Gravitational Radiation in General Spacetimes

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ADVENTURES IN GRAVITATION

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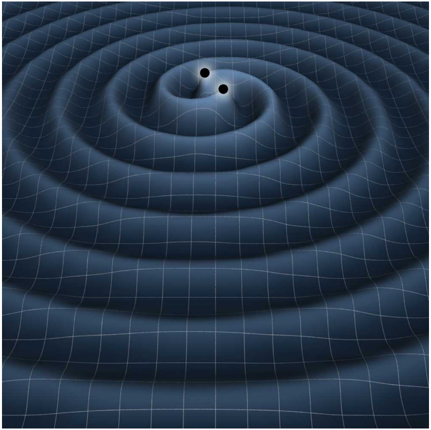
Overview

- Spacetimes and Radiation
- Investigating Spacetimes at Null Infinity
- Gravitational Waves
- New Structures
- Dynamics of General Asymptotically-Flat Systems

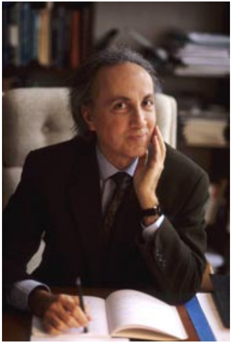


Photos: Courtesy of NASA; EHT.

GW: Measurements - Beginning of a New Era

- LIGO detected gravitational waves from binary black hole mergers for the first time in September 2015.
- Several times since then.
- LIGO and VIRGO together observed gravitational waves from a binary neutron star merger in 2017. At the same time, several telescopes registered data.
- Thibault Damour has made significant contributions to many areas of GR, one big topic is gravitational waves.

Gravitational Waves



Thibault Damour's work has been crucial to analyze the data from gravitational wave detectors.

• Indirect detection of gravitational waves from observations of the Hulse-Taylor pulsar: Thibault and Nathalie Deruelle computed the decrease in the orbital period of the binary system.

• Direct detection of gravitational waves: Thibault with Luc Blanchet described the motion of two black holes approaching each other, and Thibault with Alessandra Buonanno the final merger.

• More crucial work by Thibault including with many co-authors.

Einstein Equations

$$\mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\, \mathbf{g}_{\mu\nu}\, \mathbf{R} = \mathbf{8}\pi\, \mathbf{T}_{\mu\nu}\ , \tag{1}$$

with

$\mathbf{R}_{\mu\nu}$ the Ricci curvature tensor,

$\mathbf{R}$ the scalar curvature tensor,

$\mathbf{g}$ the metric tensor and

$\mathbf{T}_{\mu\nu}$ the energy-momentum tensor.

Investigate dynamics of spacetimes (M, g), where M a 4-dimensional manifold with Lorentzian metric g solving Einstein's equations (1).

Asymptotically Flat Spacetimes

Asymptotically Flat Spacetimes: Fall-off (in particular of metric and curvature components) towards Minkowski spacetime at infinity. Natural definition of null infinity $\Rightarrow$ understand gravitational radiation.

1. Well posedness: Celebrated results by Yvonne Choquet-Bruhat, and Choquet-Bruhat and Robert Geroch.
2. Study the Cauchy problem (i.e. initial value problem) for the Einstein equations for physical data. Asymptotically flat initial data.
3. Dynamics: Investigate open set of spacetimes resulting from open set of initial data from point 2.
4. Gravitational radiation and memory structures derived from investigations of point 3.
5. Approaching null infinity.

Gravitational Radiation

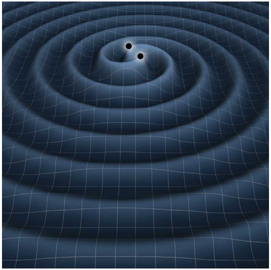
Fluctuation of curvature of the spacetime

propagating as a wave.

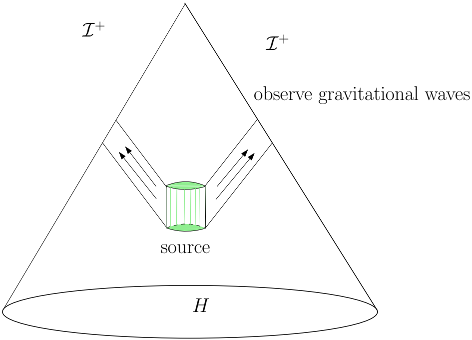
Gravitational waves:

Localized disturbances in the geometry propagate at the speed of light,

along outgoing null hypersurfaces.



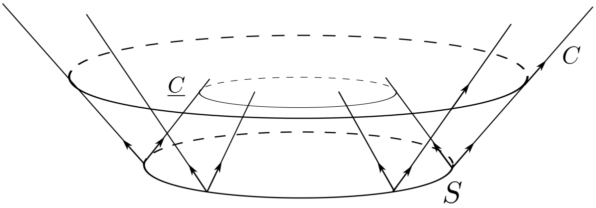
Picture: Courtesy of NASA.



Shears and Expansion Scalars

Viewing S as a hypersurface in C, respectively $\underline{C}$:

- Denote the second fundamental form of S in C by χ, and the second fundamental form of S in $\underline{C}$ by $\underline{\chi}$.
- Their traceless parts are the shears and denoted by $\hat{\chi}$, $\hat{\underline{\chi}}$ respectively.
- The traces $tr\chi$ and $tr\underline{\chi}$ are the expansion scalars.
- Null Limits of the Shears at future null infinity $\mathcal{I}^+$
  $\lim_{C_u, t \to \infty} r^2 \hat{\chi} = \Sigma(u)$ (in (A) spacetimes) and
  $\lim_{C_u, t \to \infty} r \hat{\underline{\chi}} = \Xi(u)$.



Global Solutions - Stability of Minkowski Space

The celebrated result by Demetrios Christodoulou and Sergiu Klainerman, 1991, proving the global nonlinear stability of Minkowski spacetime.

Theorem [D. Christodoulou and S. Klainerman for EV (1991)] (simplified version)

Every asymptotically flat initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.

General Spacetimes

A simplified version of the main theorem reads as follows. The original version of the theorem takes into account the detailed structures of the geometric components which requires several pages to state.

Theorem [L. Bieri (2007)]

Every asymptotically flat initial data obeying appropriate smallness assumptions (controlled via weighted Sobolev norms) gives rise to a globally asymptotically flat solution of the Einstein vacuum equations that is causally geodesically complete.

Small data ensures existence.

Large data

Main behavior along null hypersurfaces towards future null infinity

$\Rightarrow$ Largely independent from the smallness.

Asymptotic Flatness

(B) (Most general asymptotically-flat spacetimes.) Asymptotically flat initial data set in the sense of (B): an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3\,(r^{-\frac{1}{2}}) \tag{2}$$

$$k_{ij} = o_2\,(r^{-\frac{3}{2}})\,. \tag{3}$$

((CK) Christodoulou-Klainerman) Strongly asymptotically flat initial data set in the sense of (D): an initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighbourhood of infinity such that, as $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, $\bar{g}_{ij}$ and k_{ij} are:

$$\bar{g}_{ij} = (1 + \frac{2M}{r})\,\delta_{ij} + o_4\,(r^{-\frac{3}{2}}) \tag{4}$$

$$k_{ij} = o_3\,(r^{-\frac{5}{2}})\,, \tag{5}$$

where M denotes the mass.

Theorems for Large Data

Stability proofs that established the relevant properties of the spacetimes:

(CK) D. Christodoulou and S. Klainerman: 1993

(B) L. Bieri: 2007

Stability Theorems: For data as in definition (B) under a smallness condition $\Rightarrow$ established global existence and decay theorem for the Einstein vacuum equations.

Large data: It follows easily by a corollary that there exists a *complete domain of dependence of the complement of a sufficiently large compact subset of the initial hypersurface*. Thus, we have a solution spacetime with a portion of future null infinity corresponding to all values of the retarded time u not greater than a fixed constant.

$\Rightarrow$ This provides the solid foundation to investigate the asymptotic behavior at future null infinity for large data for (B) spacetimes, and to prove theorems on the nature of gravitational radiation.

Naturally, our investigations will extend to these spacetimes coupled to neutrinos via a null fluid.

Null components of the Weyl curvature W with the capital indices taking the values $1, 2$:

$$W_{A3B3} = \underline{\alpha}_{AB} \quad (6)$$
$$W_{A334} = 2\ \underline{\beta}_A \quad (7)$$
$$W_{3434} = 4\ \rho \quad (8)$$
$$^*W_{3434} = 4\ \sigma \quad (9)$$
$$W_{A434} = 2\ \beta_A \quad (10)$$
$$W_{A4B4} = \alpha_{AB} \quad (11)$$

Notation: Hodge duals $^*W$ and W^* defined as

$$^*W_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta}$$
$$W^*_{\alpha\beta\gamma\delta} = \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}$$

Let $\tau_-^2 = 1 + u^2$ and $r(t,u)$ is the area radius of the surface $S_{t,u}$, optical function u (retarded time).

Weyl curvature components

(CK)

$$
\begin{aligned}
\underline{\alpha}(W) &= O\left(r^{-1}\ \tau_-^{-\frac{5}{2}}\right)\\
\underline{\beta}(W) &= O\left(r^{-2}\ \tau_-^{-\frac{3}{2}}\right)\\
\rho(W) &= O\left(r^{-3}\right)\\
\sigma(W) &= O\left(r^{-3}\ \tau_-^{-\frac{1}{2}}\right)\\
\alpha(W),\ \beta(W) &= o\left(r^{-\frac{7}{2}}\right)
\end{aligned}
$$

(B)

$$
\begin{aligned}
\underline{\alpha} &= O\left(r^{-1}\ \tau_-^{-\frac{3}{2}}\right)\\
\underline{\beta} &= O\left(r^{-2}\ \tau_-^{-\frac{1}{2}}\right)\\
\rho,\ \sigma,\ \alpha,\ \beta &= o\left(r^{-\frac{5}{2}}\right)
\end{aligned}
$$

Structures in (B) Spacetimes

The Bianchi equations for $\not{D}_3\rho$ as well as $\not{D}_3\sigma$ are

$$\not{D}_3\rho + \frac{3}{2}tr\underline{\chi}\rho = -\not{div}\,\underline{\beta} - \frac{1}{2}\hat{\chi}\underline{\alpha} + (\varepsilon - \zeta)\underline{\beta} + 2\underline{\xi}\beta + \frac{1}{4}(D_3R_{34} - D_4R_{33}) \quad (12)$$

$$\not{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma = -\not{curl}\,\underline{\beta} - \frac{1}{2}\hat{\chi}^*\underline{\alpha} + \varepsilon^*\underline{\beta} - 2\zeta^*\underline{\beta} - 2\underline{\xi}^*\beta + \frac{1}{4}(D_\mu R_{3\nu} - D_\nu R_{3\mu})\varepsilon^{\mu\nu}{}_{34} \quad (13)$$

For small as well as large data, the following is a consequence of the relations between the shears, the shear and curvature, and the stability proof (B).

$$\hat{\chi} = [r^{-\frac{3}{2}}] + \{r^{-2}\tau_-^{+\frac{1}{2}}\} + l.o.t. \tag{14}$$

$$\underline{\hat{\chi}} = \{r^{-1}\tau_-^{-\frac{1}{2}}\} + [r^{-\frac{3}{2}}] + l.o.t. \tag{15}$$

Notation: In (B) spacetimes, we denote the part of $\hat{\chi}$ with decay $o(r^{-\frac{3}{2}})$ and which is non-dynamical (i.e. which does not evolve with u) by $[r^{-\frac{3}{2}}]$. Denote the leading order dynamical part of $\hat{\chi}$ (i.e. which evolves with u) by $\{r^{-2}\tau_-^{+\frac{1}{2}}\}$. More generally, for any of the non-peeling curvature components and any of the Ricci coefficients which have a leading order non-dynamical part, let $[\cdot]$ denote the leading order non-dynamical part (thus not evolving in u) of this component; and let $\{\cdot\}$ denote its leading order dynamical part (thus evolving in u).

By the proof (B) and the smallness conditions therein for the e_3-derivative of ρ, respectively σ:

$$\int_H r^4 |\rho_3|^2 \leqslant c\varepsilon$$

$$\int_H r^4 |\sigma_3|^2 \leqslant c\varepsilon$$

it is a consequence that

$$\rho_3 = O(r^{-3} \tau_-^{-\frac{1}{2}}) \quad , \quad \sigma_3 = O(r^{-3} \tau_-^{-\frac{1}{2}})$$

For small data it follows that

$$\rho \;=\; [r^{-\frac{5}{2}}] + \{r^{-3} \tau_-^{+\frac{1}{2}}\} + \{r^{-3}\} + \{r^{-3} \tau_-^{+\beta}\} + O(r^{-3} \omega^{-\alpha}) \quad (16)$$

and

$$\sigma \;=\; [r^{-\frac{5}{2}}] + \{r^{-3} \tau_-^{+\frac{1}{2}}\} + \{r^{-3}\} + \{r^{-3} \tau_-^{+\beta}\} + O(r^{-3} \omega^{-\alpha}) \quad (17)$$

with ω denoting r or τ_- and $\alpha > 0$, $0 < \beta < \frac{1}{2}$.
For large data, there are more terms present with a variety of decay, including terms in ρ, respectively σ, of the order $r^{-\frac{5}{2}} \tau_-^{-\alpha}$ with $\alpha > 0$.

Limits at null infinity $\mathcal{I}^+$

Limits at null infinity $\mathcal{I}^+$

More general phenomenon. Several quantities, which are defined locally on the surface $S_{t,u}$, do not attain corresponding limits on a given null hypersurface C_u as $t \to \infty$. However, the difference of their values at corresponding points on S_u and S_{u_0} does tend to a limit.

For instance, consider $\hat{\chi}$ defined locally on $S_{t,u}$. Even though $r^2 \hat{\chi}$ does not have a limit as $r \to \infty$ on a given C_u, the difference at corresponding points on S_u in C_u and on S_{u_0} in C_{u_0} does have a limit. In particular, these points being joined by an integral curve of e_3, the said difference attains the limit

$$\int_{u_0}^{u} \not{D}_3 \hat{\chi} \, du'$$

The part of $\hat{\chi}$ with slow decay of order $o(r^{-\frac{3}{2}})$ is non-dynamical, that is, it does not evolve with u. We see that this part does not tend to any limit at null infinity $\mathcal{I}^+$. Similarly, the components of the curvature that are not peeling have leading order terms that are non-dynamical (and do not attain corresponding limits at $\mathcal{I}^+$). Taking off these pieces gives us the dynamical parts of these (non-peeling) curvature components.

Theorem [L. Bieri (2007)]

For the spacetimes of types (B), the normalized curvature components $r\underline{\alpha}(W)$, $r^2\underline{\beta}(W)$ have limits on C_u as $t \to \infty$:

$$\lim_{C_u, t\to\infty} r\underline{\alpha}(W) = A_W(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^2\underline{\beta}(W) = \underline{B}_W(u,\cdot),$$

where the limits are on S^2 and depend on u. These limits satisfy

$$|A_W(u,\cdot)| \leqslant C(1+|u|)^{-3/2} \qquad |\underline{B}_W(u,\cdot)| \leqslant C(1+|u|)^{-1/2}.$$

Moreover, the following limit exists

$$-\frac{1}{2}\lim_{C_u, t\to\infty} r\underline{\widehat{\chi}} = \lim_{C_u, t\to\infty} r\hat{\eta} = \Xi(u,\cdot)$$

Further, it follows that

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{4}A_W \tag{18}$$

$$\underline{B} = -2 \,\text{d}\!\!/\text{iv}\, \Xi \tag{19}$$

Curvature Components ρ, σ and Derivatives ρ_3, σ_3

$$\rho_3 - \underbrace{A_\rho(r,u,\cdot)}_{\text{will cancel in Bianchi equ.}} = \underbrace{\rho_{\frac{1}{2}}(r,u,\cdot) + \rho_\beta(r,u,\cdot) + B_\rho(r,u,\cdot)}_{\text{will impact gravitational radiation, more structures}} + l.o.t.$$

$$\sigma_3 - \underbrace{A_\sigma(r,u,\cdot)}_{\text{will cancel in Bianchi equ.}} = \underbrace{\sigma_{\frac{1}{2}}(r,u,\cdot) + \sigma_\beta(r,u,\cdot) + B_\sigma(r,u,\cdot)}_{\text{will impact gravitational radiation, more structures}} + l.o.t.$$

Theorem [L. Bieri (2020)]

For (B) spacetimes the following holds for the domain of dependence of the complement of a sufficiently large compact subset of the initial hypersurface. The quantities $r^3\rho_{\frac{1}{2}}, r^3\rho_\beta, r^3\sigma_{\frac{1}{2}}, r^3\sigma_\beta$ have limits on any null hypersurface C_u as $t \to \infty$. Namely, for $0 < \beta < \frac{1}{2}$,

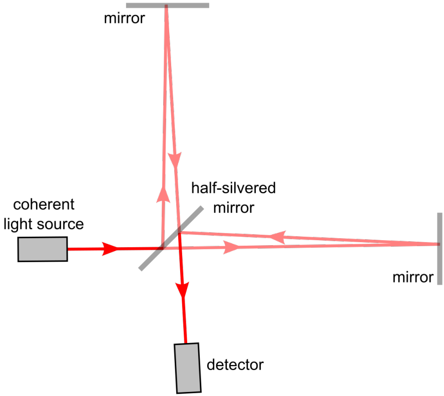
$$\lim_{C_u, t\to\infty} (r^3 \rho_{\frac{1}{2}}) = \mathcal{R}_{\frac{1}{2}}(u,\cdot) \quad , \qquad \lim_{C_u, t\to\infty} (r^3 \rho_\beta) = \mathcal{R}_\beta(u,\cdot)$$

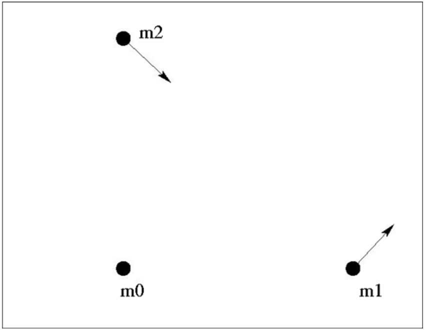
$$\lim_{C_u, t\to\infty} (r^3 \sigma_{\frac{1}{2}}) = \mathcal{S}_{\frac{1}{2}}(u,\cdot) \quad , \qquad \lim_{C_u, t\to\infty} (r^3 \sigma_\beta) = \mathcal{S}_\beta(u,\cdot)$$

$$\left|\mathcal{R}_{\frac{1}{2}}(u,\cdot)\right| \leqslant C\,(1+|u|)^{-1/2} \ , \ |\mathcal{R}_\beta(u,\cdot)| \leqslant C\,(1+|u|)^{-1+\beta}$$

$$\left|\mathcal{S}_{\frac{1}{2}}(u,\cdot)\right| \leqslant C\,(1+|u|)^{-1/2} \ , \ |\mathcal{S}_\beta(u,\cdot)| \leqslant C\,(1+|u|)^{-1+\beta}$$

From Mathematical Theory to Physics and Observation





Memory Effect of Gravitational Waves

Yakov B. Zel'dovich and Alexander G. Polnarev,

Demetrios Christodoulou,

Thibault Damour, Luc Blanchet

Early works followed up by a number of authors.

In the last few years: Contributions from various matter- and energy fields to gravitational wave memory: LB with Poning Chen, Shing-Tung Yau, David Garfinkle. Memory in ΛCDM cosmology: LB, Garfinkle, Nicolas Yunes.

Various questions investigated by many authors. Growing field of research.

A paper by P. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen suggests a method for detecting gravitational wave memory with aLIGO by stacking events.

Memory analogs in electromagnetic theory: LB and Garfinkle.

Memory - Permanent Displacement

Structures at $\mathcal{I}^+$. Intricate local structures have implications at $\mathcal{I}^+$. Certain geometric quantities take well-defined limits at $\mathcal{I}^+$ and obey specific equations.

The permanent displacement $\triangle x$ of geodesics (marked by test masses in a detector) is related to the difference $(Chi^- - Chi^+)$ at $\mathcal{I}^+$:

$$\triangle x = -\frac{d_0}{r}\left(Chi^- - Chi^+\right) , \tag{20}$$

where d_0 denotes the initial distance between the test masses, and Chi the null limit of a geometric quantity related to the shear (in spacetimes with stronger fall off it is the limit of the shear).

Contributions to the permanent displacement $\triangle x$:

AF systems with $O(r^{-1})$ fall off: "Simple" structure. The ordinary memory is sourced by the change in the radial component of the electric part of the Weyl tensor. The null memory is sourced by F, the energy per unit solid angle radiated to infinity (including shear and component of energy-momentum tensor).

(B) spacetimes: NEW and rich structures. Let's investigate these now.

Parity of Gravitational Waves and Memory

(M, g) denote our solution spacetimes.

The Weyl tensor $W_{\alpha\beta\gamma\delta}$ is decomposed into its electric and magnetic parts, which are defined by

$$E_{ab} := W_{atbt} \tag{21}$$

$$H_{ab} := \frac{1}{2}\varepsilon^{ef}{}_{a} W_{efbt} \tag{22}$$

Here ε_{abc} is the spatial volume element and is related to the spacetime volume element by $\varepsilon_{abc} = \varepsilon_{tabc}$. The electric part of the Weyl tensor is the crucial ingredient in the equation governing the distance between two objects in free fall. In particular, their spatial separation denoted by Δx^a:

$$\frac{d^2 \Delta x^a}{dt^2} = -E^a{}_b \Delta x^b \tag{23}$$

In this decomposition, it is

$$E_{NN} = \rho \quad , \quad H_{NN} = \sigma \ .$$

Electric and Magnetic Memory

Memory effect caused by the electric part of the curvature tensor
$\Rightarrow$ called *electric parity memory (i.e. electric memory)*.

Memory effect caused by the magnetic part of the curvature tensor
$\Rightarrow$ called *magnetic parity memory (i.e. magnetic memory)*.

So far

AF systems with $O(r^{-1})$ decay towards infinity
$\Rightarrow$ only electric parity memory, no magnetic memory occurs.

New (B, 2020)

AF spacetimes of slower decay like (B) spacetimes
$\Rightarrow$ magnetic memory occurs naturally.

Overall memory is growing and new structures arise.

Shown for the Einstein vacuum equations and Einstein-null-fluid equations describing neutrino radiation. The new results hold as well for the Einstein equations coupled to other fields of slow decay towards infinity and obeying other corresponding properties.

Derivation of Electric Memory

Einstein vacuum equations:

Consider the Bianchi equation for $\not{D}_3\rho$.

Notation $\rho_3 := \not{D}_3\rho \ + \ \frac{3}{2}tr\underline{\chi}\rho$.

In the Bianchi equation for $\not{D}_3\rho$

$$\not{D}_3\rho \ + \ \frac{3}{2}tr\underline{\chi}\rho \quad = \quad -di\not{v}\,\underline{\beta} - \frac{1}{2}\hat{\chi}\underline{\alpha} \ + \ (\varepsilon - \zeta)\underline{\beta} \ + \ 2\underline{\xi}\beta \qquad (24)$$

we focus on the higher order terms,

$$\rho_3 \ = \ - \quad \underbrace{di\not{v}\,\underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} \quad - \quad \underbrace{\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} \quad + \ l.o.t.$$

A short computation shows that

$$\rho_3 = - \underbrace{\not{d}iv\, \underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} - \underbrace{\frac{\partial}{\partial u}(\hat{\chi} \cdot \underline{\hat{\chi}})}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} + \underbrace{\frac{1}{4} tr\chi |\underline{\hat{\chi}}|^2}_{=O(r^{-3}\tau_-^{-1})} + l.o.t.$$

Thus it is

$$\rho_3 + \frac{\partial}{\partial u}(\hat{\chi} \cdot \underline{\hat{\chi}}) = -\not{d}iv\, \underline{\beta} + \frac{1}{4} tr\chi |\underline{\hat{\chi}}|^2 = O(r^{-3}\tau_-^{-\frac{1}{2}}) \quad (25)$$

Structures:

For large data, various terms of order $r^{-\frac{5}{2}}\tau_-^{-1-\alpha}$ with $\alpha \geqslant 0$ on the left hand side of (25), but these cancel.

Limit at $\mathcal{I}^+$ of the left hand side of (25)

$\Rightarrow$ leading order term originates from ρ_3 and is of order $O(r^{-3}\tau_-^{-\frac{1}{2}})$.

Future Null Infinity and Electric Memory

Notation for the corresponding limit of the LHS of (25):

$$\mathcal{P}_3 := \lim_{C_u, t\to\infty} r^3 \big(\rho_3 + \frac{\partial}{\partial u}(\hat{\chi}\cdot\underline{\hat{\chi}})\big) \tag{26}$$

$$\mathcal{P} := \int_u \mathcal{P}_3 \, du \tag{27}$$

Note that $\mathcal{P}$ is defined on $S^2 \times \mathbb{R}$ up to an additive function $C_{\mathcal{P}}$ on S^2 (thus the latter is independent of u). Later, when taking the integral $\int_{-\infty}^{+\infty} \mathcal{P}_3 \, du$, the term $C_{\mathcal{P}}$ will cancel.

Taking the limit of $\big(r^3 \text{ (25)}\big)$ on C_u as $t \to \infty$, each term on the right hand side takes a well-defined limit. This yields

$$\mathcal{P}_3 = -\not{d}\mathrm{iv}\, \underline{B} + 2|\Xi|^2 \tag{28}$$

Details for $\mathcal{P}$ in (27):
Taking into account all these structures we derive:
$\mathcal{P}$ has the following structure for $0 < \beta < \frac{1}{2}$ and $\gamma > 0$,

$$\mathcal{P} \quad = \quad \underbrace{\{\tau_-^{+\frac{1}{2}}\} \ + \ \{\tau_-^{\beta}\}}_{=\mathcal{P}_{\rho_1}} + \underbrace{\{\mathcal{F}(u, \cdot)\}}_{=\mathcal{P}_{\rho_2} - \frac{1}{2}D} + \{\tau_-^{-\gamma}\} \ + \ C_{\mathcal{P}} \qquad (29)$$

where $\mathcal{F}(u, \cdot) \leqslant C$. Again $\{\cdot\}$ as explained above. And $C_{\mathcal{P}}$ is an additive function on S^2 introduced above. Terms of order $O(\tau_-^{\alpha})$ with $0 < \alpha \leqslant \frac{1}{2}$ originate from the integral of the limits of the ρ_3 part. Denote this part by $\mathcal{P}_{\rho_1}$. In (29), the quantity $\mathcal{F}(u, \cdot)$ has pieces that are sourced by ρ_3 and pieces that are sourced by $\frac{\partial}{\partial u}(\hat{\chi} \cdot \underline{\hat{\chi}})$, we denote the former by $\mathcal{P}_{\rho_2}$ and the latter by $-\frac{1}{2}D$.

Next, we define

$$\begin{aligned} Chi_3 &:= \lim_{C_u, t \to \infty} \left(r^2 \frac{\partial}{\partial u} \hat{\chi} \right) && (30) \\ Chi &:= \int_u Chi_3 \; du && (31) \end{aligned}$$

We have (see before)

$$\underline{B} = -2 \not{div} \, \Xi \quad , \quad Chi_3 = -\Xi \tag{32}$$

Using these with the above we obtain

$$\mathcal{P}_3 = -2 \not{div} \not{div} \, Chi_3 + 2|\Xi|^2 \tag{33}$$

Integrating (33) with respect to u gives

$$(\mathcal{P}^- - \mathcal{P}^+) - \int_{-\infty}^{+\infty} |\Xi|^2 \; du \; = \; \not{div} \not{div} \, (Chi^- - Chi^+) \tag{34}$$

In $(\mathcal{P}^- - \mathcal{P}^+)$ an abundance of new terms, leading order $|u|^{+\frac{1}{2}}$.

Derivation of Magnetic Memory

Consider the Bianchi equation for $\not{D}_3\sigma$.

Notation $\sigma_3 = \not{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma$. In the Bianchi equation for σ_3

$$\sigma_3 \;=\; -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\;{}^*\underline{\alpha} + \varepsilon\;{}^*\underline{\beta} - 2\zeta\;{}^*\underline{\beta} - 2\underline{\xi}\;{}^*\beta$$

we concentrate on the higher order terms

$$\sigma_3 \quad = \quad -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\;{}^*\underline{\alpha} + l.o.t. \tag{35}$$

A short computation yields

$$\sigma_3 + \frac{\partial}{\partial u}(\hat{\chi}\wedge\underline{\hat{\chi}}) \quad = \quad -c\not{u}rl\,\underline{\beta} \;=\; O(r^{-3}\tau_-^{-\frac{1}{2}}) \tag{36}$$

For $\hat{\chi}\wedge\underline{\hat{\chi}}$ the orders of the terms are at the level of $\hat{\chi}\cdot\underline{\hat{\chi}}$ above.

Multiply the left hand side of (36) by r^3 and and take the limit on each C_u for $t \to \infty$ denoting this limit by $\mathcal{Q}_3$. Then introduce $\mathcal{Q}$ as follows:

$$\mathcal{Q}_3 := \lim_{C_u, t\to\infty} r^3 \big(\sigma_3 + \frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}})\big) \tag{37}$$

$$\mathcal{Q} := \int_u \mathcal{Q}_3 \, du \tag{38}$$

Note that $\mathcal{Q}$ is defined on $S^2 \times \mathbb{R}$ up to an additive function $C_{\mathcal{Q}}$ on S^2 (thus the latter is independent of u). Later, when taking the integral $\int_{-\infty}^{+\infty} \mathcal{Q}_3 \, du$, the term $C_{\mathcal{Q}}$ will cancel.

Taking the limit of $\big(r^3$ (36)$\big)$ on C_u as $t \to \infty$, the term on the right hand side takes a well-defined limit. This yields

$$\mathcal{Q}_3 = -c\not{u}rl\, \underline{B} \tag{39}$$

Details for $\mathcal{Q}$ in (38):
Taking into account all these structures we derive:
$\mathcal{Q}$ has the following structure for $0 < \beta < \frac{1}{2}$ and $\gamma > 0$,

$$\mathcal{Q} \quad = \quad \underbrace{\{\tau_-^{+\frac{1}{2}}\} \ + \ \{\tau_-^{\beta}\}}_{=\mathcal{Q}_{\sigma_1}} + \underbrace{\{\mathcal{F}(u, \cdot)\}}_{=\mathcal{Q}_{\sigma_2} - \frac{1}{2}G} + \{\tau_-^{-\gamma}\} \ + \ C_{\mathcal{Q}} \qquad (40)$$

where $\mathcal{F}(u, \cdot) \leqslant C$. Again $\{\cdot\}$ as explained above. And $C_{\mathcal{Q}}$ is an additive function on S^2 introduced above. Terms of order $O(\tau_-^{\alpha})$ with $0 < \alpha \leqslant \frac{1}{2}$ originate from the integral of the limits of the σ_3 part. Denote this part by $\mathcal{Q}_{\sigma_1}$. In (40), the quantity $\mathcal{F}(u, \cdot)$ has pieces that are sourced by σ_3 and pieces that are sourced by $\frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}})$, we denote the former by $\mathcal{Q}_{\sigma_2}$ and the latter by $-\frac{1}{2}G$.

Continue to compute using equation (39):

Consider (39) and employ the derived relations between $\hat{\chi}$, $\underline{\hat{\chi}}$ and $\underline{\beta}$ as well as the corresponding limits (19) and (32) to compute

$$\mathcal{Q}_3 \ = \ -2 \ c\not{u}rl \ di\not{v} \, Chi_3 \tag{41}$$

Integrating (41) with respect to u yields

$$(\mathcal{Q}^- - \mathcal{Q}^+) \ = \ c\not{u}rl \ di\not{v} \, (Chi^- - Chi^+) \tag{42}$$

In $(\mathcal{Q}^- - \mathcal{Q}^+)$ an abundance of new terms, leading order $|u|^{+\frac{1}{2}}$.

We obtain

$$(\mathcal{Q}^-_{\sigma_1} - \mathcal{Q}^+_{\sigma_1}) \ + \ (\mathcal{Q}^-_{\sigma_2} - \mathcal{Q}^+_{\sigma_2}) \ - \ \frac{1}{2}(G^- - G^+) \\ = \ c\not{u}rl \, di\not{v} \, (Chi^- - Chi^+) \tag{43}$$

- Behavior of $(\mathcal{Q}^- - \mathcal{Q}^+)$ as well as $c\not{u}rl \, di\not{v} \, (Chi^- - Chi^+)$:
  Fix a point on the sphere S^2 at fixed u_0 and consider $\mathcal{Q}(u_0)$. Next, take $\mathcal{Q}(u)$ at the corresponding point for some value of $u \neq u_0$. Keep u_0 fixed and let u tend to $+\infty$, respectively to $-\infty$. Then the difference $\mathcal{Q}(u) - \mathcal{Q}(u_0)$ is no longer finite, but it grows with $|u|^{+\frac{1}{2}}$. A corresponding argument holds for $Chi(u) - Chi(u_0)$.

- $(G^- - G^+)$ is finite. Contributions rooted in magnetic Weyl curvature and shears (shears: sourced by $\int_u \frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}}) \, du$).

- In AF systems with fall-off $O(r^{-1})$ towards infinity, each term in the above equation is identically zero.

- $\mathcal{Q}$ part features terms of diverging order $|u|^{+\frac{1}{2}}$, $|u|^{+\beta}$ for $0 < \beta < \frac{1}{2}$. Rooted in magnetic Weyl curvature.

Gravitational Waves: New Structures

There exist functions Φ and Ψ such that

$\not{d}iv \, (Chi^- - Chi^+) = \not\nabla \, \Phi + \not\nabla^{\perp} \Psi$.

Let $Z := \not{d}iv \, (Chi^- - Chi^+)$. Note that then the following holds:

$$\not{d}iv \, Z \; = \; \not\triangle \, \Phi \quad , \qquad c\not{u}rl \, Z \; = \; \not\triangle \, \Psi \quad .$$

We obtain the system on S^2, that is solved by Hodge theory.

$$\begin{aligned}
\not{d}iv \, (Chi^- - Chi^+) \; &= \; \not\nabla \, \Phi + \not\nabla^{\perp} \Psi && (44) \\
c\not{u}rl \, \not{d}iv \, (Chi^- - Chi^+) \; &= \; \not\triangle \, \Psi \\
&= \; (\mathcal{Q} - \bar{\mathcal{Q}})^- - (\mathcal{Q} - \bar{\mathcal{Q}})^+ && (45) \\
\not{d}iv \, \not{d}iv \, (Chi^- - Chi^+) \; &= \; \not\triangle \, \Phi \\
&= \; (\mathcal{P} - \bar{\mathcal{P}})^- - (\mathcal{P} - \bar{\mathcal{P}})^+ \\
&\quad \; -2(F - \bar{F}) && (46)
\end{aligned}$$

New quantities at diverging and finite orders in $\mathcal{Q}$ and $\mathcal{P}$ parts.

For the more general spacetimes of slow decay (like (B)) we conclude:

1. There is the new magnetic memory effect growing with $|u|^{\frac{1}{2}}$ sourced by $\mathcal{Q}$, rooted in the magnetic Weyl curvature and finite contributions rooted in curvature and the shears.

2. $\mathcal{Q}$ has further diverging terms at lower order.

3. There is the electric memory, previously established. This electric part is growing with $|u|^{\frac{1}{2}}$ sourced by $\mathcal{P}$, further lower-order growing terms and finite contributions from $\mathcal{P}$ and from F (the latter may be unbounded for systems of decay $O(r^{-\frac{1}{2}})$).

4. $c\not{u}rl\ d\not{i}v\,(Chi^- - Chi^+)$ being non-trivial allows for the magnetic structures to appear in gravitational radiation and to enter the permanent changes of the spacetime. Thus, these more general spacetimes generate memory of magnetic type.

Points 1, 2, 4 are NEW.

Point 3, the leading order behavior as well as the null memory were established in (B, 2018). The finer structures are new.

Adding Neutrinos

(B 2020) Einstein-null-fluid equations describing neutrino radiation:

$$R_{\mu\nu} \; = \; 8\pi \, T_{\mu\nu} \; .$$

Describe the neutrinos in this equation, represented via the energy-momentum tensor given by

$$T^{\mu\nu} \;\; = \;\; \mathcal{N} K^{\mu} K^{\nu} \tag{47}$$

with K being a null vector and $\mathcal{N} = \mathcal{N}(\theta_1, \theta_2, r, \tau_-)$ a positive scalar function depending on r, τ_-, and the spherical variables θ_1, θ_2.

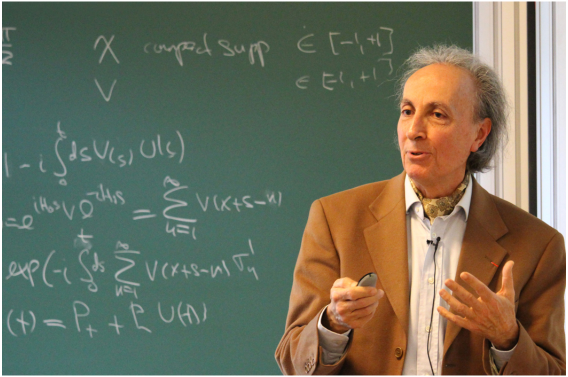
When coupled to the Einstein equations in the most general settings, the energy-momentum tensor $T^{\mu\nu}$ obeys those loose decay laws. No symmetry nor other restrictions imposed.

In particular, we do not have stationarity outside a compact set, but instead a distribution of neutrinos decaying very slowly towards infinity.

"Geometric terms": same growth rate as in EV case.

"T" terms: growing at rate $\sqrt{|u|}$.

Happy Birthday Thibault!



Joyeux Anniversaire Thibault!