An introduction to shifted quantum algebras

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[Based on [Hernandez 2010.06996], [JEB 2107.10063, 21??.?????]**]**

Outline

1. Shifted quantum affine $\mathfrak{sl}(2)$ algebra

2. Shifted quantum toroidal $\mathfrak{gl}(1)$ algebra

3. Application to string theory

1. Shifted quantum affine $\mathfrak{sl}(2)$ algebra

What is a quantum group?

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Facts:

- The coproduct is used to define an action on the tensor product of modules:

Let $\rho_1 : \mathcal{A} \to GL(V_1)$ and $\rho_2 : \mathcal{A} \to GL(V_2)$ be representations on the modules V_1 and V_2.

$\rightsquigarrow \quad (\rho_1 \otimes \rho_2)\Delta : \mathcal{A} \to GL(V_1 \otimes V_2)$ is a representation on $V_1 \otimes V_2$.

- The simplest case is the cocommutative coproduct: $\Delta(a) = a \otimes 1 + 1 \otimes a$ for $a \in \mathcal{A}$.
- The underlying mathematical structure is called a (quasi-triangular) Hopf algebra.

Integrability: In general, $(\rho_1 \otimes \rho_2)\Delta$ and $(\rho_2 \otimes \rho_1)\Delta$ are different representations!

$\rightsquigarrow$ The difference is encoded in the universal R-matrix that satisfies the **Yang-Baxter equation**: $R_{12}R_{23}R_{13} = R_{23}R_{13}R_{12}$.

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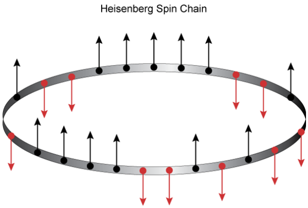
Remark: Two main types of quantum groups: **Yangians** and **quantum affine algebras**

$\rightsquigarrow$ We focus here on quantum affine algebras (Yangians follow from a degenerate limit)

Motivations



Why quantum affine $\mathfrak{sl}(2)$?

- Symmetry of quantum integrable systems: XXZ spin chain, 6 vertex model,...
- K-theoretic COhomological Hall Algebra in algebraic geometry
- Simplest non-trivial example of quantum group

  $\rightsquigarrow$ Sufficient to understand the problem and the solution!

Definition: The quantum affine $\mathfrak{sl}(2)$ algebra is generated by the modes $X_k^{\pm}$, $\Psi_{\pm l}^{\pm}$ of the currents

$$X^{\pm}(z) = \sum_{k\in\mathbb{Z}} z^{-k} X_k^{\pm}, \quad \Psi^{\pm}(z) = \sum_{l\geq 0} z^{\mp l} \Psi_{\pm l}^{\pm},$$

satisfying the algebraic relations

$$[\Psi^{\pm}(z), \Psi^{\pm}(w)] = [\Psi^{\pm}(z), \Psi^{\mp}(w)] = 0,$$

$$\Psi_0^+ \Psi_0^- = \Psi_0^- \Psi_0^+ = 1, \quad \Psi_0^+ X_k^{\pm} = q^{\pm 2} X_k^{\pm} \Psi_0^+,$$

$$(z - q^{\pm 2} w) X^{\pm}(z) X^{\pm}(w) = (q^{\pm 2} z - w) X^{\pm}(w) X^{\pm}(z),$$

$$\Psi^+(z) X^{\pm}(w) = \left[\frac{q^{\pm 2} z - w}{z - q^{\pm 2} w}\right]_+ X^{\pm}(w) \Psi^+(z),$$

$$\Psi^-(z) X^{\pm}(w) = \left[\frac{q^{\pm 2} z - w}{z - q^{\pm 2} w}\right]_- X^{\pm}(w) \Psi^-(z),$$

$$[X^+(z), X^-(w)] = \frac{\delta(z/w)}{q - q^{-1}} \left(\Psi^+(z) - \Psi^-(z)\right),$$

with $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$ and $[\cdots]_{\pm}$ denotes an expansion in powers of $z^{\mp 1}$.

Remarks:

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- We use here the presentation in terms of Drinfeld generators, for Chevalley generators,

$$e_1 \to X_0^+, \quad f_1 \to X_0^-, \quad k_1^{\pm 1} \to \Psi_0^\pm,$$
$$e_0 k_0^{-1} \to X_1^-, \quad k_0 f_0 \to X_{-1}^+, \quad k_0 k_1 \to q^c = 1.$$

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- Instead, in the RLL presentation (cf **[Frenkel, Ding 1993]**),

$$L^\pm(z) = \begin{pmatrix} 1 & 0 \\ e^\pm(z) & 1 \end{pmatrix} \begin{pmatrix} k_1^\pm(z) & 0 \\ 0 & k_2^\pm(z) \end{pmatrix} \begin{pmatrix} 1 & f^\pm(z) \\ 0 & 1 \end{pmatrix}$$

and $X^+(z) = e^+(z) - e^-(z)$, $X^-(z) = f^+(z) - f^-(z)$, $\Psi^\pm(z) = k_2^\pm(z)^{-1} k_1^\pm(z)$.

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The relation that will play a crucial role is $\Psi_0^+\Psi_0^- = \Psi_0^-\Psi_0^+ = 1$!!!

Why shifted quantum algebras?

Kirillov-Reshetikhin modules

KR module: Module of dimension $N+1$ spanned by the states $|k\rangle$ with $k=0,\cdots N$,

$$X^+(z)\,|k\rangle = \delta(\nu q^{2k}/z)q^{-N}\frac{q^{2k}-q^{2N}}{q-q^{-1}}\,|k+1\rangle\,,$$
$$X^-(z)\,|k\rangle = -\delta(\nu q^{2(k-1)}/z)q\frac{1-q^{-2k}}{q-q^{-1}}\,|k-1\rangle\,,$$
$$\Psi^\pm(z)\,|k\rangle = q^{2k-N}\left[\frac{(z-\nu q^{2N})(z-\nu q^{-2})}{(z-\nu q^{2k})(z-\nu q^{2(k-1)})}\right]_\pm |k\rangle\,.$$

Remarks:

- The representation has weight $\nu\in\mathbb{C}^\times$.
- Recover usual XXZ spin chain for $N=1$: two states at each site, $|-\rangle\equiv|0\rangle$, $|+\rangle\equiv|1\rangle$.
- $\Psi_0^\pm\,|k\rangle = q^{\pm(2k-N)}\,|k\rangle$, since

$$q^{2k-N}\left[\frac{(z-\nu q^{2N})(z-\nu q^{-2})}{(z-\nu q^{2k})(z-\nu q^{2(k-1)})}\right]_\pm = q^{\pm(2k-N)}(1+O(z^{\mp1}))$$

Thus, we have indeed $\Psi_0^+\Psi_0^- = \Psi_0^-\Psi_0^+ = 1$.

What we wish to do...

We would like to take the limit $N \to \infty$ of the KR representation:

After rescaling, $X^+(z) \to q^N X^+(z)$, $\Psi^\pm(z) \to q^N \Psi^\pm(z)$ the action has a well-defined limit for $|q| < 1$,

$$X^+(z)\,|k\rangle = \delta(\nu q^{2k}/z)\frac{q^{2k}}{q-q^{-1}}\,|k+1\rangle\,,$$

$$X^-(z)\,|k\rangle = -\delta(\nu q^{2(k-1)}/z)q\frac{1-q^{-2k}}{q-q^{-1}}\,|k-1\rangle\,,$$

$$\Psi^\pm(z)\,|k\rangle = q^{2k}\left[\frac{z(z-\nu q^{-2})}{(z-\nu q^{2k})(z-\nu q^{2(k-1)})}\right]_\pm |k\rangle\,.$$

$\rightsquigarrow$ We no longer have $X^+(z)\,|N\rangle = 0$, the module is infinite dimensional.

The problem

All the algebraic relations of the quantum affine $\mathfrak{sl}(2)$ algebra are satisfied, but one!

Since

$$q^{2k}\left[\frac{z(z-\nu q^{-2})}{(z-\nu q^{2k})(z-\nu q^{2(k-1)})}\right]_{+} = q^{2k} + O(z^{-1})$$

$$q^{2k}\left[\frac{z(z-\nu q^{-2})}{(z-\nu q^{2k})(z-\nu q^{2(k-1)})}\right]_{-} = -\nu^{-1}q^{-2k}z + O(z^2).$$

we have $\Psi_0^- \left|k\right\rangle = 0$ and thus $\Psi_0^+\Psi_0^- = \Psi_0^-\Psi_0^+ = 1$ does not hold.

The solution

[Hernandez-Jimbo 2011] Relax the condition $\Psi_0^+\Psi_0^- = \Psi_0^-\Psi_0^+ = 1$, define the **asymptotic** quantum affine $\mathfrak{sl}(2)$ algebra...

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We can do better! Notice that

$$\Psi_0^+ \left|k\right\rangle = q^{2k} \left|k\right\rangle , \quad \Psi_{-1}^- \left|k\right\rangle = -\nu^{-1} q^{-2k} \left|k\right\rangle .$$

Thus, $\Psi_0^+\Psi_{-1}^- = -\nu^{-1}$ is central! Up to a rescaling, we can define

$$\Psi^+(z) = \sum_{l\geq 0} z^{-l}\Psi_l^+, \quad \Psi^-(z) = \sum_{l\geq 1} z^{l}\Psi_{-l}^+, \quad \Psi_0^+\Psi_{-1}^- = \Psi_{-1}^-\Psi_0^+ = 1.$$

This is the main idea behind the definition of **shifted quantum algebras**!

Remark:

This representation, denoted L_ν^- is called the **prefundamental representation**. It is equivalent to the q-oscillator algebra of **[Bazhanov-Lukyanov-Zamolodchikov]** introduced to describe the integrable structure of 2D Conformal Field Theories. It also plays an important role in the definition of Baxter's Q-operator.

Shifted quantum algebras

Definition: The shifted quantum affine $\mathfrak{sl}(2)$ algebra $\mathcal{U}^{\boldsymbol{\mu}}$ with $\boldsymbol{\mu} = (\mu_+, \mu_-) \in \mathbb{Z} \times \mathbb{Z}$ is generated by the modes of the currents $X^{\pm}(z)$ and

$$\Psi^+(z) = \sum_{l \geq -\mu_+} z^{-l} \Psi_l^+ = z^{\mu_+} \Psi_{-\mu_+}^+ + O(z^{\mu_+ - 1}),$$

$$\Psi^-(z) = \sum_{l \geq -\mu_-} z^{l} \Psi_{-l}^- = z^{-\mu_-} \Psi_{\mu_-}^- + O(z^{-\mu_- + 1}),$$

satisfying the same algebraic relations as before, replacing $\Psi_0^{\pm}$ with $\Psi_{\mp\mu_{\pm}}^{\pm}$, i.e.

$$\Psi_{-\mu_+}^+ \Psi_{\mu_-}^- = \Psi_{\mu_-}^- \Psi_{-\mu_+}^+ = 1.$$

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satisfying the same algebraic relations as before, replacing $\Psi_0^{\pm}$ with $\Psi^{\pm}_{\mp\mu_\pm}$, i.e.

$$\Psi^+_{-\mu_+} \Psi^-_{\mu_-} = \Psi^-_{\mu_-} \Psi^+_{-\mu_+} = 1.$$

Remarks:

- When $\mu_\pm \leq 0$, we find a special case of the asymptotic algebra.
- Depends only on $\mu_+ + \mu_-$ up to automorphism $(X^+(z) \to z^p X^+(z),\ \Psi^\pm(z) \to z^p \Psi^\pm(z))$

$\rightsquigarrow$ Finite dimensional representation only if $\mu_+ + \mu_- \geq 0$ **[Hernandez 2020]**.

- For a general quantum affine algebra, $\Psi_i^{\pm}(z) = z^{\mp\alpha_i(\mu_\pm)} \Psi_{i, \mp\alpha_i(\mu_\pm)} + \cdots$ with $\mu_\pm$ in the coroot lattice (α_i simple root).

Examples

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Example 2. Let $P(z)$ be the polynomial

$$P(z) = m^{1/2} \prod_{a=1}^{d} (1 - z/m_a), \quad m = \prod_{a=1}^{d} (-m_a).$$

We define the one-dimensional representation ρ_P of $\mathcal{U}^{(d,0)}$ as

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Remark:

- When $d = 1$, we recover representation L_ν^+ of **[Hernandez-Jimbo]**.
- We could also use $P(z) = m^{-1/2} \prod_{a=1}^{d} (1 - m_a/z)$ to define a representation of $\mathcal{U}^{(0,d)}$.

Where it gets interesting...

The Drinfeld coproduct Δ becomes a homomorphism of algebras $\Delta : \mathcal{U}^{\mu+\mu'} \to \mathcal{U}^{\mu} \otimes \mathcal{U}^{\mu'}$,

$$\Delta(X^+(z)) = X^+(z) \otimes 1 + \Psi^-(z) \otimes X^+(z),$$

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$\rightsquigarrow$ From a representation ρ of $\mathcal{U}^{\boldsymbol{\mu}}$ and a polynomial $P(z)$ as before, we can define two representations of $\mathcal{U}^{\boldsymbol{\mu}+(d,0)}$ **on the same module**,

$$\iota_P \rho = (\rho_P \otimes \rho)\Delta, \qquad \iota_P^* \rho = (\rho \otimes \rho_P)\Delta.$$

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We can start playing with the zeros of $P(z)$!

Example: For instance, let ρ be the prefundamental representation of $\mathcal{U}^{(0,-1)}$ and take $P(z) = (-\nu)^{-1/2} q^{-N}(1 - \nu q^{2N}/z)$, then we find a representation $\iota_P \rho$ of $\mathcal{U}$ that reads

$$X^+(z)\,|k\rangle = \delta(\nu q^{2k}/z) P(z) \frac{q^{2k}}{q - q^{-1}}\,|k+1\rangle\,,$$

$$X^-(z)\,|k\rangle = -\delta(\nu q^{2(k-1)}/z) q \frac{1 - q^{-2k}}{q - q^{-1}}\,|k-1\rangle\,,$$

$$\Psi^\pm(z)\,|k\rangle = q^{2k} P(z) \left[\frac{z(z - \nu q^{-2})}{(z - \nu q^{2k})(z - \nu q^{2(k-1)})}\right]_\pm |k\rangle\,.$$

$\rightsquigarrow$ Due to the zero of $P(z)$, we observe that $X^+(z)\,|N\rangle = 0$. Thus, we can define a subrepresentation on the submodule spanned by $|k\rangle$ with $k \leq N$. This subrepresentation is isomorphic to the Kirillov-Reshetikhin representation.

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This kind of trick will be very useful for the toroidal algebra!

2. Shifted quantum toroidal $\mathfrak{gl}(1)$ algebra

Definition: The shifted quantum toroidal $\mathfrak{gl}(1)$ algebra $\mathcal{E}^{\boldsymbol{\mu}}$ is generated by the modes $x_k^\pm$, $\psi_{\pm l}^\pm$ of the currents

$$x^\pm(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_k^\pm, \quad \psi^\pm(z) = \sum_{l\geq-\mu_\pm} z^{\mp l}\psi_{\pm l}^\pm,$$

satisfying the algebraic relations

$$[\psi^\pm(z),\psi^\pm(w)] = [\psi^\pm(z),\psi^\mp(w)] = 0, \quad \psi^+_{-\mu_+}\psi^-_{\mu_-} = \psi^-_{\mu_-}\psi^+_{-\mu_+} = 1, \quad \psi^\pm_{\mp\mu_\pm} \text{ central},$$

$$(z-q_1^{\pm1}w)(z-q_2^{\pm1}w)(z-q_3^{\pm1}w)x^\pm(z)x^\pm(w) = (q_1^{\pm1}z-w)(q_2^{\pm1}z-w)(q_3^{\pm1}z-w)x^\pm(w)x^\pm(z),$$

$$\psi^+(z)x^\pm(w) = \left[\frac{(z-q_1^{\pm1}w)(z-q_2^{\pm1}w)(z-q_3^{\pm1}w)}{(q_1^{\pm1}z-w)(q_2^{\pm1}z-w)(q_3^{\pm1}z-w)}\right]_+ x^\pm(w)\psi^+(z),$$

$$\psi^-(z)x^\pm(w) = \left[\frac{(z-q_1^{\pm1}w)(z-q_2^{\pm1}w)(z-q_3^{\pm1}w)}{(q_1^{\pm1}z-w)(q_2^{\pm1}z-w)(q_3^{\pm1}z-w)}\right]_- x^\pm(w)\psi^-(z),$$

$$[x^+(z),x^-(w)] = \frac{(1-q_1)(1-q_2)}{(1-q_1q_2)}\delta(z/w)\left(\psi^+(z)-\psi^-(z)\right),$$

It has two parameters $q_1, q_2 \in \mathbb{C}^\times$ ($q_3 = q_1^{-1}q_2^{-1}$), and $\boldsymbol{\mu} = (\mu_+,\mu_-) \in \mathbb{Z}\times\mathbb{Z}$ as before.

Properties:

i. $\mathcal{E}^{\mu}$ with $\mu = (0,0)$ is the usual quantum toroidal $\mathfrak{gl}(1)$ algebra (at $c = 0$).
(It is also called Ding-Iohara-Miki, quantum continuous $\mathfrak{gl}(\infty)$,...)

ii. The Drinfeld coproduct provides an homomorphism $\Delta : \mathcal{E}^{\mu+\mu'} \to \mathcal{E}^{\mu} \otimes \mathcal{E}^{\mu'}$.

iii. We also have the one-dimensional representations ρ_P defined by the polynomials $P(z)$,

$$\rho_P(x^{\pm}(z)) = 0, \quad \rho_P(\psi^{\pm}(z)) = P(z), \quad P(z) = m^{1/2}\prod_{a=1}^{d}(1 - z/m_a).$$

iv. We can shift any representation ρ into $\iota_P\rho$ (or $\iota_P^*\rho$) by multiplying $\rho(x^+(z))$ (or $\rho(x^-(z))$) and $\rho(\psi^{\pm}(z))$ by $P(z)$.

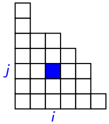
Fock representation The (vertical) Fock representation acts on states $|\lambda\rangle$ parameterized by Young diagrams λ,

$$x^+(z)\,|\lambda\rangle = \sum_{\square\in A(\lambda)} \delta(z/\chi_\square)A_\lambda^+(\square)\,|\lambda+\square\rangle\,,$$

$$x^-(z)\,|\lambda\rangle = \sum_{\square\in R(\lambda)} \delta(z/\chi_\square)A_\lambda^-(\square)\,|\lambda-\square\rangle\,,$$

$$\psi^\pm(z)\,|\lambda\rangle = q_3^{-1/2}\left[\frac{\prod_{\square\in A(\lambda)}(1-q_3\chi_\square/z)\prod_{\square\in R(\lambda)}(1-\chi_\square/(q_3 z))}{\prod_{\square\in A(\lambda)}(1-\chi_\square/z)\prod_{\square\in R(\lambda)}(1-\chi_\square/z)}\right]_\pm |\lambda\rangle$$

where $A_\lambda^\pm(\square)$ are known coefficients (but not enlightening). $x^+(z)$ add boxes ($A(\lambda)$ is the set of addable boxes), while $x^-(z)$ removes boxes ($R(\lambda)$ is the set of removable boxes). For a box $\square$ of coordinates (i,j), we set $\chi_\square = vq_1^{i-1}q_2^{j-1}$ with the weight $v\in\mathbb{C}^\times$.



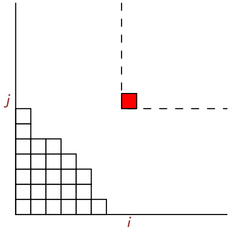
Shifted Fock representation We define the shifted Fock representation $\iota_P \rho_F$ of $\mathcal{E}^{(d,0)}$ by

$$x^+(z)\,|\lambda\rangle = P(z)\sum_{\square\in A(\lambda)} \delta(z/\chi_\square)A_\lambda^+(\square)\,|\lambda+\square\rangle\,,$$

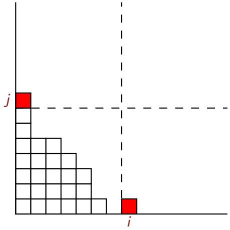
$$x^-(z)\,|\lambda\rangle = \sum_{\square\in R(\lambda)} \delta(z/\chi_\square)A_\lambda^-(\square)\,|\lambda-\square\rangle\,,$$

$$\psi^\pm(z)\,|\lambda\rangle = q_3^{-1/2}\left[P(z)\frac{\prod_{\square\in A(\lambda)}(1-q_3\chi_\square/z)\prod_{\square\in R(\lambda)}(1-\chi_\square/(q_3 z))}{\prod_{\square\in A(\lambda)}(1-\chi_\square/z)\prod_{\square\in R(\lambda)}(1-\chi_\square/z)}\right]_\pm |\lambda\rangle$$

$\rightsquigarrow$ We define a **pit (sub)representation** if $P(z)$ as a zero at $z = vq_1^{i-1}q_2^{j-1}$: the box $\square = (i,j)$ is forbidden and Young diagrams are restricted to a fat hook



$\rightsquigarrow$ With two pits, we can define **finite dimensional representations**:



Here $P(z)$ has two zeros at $z = vq_1^{i-1}$ and $z = vq_2^{j-1}$, Young diagrams are restricted to the rectangle $i \times j$: there are $\binom{i+j}{i}$ states.

Main message

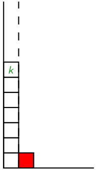
- Shifted quantum affine algebras admit **infinite dimensional** irreducible highest ℓ-weight representations.
- Shifted quantum toroidal algebras admit **finite dimensional** irreducible highest ℓ-weight representations.

Redde Cæsari quæ sunt Cæsaris...

- Shifted Yangian: **[Brundan, Kleshchev 2008]** in the representation theory of finite W-algebras.
- Asymptotic quantum affine algebra: **[Hernandez-Jimbo 2011]** in the study of the prefundamental representation.
- Shifted quantum affine algebras: **[Finkelberg, Tsymbaliuk 2017]** in the study of K-theoretic Coulomb branches of 3D $\mathcal{N} = 4$ supersymmetric gauge theories.
- Shifted quantum toroidal algebras: **[Finkelberg, Tsymbaliuk 2017]** (same context)

Limit of the quantum toroidal algebra

Remark: If $P(z)$ as a single zero at $z = vq_1$, Young diagrams are restricted to single columns,



The submodule can be mapped to the prefundamental module spanned by states $|k\rangle$ with $k \in \mathbb{Z}^{\geq 0}$ being the number of boxes.

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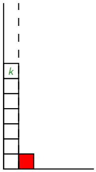

The submodule can be mapped to the prefundamental module spanned by states $|k\rangle$ with $k \in \mathbb{Z}^{\geq 0}$ being the number of boxes.

Is there any relation to the prefundamental representation?

Motivation: Study of the limit $q_1 \to \infty$ (or equivalently $q_1 \to 0$) of the quantum toroidal $\mathfrak{gl}(1)$ algebra. There are two interesting cases:[1]

$$\text{LI:} \quad q_1 \to \infty, \quad q_2 \to 0, \quad q_1 q_2 \text{ fixed},$$

$$\text{LII:} \quad q_1 \to \infty, \quad q_2 \text{ fixed}, \quad q_3 \to 0.$$

In both cases,

$$\frac{(z - q_1 w)(z - q_2 w)(z - q_3 w)}{(z - q_1^{-1} w)(z - q_2^{-1} w)(z - q_3^{-1} w)} \to \frac{q^{-2} z - w}{z - q^{-2} w}, \quad q^2 = q_3 \text{ or } q_2.$$

$\rightsquigarrow$ The algebraic relations of the currents reduce to those of quantum affine $\mathfrak{sl}(2)$!

[1] The symmetry under permutation of (q_1, q_2, q_3) is broken by the choice of central element $C = q_3^c$ or the choice of representation.

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⇝ The algebraic relations of the currents reduce to those of quantum affine $\mathfrak{sl}(2)$!

⚠ However:

⇝ The relation $\Psi_0^+\Psi_0^- = \Psi_0^-\Psi_0^+ = 1$ does not necessarily hold! ⇒ **asymptotic algebra**.

⇝ It is a **formal** limit: it holds for the currents, not their modes (VOA?).

[1] The symmetry under permutation of (q_1, q_2, q_3) is broken by the choice of central element $C = q_3^c$ or the choice of representation.

Limits of the toroidal algebra:

- In the limit LII, the representation $\iota_P \rho_F$ with where ρ_F is the Fock representation and $P(z) \propto (1 - z/(vq_1))$ tends to the prefundamental representation of $\mathcal{U}^{(0,-1)}$.
- The Fock representation (without shifts) tends to an infinite sum of prefundamental representations,

$$\rho_{\mathsf{F}} \to \bigoplus_{\mu} \rho_{\mathsf{prefund.}}$$

$\rightsquigarrow$ highest weights states: $|\lambda\rangle$ with $\lambda_1 = \lambda_2$ ($\mu = (\lambda_2, \lambda_3, \cdots)$ and $k = \lambda_1 - \lambda_2$).

- In the limit LI, the Fock representation tends to a sum of one dimensional and two-dimensional representations (add a box on the diagonal). $\rightsquigarrow$ not so interesting...

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Related result:

- Vector representation ρ_V of the toroidal algebra on $|k\rangle$, $k \in \mathbb{Z}$

$\rightsquigarrow$ $\iota_P^* \rho_V$ becomes highest weight (zeros of $P(z)$ in $x^-(z)$).

A new vertex representation!

Theorem**[JEB 2107.10063]** Let $\mathcal{F}$ be the 2D free boson Fock space spanned by the states $J_{-\lambda_1}\cdots J_{-\lambda_n}|\alpha\rangle$ where $|\alpha\rangle = e^{\alpha Q}|0\rangle$ is the charged vacuum annihilated by $J_{k>0}$ (recall $[J_k, J_l] = k\delta_{k+l}$, $[J_0, Q] = 1$). The map $\rho : \mathcal{U}^{(0,-\infty)} \to End(\mathcal{F})$ defined by

$$\rho(X^+(z)) = e^{Q} e^{\sum_{k>0} \frac{z^k}{k}(1-q^{2k})J_{-k}} e^{-\sum_{k>0} \frac{z^{-k}}{k} J_k} q^{-2J_0},$$

$$\rho(X^-(z)) = e^{-Q} e^{-\sum_{k>0} \frac{z^k}{k}(1-q^{-2k})J_{-k}} e^{\sum_{k>0} \frac{z^{-k}}{k} J_k},$$

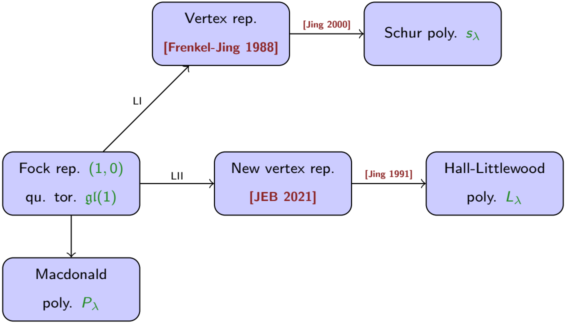
$$\rho(\Psi^+(z)) = 0, \quad \rho(\Psi^-(z)) = e^{\sum_{k>0} \frac{z^k}{k}(q^{-2k}-q^{2k})J_{-k}} q^{-2J_0},$$

is a representation of the shifted quantum affine $\mathfrak{sl}(2)$ algebra.

Remark: Analogue of vertex representations of quantum affine algebras (level $c = 1$).

[Frenkel, Jing 1988]

Action on symmetric polynomials

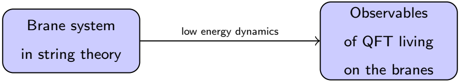


A few open questions:

- Classify the highest ℓ-weight representations of shifted quantum toroidal algebras.
- Rigorous derivation of the crystal limits LI and LII.
- General study of vertex representations for shifted quantum algebras.
- Drinfeld-Jimbo coproduct still a conjecture in general
  - $\rightsquigarrow$ R-matrix and shifted Yangians **[Hernandez, Zhang 2021]**
- Application to integrable systems? (finite dimensional representations of $\mathcal{E}^{\mu}$)

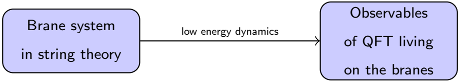
3. Application to string theory

Main problem



$\rightsquigarrow$ In general, very difficult but manageable when the system is supersymmetric!

Main problem



$\rightsquigarrow$ In general, very difficult but manageable when the system is supersymmetric!

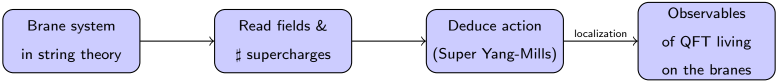
Gain insight on:

- branes dynamics
- non-perturbative phenomena in Quantum Field Theory (vortex/instantons)
- dualities (e.g. check by comparing index/partition functions)
- correspondences like AdS/CFT
- black hole physics ($\rightsquigarrow$ microstates counting)

$\oplus$ Observe correspondences with integrable systems (Bethe/gauge), 2D Conformal Field Theories (AGT or BPS/CFT), knot invariants,...

Algebraic engineering

Typical approach



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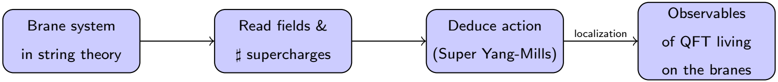
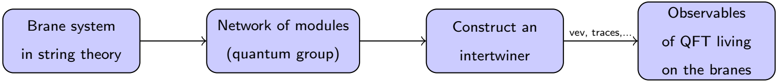


Algebraic approach



Main advantages:

- A gluing technique simplifies exact calculations

  (like the topological vertex in topological strings theory / “CFT methods”)
- Correspondence with W-algebras ($\rightsquigarrow$ AGT correspondence with 2D CFT)
- Connection with COHA of quantum Coulomb branches
- Description of non-perturbative dualities (using automorphisms)
- Integrable properties (Bethe/gauge correspondence?)

Bonus:

$\rightsquigarrow$ New results on representation theory of quantum groups, integrable systems,...

Summary of recent results

- Over the last few years, the algebraic engineering technique has been applied successfully to several supersymmetry QFT and their brane systems:

Theories	Algebra	Reference
5D $\mathcal{N}=1$	quantum toroidal $\mathfrak{gl}(1)$	**[Awata, Feigin, Shiraishi 2011]**
5D $\mathcal{N}=1$ on $\mathbb{Z}_p$-orbifold	quantum toroidal $\mathfrak{gl}(p)$	**[Awata, Kanno, et. al. 2017]**
5D $\mathcal{N}=1$ on $\mathbb{Z}_p$-orbifold	new quantum toroidal algebras!	**[JEB, Jeong 2019]**
4D $\mathcal{N}=2$	affine Yangian $\mathfrak{gl}(1)$	**[JEB, Zhang 2018]**
6D $\mathcal{N}=(1,0)$	elliptic toroidal $\mathfrak{gl}(1)$	**[Foda, Zhu 2018]**
3D $\mathcal{N}=2^*$	quantum toroidal $\mathfrak{gl}(1)$	**[Zenkevich 2018]**
3D $\mathcal{N}=2$	quantum affine $\mathfrak{sl}(2)$	**[JEB 2107.10063]**

- Other results:

⇝ D-type quiver gauge theories **[JEB, Fukuda, Matsuo, Zhu 2017]**

⇝ qq-characters observables (generating function of Wilson loops)

[JEB, Fukuda, Harada, Matsuo, Zhu 2017]

In a nutshell...

To apply the technique, we need the following ingredients:

1. A quantum group (mostly) determined by the spacetime
2. A *vertical* representation for Dp-branes

   (COHA of instanton/vortex moduli space)
3. A *horizontal* representation for NS5-branes
4. The intertwiners $\Phi : V \otimes H \to H$ and $\Phi^* : H \to V \otimes H$.

$\rightsquigarrow$ Glue the intertwiners to produce an operator $\mathcal{T}$, $\mathcal{Z} = \langle \mathcal{T} \rangle$.

In the case of 3D $\mathcal{N} = 2$ theories, we used:

1. Shifted quantum affine $\mathfrak{sl}(2)$ algebra
2. Prefundamental representation $k \equiv$ vortex charge
3. New vertex representation $\rho(\Psi^{-}(z)) = 0$
4. Custom made intertwiners
5. Obtain operator $\mathcal{T}$, deduce partition function and qq-characters.

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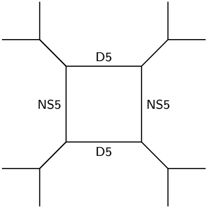
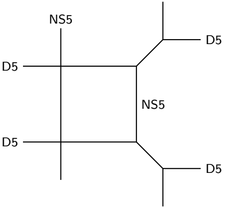
Remarks:

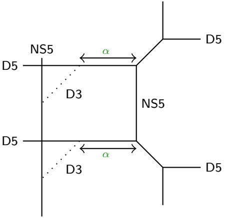
- It can be seen as the limit LII of the original 5D $\mathcal{N} = 1$ construction.

(based on the quantum toroidal $\mathfrak{gl}(1)$ algebra)

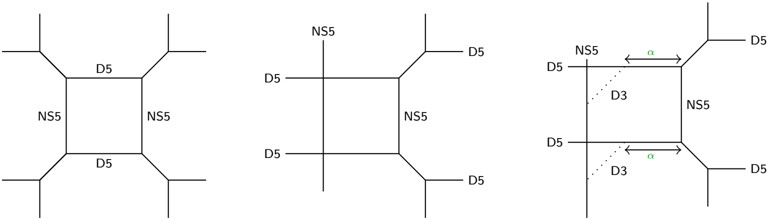
- The shift and the limit have an interpretation as Higgsing phenomena.







	0	1	2	3	4	5	6	7	8	9
Ω-bg		ϵ_2	ϵ_2	ϵ_1	ϵ_1				ϵ_3	ϵ_3
(p,q)	x	x	x	x	x	θ	θ			
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
D3	x	x	x					x		



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Ω-bg		ϵ_2	ϵ_2	ϵ_1	ϵ_1				ϵ_3	ϵ_3
(p,q)	×	×	×	×	×	θ	θ			
NS5	×	×	×	×	×	×				
D5	×	×	×	×	×		×			
D3	×	×	×					×		

Thank you!