V : \text{rank}(f) \leq r\}$$

From the practical point of view, it is not a nice format. In particular, $\mathcal{R}_r(V)$ is not closed for $d \geq 3$ and $r \geq 2$.

For any continuous parametrization $\mathcal{R}_r(V) = \{v = R(p) : p \in P\}$, and for any tensor of $v \in \overline{\mathcal{R}_r(V)} \setminus \mathcal{R}_r(V)$ of border rank r, the quantity

$$\delta(v, \epsilon) = \inf\{\|p\| : \|v - R(p)\| < \epsilon\}$$

diverges as $\epsilon \to 0$ [Hackbusch 2021].

α-ranks of multivariate functions

A multivariate function $f(x_1, \ldots, x_d)$, for any set $\alpha \subset \{1, \ldots, d\}$, can be identified with a bivariate function $f(x_\alpha, x_{\alpha^c})$ of two complementary subsets of variables.

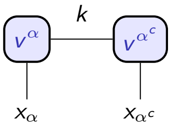
The rank of the bivariate function $f(x_\alpha, x_{\alpha^c})$ is the α-rank of f, denoted $\text{rank}_\alpha(f)$.

A function with α-rank bounded by r_α admits a representation

$$f(x) = \sum_{k=1}^{r_\alpha} v_k^\alpha(x_\alpha) v_k^{\alpha^c}(x_{\alpha^c})$$

or using tensor diagram notations

$f(x) =$



where a connection between two tensors represents a contraction along one mode of each tensor.

α-rank

Example

- $u(x) = u^1(x_1)\dots u^d(x_d)$ can be written $u(x) = u^\alpha(x_\alpha)u^{\alpha^c}(x_{\alpha^c})$, with $u^\alpha(x_\alpha) = \prod_{\nu\in\alpha} u^\nu(x_\nu)$. Therefore, for any α, $\text{rank}_\alpha(u) = 1$.

- $u(x) = \sum_{k=1}^r u_k^1(x_1)\dots u_k^d(x_d)$ can be written $\sum_{k=1}^r u_k^\alpha(x_\alpha)u_k^{\alpha^c}(x_{\alpha^c})$ with $u_k^\alpha(x_\alpha) = \prod_{\nu\in\alpha} u_k^\nu(x_\nu)$. Therefore, for any α, $\text{rank}_\alpha(u) \le r$, with equality if the functions $\{u_k^\alpha(x_\alpha)\}$ and the functions $\{u_k^{\alpha^c}(x_{\alpha^c})\}$ are linearity independent.

  We deduce the following relation between α-ranks and canonical rank:

  $$\text{rank}_\alpha(u) \le \text{rank}(u), \quad \text{for any } \alpha.$$

- $u(x) = u^1(x_1) + \dots + u^d(x_d)$ can be written $u(x) = u^\alpha(x_\alpha) + u^{\alpha^c}(x_{\alpha^c})$, with $u^\alpha(x_\alpha) = \sum_{\nu\in\alpha} u^\nu(x_\nu)$. Therefore, $\text{rank}_\alpha(u) \le 2$.

Low-rank tensor format

Given

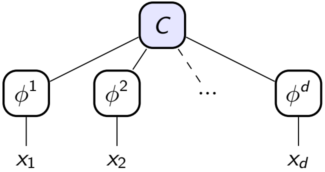
- a finite-dimensional tensor space $V = V_1 \otimes \ldots \otimes V_d$ of multivariate functions
- a collection T of subsets in $\{1, \ldots, d\}$,
- a tuple of ranks $r = (r_\alpha)_{\alpha \in T}$,

we define a low-rank tensor format in V as a set of functions

$$\mathcal{T}_r^T(V) = \{f \in V : \text{rank}_\alpha(f) \leq r_\alpha, \alpha \in T\}$$

with representation

$$f(x) = \sum_{i_1 \in I_1} \ldots \sum_{i_d \in I_d} C(i_1, \ldots, i_d) \phi^1(x_1)_{i_1} \ldots \phi^d(x_d)_{i_d} =$$



where ϕ^ν is a feature map associated with V^ν and $C \in \mathbb{R}^{I_1 \times \ldots \times I_d}$ is a rank-structured algebraic tensor.

Tensor train format

With

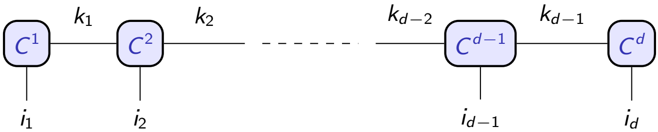
$$T = \{\{1\}, \{1,2\}, ..., \{1, \dots, d\}\},$$

$\mathcal{T}_r^T(V)$ coincides with the tensor train format.

A function f in $\mathcal{T}_r^T(V)$ has coefficients

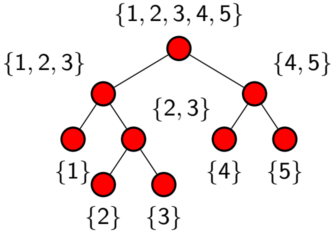
$$C(i_1, \dots, i_d) = \sum_{k_1=1}^{r_1} \dots \sum_{k_{d-1}=1}^{r_{d-1}} C^1(i_1, k_1) C^2(k_1, i_2, k_2) \dots C^d(k_{d-1}, i_d).$$

$=$

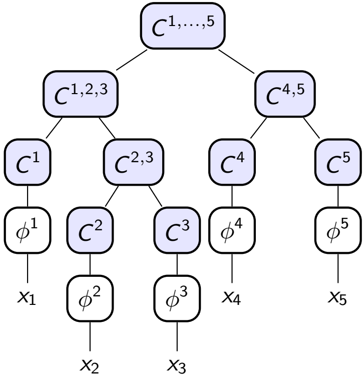


Hierarchical Tucker format (Tree tensor networks)

If T is a dimension partition tree, $\mathcal{T}_r^T(V)$ is a tree-based (or hierarchical) tensor format and a function in $\mathcal{T}_r^T(V)$ admits a multilinear parametrization with a collection of parameters $\{C^\alpha : \alpha \in T\}$ forming a tree tensor network.



Dimension tree T



Tree tensor network

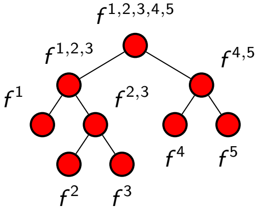
Tree tensor networks as a compositional function network

By identifying a tensor $C^{(\alpha)} \in \mathbb{R}^{n_1 \times \dots \times n_s \times r_\alpha}$ with a $\mathbb{R}^{r_\alpha}$-valued multilinear function

$$f^{(\alpha)} : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_s} \to \mathbb{R}^{r_\alpha},$$

a function v in $\mathcal{T}_r^T(V)$ admits a representation as a tree-structured composition of multilinear functions $\{f^{(\alpha)}\}_{\alpha \in T}$, e.g.

$$v(x) = f^D(f^{1,2,3}(f^1(\phi^1(x_1)), f^{2,3}(f^2(\phi^2(x_2)), f^3(\phi^3(x_3))), f^{4,5}(f^4(\phi^4(x_4)), f^5(\phi^5(x_5))))$$



Tree tensor networks as a compositional function network

A multilinear map f^α can also be written

$$f^\alpha(z_1, \dots, z_s) = A^\alpha \sigma(z_1, \dots, z_d), \quad z_k \in \mathbb{R}^{n_k},$$

with a matrix

$$A^\alpha \in \mathbb{R}^{r_\alpha \times N}, \quad N = n_1 \dots n_s$$

and a fixed multilinear function

$$\sigma(z_1, \dots, z_s) = vec(z_1 \otimes \dots \otimes z_s) \in \mathbb{R}^N$$

Tree tensor networks as feed-forward neural networks

It corresponds to a sum-product feed forward neural network with a sparse architecture (given by T), a number of hidden layers equal to $depth(T) + 1$ (including a featuring layer), and width at level ℓ related to the α-ranks of the nodes α of level ℓ.

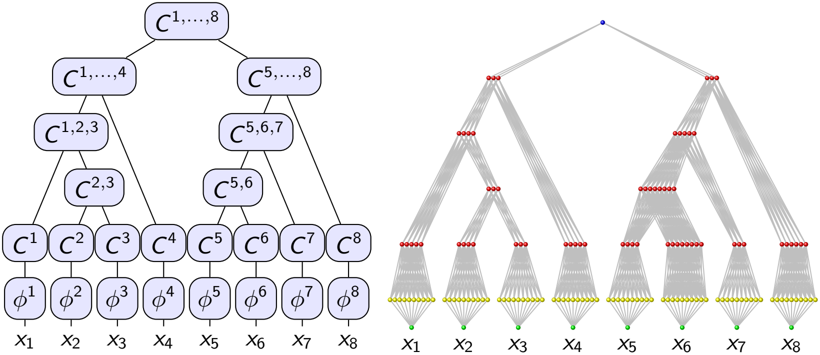


Figure: Tree tensor network and corresponding feed-forward sum-product neural network with 10 features per variable x_ν (right)

Approximation tools based on tree tensor networks

For the approximation of a function, a first approach is to introduce subspaces $V^{\nu}_{N_\nu}$ of finite dimension (e.g. polynomials, splines, wavelets, RKHS...) and consider tree tensor networks $f \in \mathcal{T}_r^T(V_N)$ where

$$V_N = V^1_{N_1} \otimes \ldots \otimes V^d_{N_d},$$

with variable N and r.

Spaces $V^{\nu}_{N_\nu}$ have to be well chosen, e.g. polynomials for analytic functions, splines with a degree adapted to the regularity of the function...

Approximation tools based on tree tensor networks

An approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ is then defined by

$$\Phi_n = \{f \in \mathcal{T}_r^T(V_N) : N \in \mathbb{N}^d, r \in \mathbb{N}^T, \mathit{compl}(f) \leq n\}.$$

The dimensions N and the ranks r are free parameters, and $\mathit{compl}(\cdot)$ is some complexity measure.

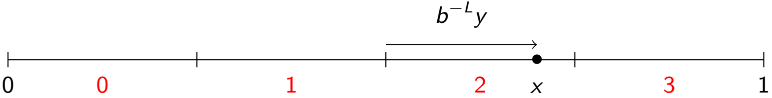
An alternative approach is to rely on tensorization of functions (specific featuring step).

Tensorization of univariate functions

Consider a function $f \in \mathbb{R}^{[0,1)}$ defined on the interval $[0,1)$.

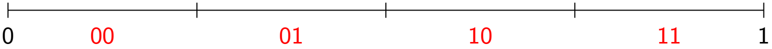
- For $b, L \in \mathbb{N}$, we subdivide uniformly the interval $[0,1)$ into b^L intervals. Any $x \in [0,1)$ can be written

$$x = b^{-L}(i + y), \quad i \in \{0, \dots, b^L - 1\}, \quad y \in [0,1).$$



- The integer i admits a representation in base b

$$i = \sum_{k=1}^{L} i_k b^{L-k} = [i_1 \dots i_L]_b, \quad i_k \in \{0, \dots, b-1\}$$

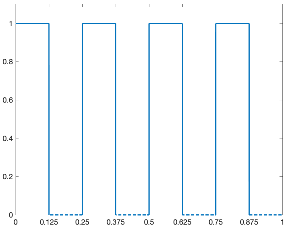


- f is thus identified with a multivariate function (tensor of order $L+1$)

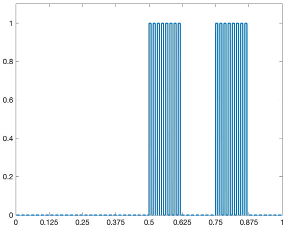
$$\boldsymbol{f} \in (\mathbb{R}^b)^{\otimes L} \otimes \mathbb{R}^{[0,1)} \quad \text{such that} \quad f(x) = \boldsymbol{f}(i_1, \dots, i_L, y)$$

Tensorization of univariate functions

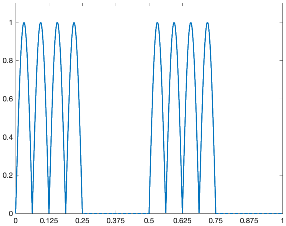
Examples of rank one functions $f(x) = v^1(i_1) ... v^L(i_L) v^{L+1}(y)$ $(b = 2)$



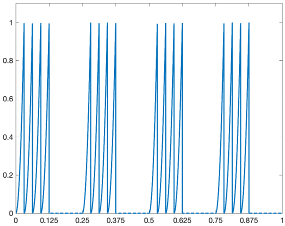
(a) $\delta_0(i_3)$



(b) $\delta_1(i_1)\delta_0(i_3)\delta_0(i_7)$



(c) $\delta_0(i_2) sin(\pi y)$ $(L = 4)$



(d) $\delta_0(i_3) y^2$ $(L = 5)$

Tensorization of multivariate functions

A function $f(x_1, \dots, x_d)$ defined on $[0,1)^d$ can be similarly identified with a tensor of order $(L+1)d$

$$\boldsymbol{f} \in (\mathbb{R}^b)^{\otimes Ld} \otimes (\mathbb{R}^{[0,1)})^{\otimes d}$$

such that

$$f(x_1, \dots, x_d) = \boldsymbol{f}(i_1^1, \dots, i_d^1, \dots, i_1^L, \dots, i_d^L, y_1, \dots, y_d)$$

where

$$x_\nu = b^{-L}(\sum_{k=1}^{L} i_\nu^k b^{L-k} + y_\nu) = [0.i_\nu^1 \dots i_\nu^L]_b + b^{-L} y_\nu$$

or equivalently, using a different ordering of variables,

$$f(x_1, \dots, x_d) = \boldsymbol{f}(i_1^1, \dots, i_1^L, y_1, \dots, i_d^1, \dots, i_d^L, y_d)$$

The map $T_{b,L}$ which associates to a function f its tensorization $\boldsymbol{f}$ is a linear isometry from $L^p([0,1)^d)$ to $L^p(\{0, \dots, b-1\}^{Ld} \times [0,1)^d)$ for any $0 < p \leq \infty$.

Approximation tools based on tree tensor networks

We consider functions whose tensorization at resolution L are in the tensor space

$$\boldsymbol{V}_L = (\mathbb{R}^b)^{\otimes Ld} \otimes S^{\otimes d}$$

with $S \subset \mathbb{R}^{[0,1)}$ some subspace of univariate functions, invariant through b-adic dilation.

If $S = \mathbb{P}_m$, $V_L = T_{b,L}^{-1}(\boldsymbol{V}_L)$ is identified with the space of multivariate splines of degree m over a uniform partition with b^{dL} elements, i.e.

$$V_L = V^1_{N_1} \otimes \ldots \otimes V^d_{N_d}$$

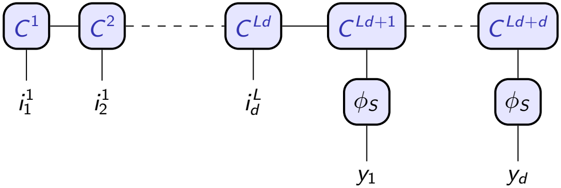
with $N_1 = ... = N_d = b^L$ and $V^\nu_{N_\nu}$ a space of univariate splines of degree m over a uniform partition with $N_\nu = b^L$ intervals.

Approximation tools based on tree tensor networks

Then as an approximation tool, we consider functions f whose tensorization is a tensor network in $\mathcal{T}_r^{T_L}(\boldsymbol{V}_L)$, with T_L a dimension tree over $\{1, \dots, Ld + d\}$.

Using the tensor train format, the corresponding function $f(x_1, \dots, x_d)$ has the representation

$f(x_1, \dots, x_d) =$



with ϕ_S the feature map associated with S. This is closely related to the quantized tensor train (QTT) format [Kazeev, Khoromskij, Oseledets, Schwab, ...]

Later on, we consider $S = \mathbb{P}_m$ and $\phi_S(y) = (1, y, ..., y^{m+1})$ or any other polynomial basis.

Approximation tools based on tree tensor networks

An approximation tool $\Phi = (\Phi_n)_{n\in\mathbb{N}}$ is then defined by

$$\Phi_n = \{f \in \Phi_{L,T_L,r} : L \in \mathbb{N}_0, r \in \mathbb{N}^{T_L}, compl(f) \leq n\}$$

with $\Phi_{L,T_L,r}$ the functions whose tensorization at resolution L is in $\mathcal{T}_r^{T_L}(\boldsymbol{V}_L)$.

The resolution L and ranks r are free parameters, and $compl(\cdot)$ is some complexity measure.

The complexity $compl(f)$ of f is defined as the complexity of the associated tensor network $\{C^\alpha\}_{\alpha \in T}$.

- Number of parameters (full tensor network)

$$compl_{\mathcal{F}}(f) = \sum_{\alpha} \text{number\_of\_entries}(C^\alpha)$$

- Number of non-zero parameters (sparse tensor network)

$$compl_{\mathcal{S}}(f) = \sum_{\alpha} \|C^\alpha\|_0$$

Complexity measures $compl_{\mathcal{F}}$ and $compl_{\mathcal{S}}$ yield two different approximation tools

$$\Phi_n^{\mathcal{F}} \quad \text{and} \quad \Phi_n^{\mathcal{S}}$$

such that

$$\Phi_n^{\mathcal{F}} \subset \Phi_n^{\mathcal{S}} \subset \Phi_{a+bn^2}^{\mathcal{F}}$$

Approximation theory of tree tensor networks

Given a function f from a Banach space X, the best approximation error of f by an element of Φ_n is

$$E(f, \Phi_n)_X := \inf_{g \in \Phi_n} \|f - g\|_X$$

Fundamental questions are:

- does $E(f, \Phi_n)_X$ converge to 0 for any f ?
  (universality)
- does a best approximation exist ?
  (proximinality)
- how fast does it converge for functions from classical function classes ?
  (expressivity)
- what are the functions for which $E(f, \Phi_n)_X$ converges with some given rate ?
  (characterization of approximation classes)

Outline

Universality

First note that for any algebraic feature tensor space V, and any tree T,

$$\bigcup_r \mathcal{T}_r^T(V) = V.$$

so the question of universality of tree tensor networks boils down to conditions on the tensor feature spaces.

- Consider the first family of approximation tools with variable feature spaces V_N, $N \in \mathbb{N}^d$.

  If $\bigcup_N V_N$ is dense in X, then the tools are universal for functions in X.

  In particular, this is true for $X = L^p((0,1)^d)$, $p < \infty$, and for polynomial or splines spaces V_N.

- Consider the second family of approximation tools using tensorization.

  If $\bigcup_L V_L$ is dense in X, then the tools are universal for functions in X.

  In particular, this is true for $X = L^p((0,1)^d)$, $p < \infty$, assuming that S contains the function one.

Proximinality

For any tree T, any T-rank r, and any finite dimensional tensor space V of X, $\mathcal{T}_r^T(V)$ is a closed set in V.

Φ_n is a finite union of such sets, all contained in a single finite dimensional space V^*. Then Φ_n is a closed set of a finite dimensional space V^* and is therefore proximinal in X.

Expressivity

Different ways to analyse the expressivity of tree tensor networks

- Exploit known results on other approximation tools and estimate the complexity to encode these tools using tree tensor networks.
- Directly encode a function using tree tensor networks (with controlled errors)
- Analyse the convergence of bilinear approximations

$$u(x_\alpha, x_{\alpha^c}) \approx \sum_{k=1}^{r_\alpha} u_k^\alpha(x_\alpha) u_k^{\alpha^c}(x_{\alpha^c})$$

  or the approximability of partial evaluations $u(\cdot, x_{\alpha^c})$ by linear approximation spaces of dimension r_α.

Encoding polynomials and splines

Polynomials

The tensorization of a polynomial of degree p has all ranks bounded by $p+1$.

Trigonometric polynomials

The tensorization of the function $\cos(\omega x + \varphi)$ has all ranks equal to 2.

Then the tensorization of a trigonometric polynomial of degree p has all ranks bounded by $2p+1$.

Free knot splines

A spline φ of degree p over N b-adic intervals forming a partition of $[0,1)$ is such that

$$\text{rank}_{\{1,\dots,\nu\}}(\boldsymbol{\varphi}) \le \begin{cases} p+N, & 1 \le \nu < \ell. \\ p+1, & \ell \le \nu \le L. \end{cases}$$

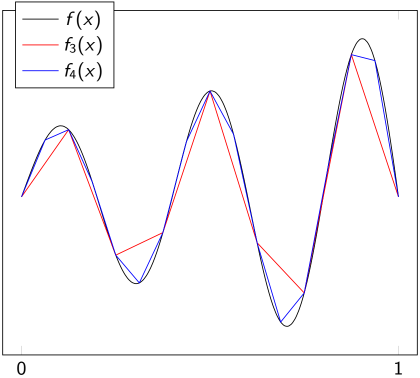
where $b^{-\ell}$ is the minimal length of intervals.

Encoding polynomials and splines

Ranks of interpolants

For a function f and its interpolation f_L onto V_L, the space of piecewise polynomials of degree m on a uniform partition of b^L intervals, it holds

$$\text{rank}_\alpha(\boldsymbol{f}_L) \leq \text{rank}_\alpha(\boldsymbol{f})$$



Encoding multi-resolution analysis

For a function $\psi : \mathbb{R} \to \mathbb{R}$ supported on $[0,1]$, we define its level ℓ *b*-adic dilation, shifted by $j = 0, \dots, b^L - 1$,

$$\psi_{\ell,j}(x) = \psi(b^\ell x - j)$$

Its tensorization at level ℓ is an elementary (rank-one) tensor

$$T_{b,\ell}\psi_{\ell,j} = e_{j_1} \otimes \dots e_{j_\ell} \otimes \psi$$

with $j = [j_1, \dots, j_\ell]_b$ and e_k the canonical basis vectors in $\mathbb{R}^b$.

Its tensorization at level $L \geq l$ is

$$T_{b,L}\psi_{\ell,j} = e_{j_1} \otimes \dots e_{j_\ell} \otimes (T_{b,L-\ell}\psi)$$

The (approximate) encoding of $\psi_{\ell,j}$ boils down to the (approximate) encoding of the mother function ψ with tensor networks.

In particular, if ψ is a (piecewise) polynomial, $\psi_{\ell,j}$ is encoded at precision ϵ using tensorization at level $L = \ell + O(\log(\epsilon^{-1}))$.

This yields a very efficient encoding of piecewise polynomial MRAs (B-spline wavelets).

Approximation of functions from Besov spaces $B^\alpha_q(L^p)$

From results on spline approximation and their encoding with tensor networks, we obtain

Theorem

Let $f \in B^\alpha_q(L^p)$ with $\alpha > 0$ and $0 < p, q \leq \infty$. Then

$$E(f, \Phi_n^{\mathcal{F}})_{L^p} \leq C n^{-\alpha/d} |f|_{B^\alpha_\infty(L^p)}$$

- Tensor networks achieve optimal rates for any Besov regularity order (measured in L^p norm).
- They perform as well as optimal linear approximation tools (e.g. splines), without requiring to adapt the tool to the regularity order α.
- The depth (resolution L) of the network is crucial to capture extra regularity ($\alpha > m + 1$).

Approximation of functions from Besov spaces $B^\alpha_q(L^\tau)$

Now consider the harder problem of approximating functions from Besov spaces $B^\alpha_q(L^\tau)$ where regularity is measured in a L^τ-norm weaker than L^p-norm.

From results on best n-term approximation using dilated splines, we obtain

Theorem

Let $f \in B^\alpha_q(L^\tau)$ with $\alpha > 0$, $0 < q \le \tau < p < \infty$, $1 \le p < \infty$ and

$$\frac{\alpha}{d} > \frac{1}{\tau} - \frac{1}{p}.$$

Then

$$E(f, \Phi_n^{\mathcal{S}})_{L^p} \le C n^{-\tilde{\alpha}/d} |f|_{B^\alpha_q(L^\tau)}, \quad E(f, \Phi_n^{\mathcal{F}})_{L^p} \le C n^{-\tilde{\alpha}/(2d)} |f|_{B^\alpha_q(L^\tau)},$$

for arbitrary $\tilde{\alpha} < \alpha$.

- Sparse tensor networks achieve arbitrarily close to optimal rates in $O(n^{-\alpha/d})$ for functions with any Besov smoothness α (measured in L^τ norm), without the need to adapt the tool to the regularity order α.
- Here depth and sparsity are crucial for obtaining near to optimal performance.
- Full tensor networks have slightly lower performance in $O(n^{-\alpha/(2d)})$.

High-dimensional approximation

- For Besov spaces $B^\alpha_q(L^p)$, tensor networks achieve (near to) optimal rate in $O(n^{-\alpha/d})$ which deteriorates with d, that is the curse of dimensionality.
- For Besov spaces with mixed smoothness $MB^\alpha_q(L^p)$, sparse tensor networks achieve near to optimal performance in $O(n^{-\alpha}\log(n)^d)$. But still the curse of dimensionality.
- For Besov spaces with anisotropic smoothness $AB^{\boldsymbol{\alpha}}_q(L^p)$, sparse tensor networks also achieve near to optimal rates in $O(n^{-s(\boldsymbol{\alpha})/d})$ with

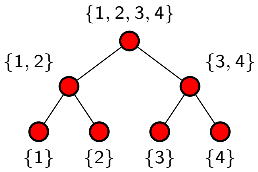
$$s(\boldsymbol{\alpha})/d = (\alpha_1^{-1} + \ldots + \alpha_d^{-1})^{-1}$$

  the aggregated smoothness. Curse of dimensionality can be circumvented with sufficient anisotropy.
- Curse of dimensionality can be circumvented for non usual function classes such as compositions of smooth functions

Compositional functions

Consider a tree-structured composition of smooth functions $\{f_\alpha : \alpha \in T\}$, see [Mhaskar, Liao, Poggio 2016] for deep neural networks, and [Bachmayr, Nouy and Schneider 2021] for tree tensor networks.

$$f_{1,2,3,4}\left(f_{1,2}\left(f_1(x_1), f_2(x_2)\right), f_{3,4}\left(f_3(x_3), f_4(x_4)\right)\right)$$



Assuming that the functions $f_\alpha \in W^{k,\infty}$ with $\|f_\alpha\|_{L^\infty} \leq 1$ and $\|f_\alpha\|_{W^{k,\infty}} \leq B$, the complexity to achieve an accuracy ϵ

$$n(\epsilon) \lesssim \epsilon^{-3/k}(L+1)^3 B^{3L} d^{1+3/2k}$$

with $L = \log_2(d)$ for a balanced tree and $L + 1 = d$ for a linear tree.

- Bad influence of the depth through the norm B of functions f_α (roughness).
- For a balanced tree, complexity scales polynomially in d: no curse of dimensionality !
- For $B \leq 1$ (and even for 1-Lipschitz functions), the complexity only scales polynomially in d whatever the tree: no curse of dimensionality !

More regularity, analytic functions

For function $f : [0,1]$ with analytic extension on an open complex domain

$$D_\rho = \{z \in \mathbb{C} : dist(z, [0,1]) < \frac{\rho - 1}{2}\}, \quad \rho > 1,$$

we obtain an exponential convergence

$$E(f, \Phi_n^{\mathcal{F}})_{L^\infty} \leq C\gamma^{-n^{1/3}},$$

with $\gamma = \min\{\rho, b^{(m+1)/b}\}$.

The proof relies on the approximation of analytic functions with polynomials and the encoding of polynomials with tree tensor networks: a chebychev polynomial p of deree $\bar{m}$ is such that

$$\|f - p\|_{L^\infty} \leq \frac{2}{\rho - 1}\|f\|_{L^\infty(D_\rho)}\rho^{-\bar{m}}$$

A polynomial of degree $\bar{m}$ can be approximated by φ in $\Phi_{L,r,m}$ with an error in $O(b^{-L(m+1)})$, so that

$$\|f - \varphi\|_{L^\infty} \lesssim \rho^{-\bar{m}} + b^{-L(m+1)}$$

We obtain the result by choosing $\bar{m} \sim n^{1/3}$ and $L \sim b^{-1}n^{1/3}$, so that $compl_{\mathcal{F}}(\varphi) \leq n$.

Analytic functions with singularities

Consider the approximation of $u(x) = x^\alpha$, $0 < \alpha \leq 1$, in L^∞.

- Piecewise constant linear approximation.

$$u \in B^\alpha_\infty(L^\infty), \quad u \notin B^\beta_\infty(L^\infty) \quad \text{for } \beta > \alpha,$$

  and a piecewise constant approximation on a uniform mesh with n elements gives a convergence in $O(n^{-\alpha})$ in L^∞,

- Piecewise constant nonlinear approximation.

$$u \in BV \subset B^1_\infty(L^1),$$

  and a piecewise constant approximation on an optimal mesh with n elements gives a convergence in $O(n^{-1})$ in L^∞,

- Piecewise constant approximation and tensor networks.
  A piecewise constant approximation on a uniform mesh with 2^L elements exploiting low-rank structures gives an exponential convergence

$$E(f, \Phi_n^{\mathcal{F}}) \leq C\beta^{-n^\gamma}$$

  Achieves almost the performance of *h-p* methods [Kazeev and Schwab].

Beyond smoothness

Consider the Weierstrass function, continuous but nowhere differentiable

$$f(x) = \sum_{k=0}^{\infty} a^{-\alpha k} \cos(a^k \pi x), \quad a > 0, \quad 0 < \alpha \leq 1,$$

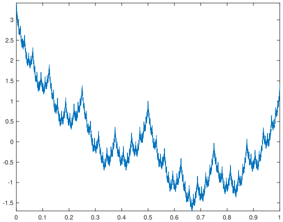


Figure: Weierstrass function for $\alpha = 1/2$, $a = 2$

We have an exponential convergence in L^∞-norm

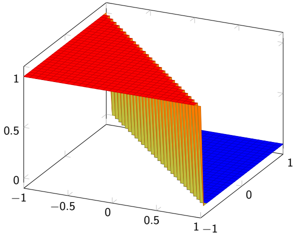
$$E(f, \Phi_n^{\mathcal{F}})_{L^\infty} \lesssim \beta^{-n^{1/3}}$$

An error ϵ is achieved with resolution $L \sim \log(\epsilon^{-1})$, ranks $\sim \log(\epsilon^{-1})$ and complexity $n \sim \log(\epsilon^{-1})^3$

Discontinuous functions: the power of tensorization

Consider the problem of approximating the bivariate function on $(-1,1)^2$

$$u(x,t) = \begin{cases} 1 & \text{if } x+t<0 \\ 0 & \text{if } x+t \geq 0 \end{cases}$$



The manifold $K = \{u(\cdot,t) : t \in (-1,1)\}$ contains the indicator functions $1_{[-1,x_i]}(x)$, $x_i = -1 + 2i/m$. Therefore the balanced convex hull of K contains the orthogonal system $S = \{\psi_i(x) = \frac{1}{2}1_{(x_i,x_{i+1}]}(x) : 1 \leq i \leq m\}$ with $\|\psi_i\|_{L^2} = (2m)^{-1/2}$ and by taking $m = 2n$, we deduce

$$d_n(K)_{L^2} \geq 1/(2\sqrt{2})n^{-1/2},$$

so that the best rank-n approximation

$$u_r(x,t) = \sum_{i=1}^{r} v_i(x)w_i(t)$$

does not converge better than $\|u - u_r\|_{L^2} \gtrsim n^{-1/2}$.

Discontinuous functions: the power of tensorization

A piecewise constant interpolant u^L on a uniform grid with mesh size 2^{-L} is such that

$$\|u - u^L\|_{L^2} = meas(\{(x,t) : u \neq u^L\})^{1/2} \leq 2^{1/2} 2^{-L/2}$$

Using a tensorization $\tilde{\boldsymbol{u}}^L(i_1^x, ..., i_L^x, i_1^t, ..., i_L^t)$, we have

$$rank_{\{1,\dots,L\}}(\tilde{\boldsymbol{u}}^L) = \text{rank}\, u_L \sim 2^L$$

that means an encoding complexity in tensor train format $compl(\tilde{\boldsymbol{u}}^L) \gtrsim 2^{2L}$, which yields an approximation error $\gtrsim n^{-1/4}$.

However, the tensorization $\boldsymbol{u}^L(i_1^x, i_1^t, ..., i_L^x, i_L^t)$ of $u^L(x,t)$ satisfies

$$rank_{\{1,\dots,\nu\}}(\boldsymbol{u}^L) \leq 3$$

for all ν. Therefore, using tensor train format, $compl(u^L) \leq 36L$ and

$$E(u, \Phi_n^{\mathcal{F}})_{L^2} \leq 2^{1/2} 2^{-n/72}.$$

Outline

Canonical versus tree-based format

Consider a finite dimensional tensor space $V = V^1 \otimes \ldots \otimes V^d$ with $\dim(V_\nu) = \mathbb{R}^N$, which is identified with $\mathbb{R}^{N\times\cdots\times N}$. Denote by $\mathcal{R}_r = \{v : \text{rank}(v) \leq r\}$ and $\mathcal{T}_r^T = \{v : \text{rank}_\alpha(v) \leq r, \alpha \in T\}$.

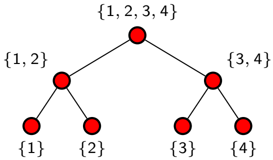
- From canonical format to tree-based format.
  For any v in V and any $\alpha \subset D$, the α-rank is bounded by the canonical rank:
  $$\text{rank}_\alpha(v) \leq \text{rank}(v).$$
  Therefore, for any tree T,
  $$\mathcal{R}_r \subset \mathcal{T}_r^T,$$
  so that an element in $\mathcal{R}_r$ with storage complexity $O(dNr)$ admits a representation in $\mathcal{T}_r^T$ with a storage complexity $O(dNr + dr^{s+1})$ where s is the arity of the tree T.

- From tree-based format to canonical format. For a balanced or linear binary tree, the subset
  $$S = \{v \in \mathcal{T}_r^T : \text{rank}(v) < q^{d/2}\}, \quad q = \min\{N, r\},$$
  is of Lebesgue measure 0.

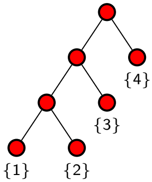
  Then a typical element $v \in \mathcal{T}_r^T$ with storage complexity of order $dNr + dr^3$ admits a representation in canonical format with a storage complexity of order $dNq^{d/2}$.

Influence of the tree

- For some functions, the choice of tree is not crucial. For example, an additive function
$$u_1(x_1) + \ldots + u_d(x_d)$$
has α-ranks equal to 2 whatever $\alpha \subset D$.
- But usually, different trees lead to different complexities of representations.



T^B (Balanced tree)



T^L (Linear tree)

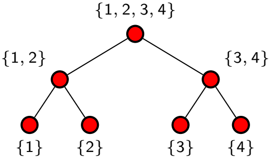
- If $\text{rank}_{T^L}(u) \leq r$ then $\text{rank}_{T^B}(u) \leq r^2$
- If $\text{rank}_{T^B}(u) \leq r$ then $\text{rank}_{T^L}(u) \leq r^{\log_2(d)/2}$

Influence of the tree

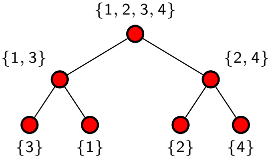
Given a tree T and a permutation σ of $D = \{1, \dots, d\}$, we define a tree T_σ

$$T_\sigma = \{\sigma(\alpha) : \alpha \in T\}$$

having the same structure as T but different nodes.



T



T_σ with $\sigma = (3, 1, 2, 4)$

If $\text{rank}_T(u) \leq r$ then $\text{rank}_{T_\sigma}(u)$ typically depends on d.

Influence of the tree

- Consider the Henon-Heiles potential

$$u(x) = \frac{1}{2}\sum_{i=1}^{d} x_i^2 + 0.2\sum_{i=1}^{d-1}(x_i x_{i+1}^2 - x_i^3) + \frac{0.2^2}{16}\sum_{i=1}^{d-1}(x_i^2 + x_{i+1}^2)^2$$

  Using a linear tree $T = \{\{1\},\{2\},\ldots,\{d\},\{1,2\},\{1,2,3\},\ldots,\{1,\ldots,d-1\},D\}$,

$$\mathrm{rank}_T(u) \leq 4, \quad storage(u) = O(d)$$

  but for the permutation

$$\sigma = (1,3,\ldots,d-1,2,4,\ldots,d) \tag{$\star$}$$

  and the corresponding linear tree T_σ,

$$\mathrm{rank}_{T_\sigma}(u) \leq 2d+1, \quad storage(u) = O(d^3).$$

- For a typical tensor in $\mathcal{T}_r^T$ with T a binary tree, its representation in tree based format with tree T_σ, with σ as in ($\star$), has a complexity scaling exponentially with d.
- As an example, consider the function $u(x,t) = 1_{x+t<0}$identified (through tensorization) with tensors $\boldsymbol{u}(i_1^x,\ldots,i_L^x,y^x,i_1^t,\ldots,i_L^t,y^t)$ and $\boldsymbol{u}(i_1^x,i_1^y,\ldots,i_L^x,i_L^y,y^x,y^t)$. Huge impact of the ordering !

Influence of the tree

- Consider the probability distribution $f(x) = \mathbb{P}(X = x)$ of a Markov chain $X = (X_1, \dots, X_d)$ given by

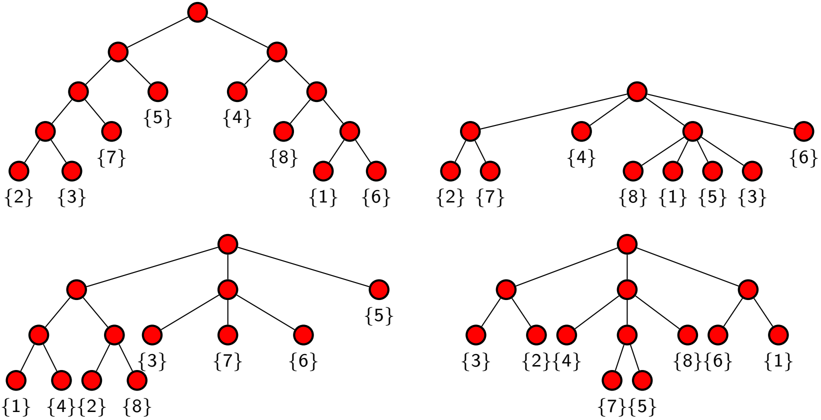
$$f(x) = f_1(x_1) f_{2|1}(x_2|x_1) \dots f_{d|d-1}(x_d|x_{d-1})$$

where bivariate functions $f_{i|i-1}$ have a rank r.

  - With the linear tree T containing interior nodes $\{1, 2\}, \{1, 2, 3\}, \dots, \{1, \dots, d-1\}$, f admits a representation in tree-based format with storage complexity in r^4.
  - The canonical rank of f is exponential in d.
  - But when considering the linear tree T_σ obtained by applying permutation $\sigma = (1, 3, \dots, d-1, 2, 4, \dots, d)$ to the tree T, the storage complexity in tree-based format is also exponential in d.

How to choose a good tree ?

A combinatorial problem...



Outline

Properties of tree tensor networks

We here consider approximation tools $(\Phi_n)_{n\geq 1}$ based on tensorization and tensor train format (with or without sparsity).
They satisfy

(P1) $\Phi_0 = \{0\}$, $0 \in \Phi_n$

(P2) $a\Phi_n = \Phi_n$ for any $a \in \mathbb{R} \setminus \{0\}$ (cone)

(P3) $\Phi_n \subset \Phi_{n+1}$ (nestedness)

(P4) $\Phi_n + \Phi_n \subset \Phi_{cn}$ for some constant c (not too nonlinear)

For $X = L^p$, they further satisfy

(P5) $\bigcup_n \Phi_n$ is dense in L^p for $0 < p < \infty$ (universality),

(P6) for each $f \in L^p$ for $0 < p \leq \infty$, there exists a best approximation in Φ_n (proximinal sets).

Approximation classes

For an approximation tool $\Phi = (\Phi_n)_{n \in \mathbb{N}}$, we define for any $\alpha > 0$ the approximation class

$$A^{\alpha}_{\infty}(L^p) := A^{\alpha}_{\infty}(L^p, \Phi)$$

of functions $f \in L^p$ such that

$$E(f, \Phi_n)_{L^p} \leq C n^{-\alpha}$$

- Properties (P1)-(P4) of Φ imply that $A^{\alpha}_{\infty}(L^p)$ is a quasi-Banach space with quasi-semi-norm

$$|f|_{A^{\alpha}_{\infty}} := \sup_{n \geq 1} n^{\alpha} E(f, \Phi_n)_{L^p}$$

- Full and sparse complexity measures yield two different approximation spaces

$$\mathcal{F}^{\alpha}_{\infty}(L^p) = A^{\alpha}_{\infty}(L^p, \Phi^{\mathcal{F}}), \quad \mathcal{S}^{\alpha}_{\infty}(L^p) = A^{\alpha}_{\infty}(L^p, \Phi^{\mathcal{S}})$$

such that

$$\mathcal{F}^{\alpha}_{\infty}(L^p) \hookrightarrow \mathcal{S}^{\alpha}_{\infty}(L^p) \hookrightarrow \mathcal{F}^{\alpha/2}_{\infty}(L^p)$$

Direct embeddings

From results on the approximation properties for Besov spaces, we have the following results.

- (Linear approximation) For $\alpha > 0$ and $0 < p \leq \infty$,

$$B_q^\alpha(L^p) \hookrightarrow \mathcal{F}_\infty^{\alpha/d}(L^p),$$

$$MB_q^\alpha(L^p) \hookrightarrow \mathcal{S}_\infty^\alpha(L^p),$$

$$AB_q^{\boldsymbol{\alpha}}(L^p) \hookrightarrow \mathcal{S}_\infty^{s/d}(L^p)$$

  with $s(\boldsymbol{\alpha}) := d(\alpha_1^{-1} + \ldots + \alpha_d^{-1})^{-1}$.

- (Nonlinear approximation) For $\alpha > 0$, $1 \leq p < \infty$, $0 < q \leq \tau < p < \infty$ and $\frac{\alpha}{d} > \frac{1}{\tau} - \frac{1}{p}$,

$$B_q^\alpha(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}/d}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/(2d)}(L^p),$$

$$MB_q^\alpha(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/2}(L^p)$$

  for arbitrary $\tilde{\alpha} < \alpha$, and

$$AB_q^{\boldsymbol{\alpha}}(L^\tau) \hookrightarrow \mathcal{S}_\infty^{\tilde{\alpha}/d}(L^p) \hookrightarrow \mathcal{F}_\infty^{\tilde{\alpha}/(2d)}(L^p)$$

  for arbitrary $\tilde{\alpha} < s(\boldsymbol{\alpha})$.

Interpolation family

The properties of Φ_n allow to apply classical results from approximation theory, in particular to deduce from embedding results on $A^\alpha_\infty(L^p)$ embedding results on interpolation spaces

$$A^\beta_q(L^p) = (L^p, A^\alpha_\infty(L^p))_{\beta/\alpha,q}, \quad 0 < \beta < \alpha, \qquad 0 < q \leq \infty$$

that are quasi-Banach spaces with quasi-norm

$$\|f\|_{A^\alpha_q} = \|f\|_{L^p} + |f|_{A^\alpha_q}, \quad |f|_{A^\alpha_q} = \left(\sum_{n=1}^\infty n^{-1} \left(n^\alpha E(f, \Phi_n)_X\right)^q \right)^{1/q}$$

(functions with faster convergence than those of $A^\alpha_\infty(L^p)$).

No inverse embedding

For any $\alpha > 0$, $q \leq \infty$, and any β,

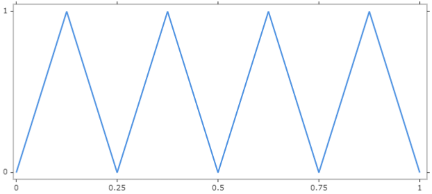
$$\mathcal{F}_q^{\alpha}(L^p) \not\hookrightarrow B_q^{\beta}(L^p).$$

That means that approximation classes contain functions that have no smoothness in a classical sense.

Tree tensor networks may be useful for the approximation of functions beyond standard smoothness classes.

No inverse embedding

This is proved by contradiction by considering the sawtooth function φ_L with 2^L teeth such that $\varphi_L \in \Phi_n$ with $n \sim L$.



From properties (P1)-(P6), $\mathcal{F}_q^\alpha(L^p)$ satisfies the Berstein inequality, that is

$$\|\varphi\|_{\mathcal{F}_q^\alpha(L^p)} \lesssim n^\alpha \|\varphi\|_{L^p} \quad \forall \varphi \in \Phi_n.$$

Moreover, $\|\varphi_L\|_{L^p} \sim 1$ and $\|\varphi_L\|_{B_q^\beta(L^p)} \gtrsim 2^{\beta L}$. If the embedding were true, we would have

$$2^{\beta n} \lesssim \|\varphi_L\|_{B_q^\beta(L^p)} \lesssim \|\varphi_L\|_{\mathcal{F}_q^\alpha(L^p)} \lesssim n^\alpha,$$

a contradiction.

The role of depth

Consider the approximation with restricted resolution

$$\Phi_n^{\mathcal{L}} = \{f \in \Phi_n : L(f) \leq \mathcal{L}(n)\}$$

where $L(f)$ is the minimal resolution L such that $f \in V_L$, and $\mathcal{L}$ some growth function.

Since $L(f) \leq n$ for $f \in \Phi_n$, $\Phi_n^{\mathcal{L}} = \Phi_n$ for $\mathcal{L} = n$.

In dimension $d = 1$, for $\mathcal{L}(n) = r \log_b(n) + c$, the following Bernstein inequality holds

$$|f|_{B_\tau^{m+1}(L^\tau)} \lesssim \|f\|_{L^p} b^{c(m+1)} n^{r(m+1)}$$

with τ the Sobolev embedding number, and m the local polynomial degree. This implies the inverse embedding of the corresponding approximation class

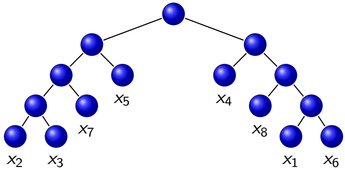
$$A_\infty^\alpha(L^p; (\Phi_n^{\mathcal{L}})) \hookrightarrow B_\tau^{\alpha/(m+1)}(L^\tau)$$

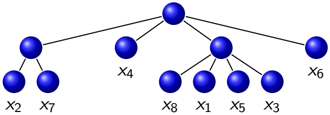
Hence the importance of depth L for going beyond standard regularity classes.

Some open questions

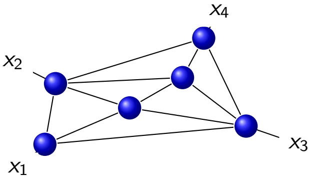
- What are the properties of the approximation tool with free tree T over $\{1, \ldots, (L+1)d\}$

$$\Phi_n = \{f \in \Phi_{L,T,r,m} : L \in \mathbb{N}_0, T \subset 2^{\{1,\ldots,(L+1)d\}}, r \in \mathbb{N}^{\#T}, compl(f) \leq n\} \quad ?$$





- What about approximation classes of more general tensor networks ?



Some open questions

- Algorithms to practically compute approximations achieving a certain precision with almost optimal complexity, using available information on the function (model equations, point samples...)
- Computational complexity of (deterministic or randomized) algorithms based on point samples for functions from approximation classes of tensor networks ?
- Theory to practice gap ?

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