MPM-PN explorations of inspiraling compact binaries
Down Indo-French Memory Lane

Bala Iyer

ICTS-TIFR,
Bangalore, India

Damour Fest: Adventures in Gravitation

IHES, France, 14 October 2021

INTERNATIONAL CENTRE for THEORETICAL SCIENCES
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Dear Thibault,
Wish I could be there at IHES in person
to recall some fond memories
and convey warm regards to you
from me and my family
on your seventieth birthday.
Thank you
for sharing your deep insights in physics,
Being there for me always
and
your Deep Friendship over the years.

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- Applied in 1989 to seek opportunities in the three areas.. Had met Thibault at Cargese and Goa: Applied to him for GW.. Thrilled it succeeded. Strangely had a Chandra connection (PN and GW)

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- By Christmas had the first cut results.. When finalized noticed corresponding results using tensor spherical harmonics (Campbell, Macek and Morgan) in Thorne's classic RMP on `Multipole Expansions of gravitational radiation` had incorrect coefficients.

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- On my return to India to keep in touch with the topic, I generalized it to the 1PN Current octupole and collaborated remotely with Thibault

Theme I

Quasi-circular Inspiral of Compact Binaries

Last three minutes.. Cutler et al PRL 1993

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- (iii) Inspiral can be treated in the adiabatic approximation as sequence of circular orbits.. Allows one to treat separately the conservative EOM problem, the radiation problem, and radiation reaction (phasing)

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- 2PN phasing: [Blanchet Damour Iyer, Will Wiseman, PRL 1995]
- The Multipolar Post Minkowskian (MPM) formalism matching to a PN source that may have appeared like an overkill for the Binary Pulsar Analysis, is a good example of the advantage a complete and mathematically rigorous treatment of a problem can bring for more demanding applications that could be around the corner

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  EOM: Harmonic [Luc Blanchet, G. Faye];
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- Breakthrough; Use of the gauge invariant *dimensional regularisation* in ADM EOM determined the unknown coeffs and solved the problem of EOM in ADM approach ([Damour, Jaranowski, Schäfer 2001]) - Piotr's TALK

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- DimReg of UV divergences. Hadamard Reg for IR.
  Use of DimReg in 3PN wave generation determined the unknown coeffs in MQ/GW flux ([Blanchet, Damour, Esposito-Farese, BI 2005]) with same renormalization of WL.
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- Computation of Spin-Orbit and Spin Spin terms in EOM and GW Flux - Waveform Phase and Amplitude ([Arun, Bohe, Blanchet, Faye, Marsat, Buonanno..]

MPM-PN formalism Today

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- Currently MPM-PN and Numerical Relativity are the scaffolds underlying the impressive Arch of EOB and Phenom template families used for detection, parameter estimation, TOG

Theme II

Applications to GW Data Analysis

GWDA Applications

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- Foundation of current Indian presence in TOG activity in the LIGO-Virgo collaboration.

Theme III

Beyond PN expansions by Resummation

Extending domains of PN expansions

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- Two EOB streams. Nagar, Damour et al at IHES; Buonanno et al at Maryland AEI
- Improved resummed templates based on multiplicative decomposition of the complex multipolar waveform [2008 Damour + BI + Nagar].
- Involved revisit and Completion of the 1990 work on current moments with Thibault to provide closed form expression for 1PN current moment and associated h_{lm} for $l + m$ odd. Kidder had provided the corresponding results for the 1PN mass moment and associated h_{lm} for $l + m$ even.
- 5.5PN Improved GW polarizations in test particle limit ([Fujita, BI 2010])

Theme IV

Quasi-Eccentric Inspiral of Compact Binaries

Compact Binaries in Eccentric orbits

- Gravitational Radiation Reaction (GRR) decreases the orbital eccentricity to negligible values by the epoch the emitted gravitational radiation enters the sensitive bandwidth of the interferometers. For an isolated binary, the eccentricity goes down roughly by a factor of three, when its semi-major axis is halved - ([Peters, 1964])

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- Even $e \sim 10^{-3} - 10^{-2}$ can produce parameter estimation biases ([Favata 2014]).

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- Dynamical formation predicts $\sim 5 - 10\%$ of BBH merging in globular clusters have $e > 0.1$ in LIGO freq. band ($> 10Hz$).

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- 3PN angular momentum flux and the secular evolution of orbital elements for ICB in quasi-elliptical orbits: [Sinha et al (2009)]

Instantaneous vs Hereditary Contributions

- For instantaneous terms in the energy flux, explicit closed form analytical expressions can be given in terms of dynamical variables related to relative speed v and relative separation r. These expressions can be conveniently averaged in the time domain over an orbit using its quasi-Keplerian representation. For 3PN fluxes we need the 3PN GQKR.

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- For hereditary contribution one can only write down formal analytical expressions as integrals over the past. More explicit expressions in terms of the dynamical variables require in addition a *model* of the binary's orbit to implement the integration over the past history.

Hereditary Contributions

- For circular orbits the tail integrals are evaluated using standard integrals for a *fixed* non-decaying circular orbit directly in the time-domain. ‘Remote-past’ contribution to the tail integrals can be proved to be negligible and errors due to inspiral by gravitation radiation reaction to be at least 4PN.

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- In the elliptic orbit case situation is more involved. Even after using the quasi-Keplerian parametrization, one cannot perform the integrals in the time domain (as for the circular orbit case), since the multipole moments have a more complicated dependence on time so that the integrals are not analytically solvable in simple closed forms. Consequently, one goes over to the Frequency domain..

1PN Quasi-Keplerian Reprn (QKR)

- To tackle hereditary terms at 2.5PN and 3PN we need to go beyond the (Newtonian) Keplerian representation to a 1PN QKR of the orbit ([Damour and Deruelle, 1985]). r, t and ϕ are expressed in terms of the eccentric anomaly u and true anomaly V as (ℓ is mean anomaly)

$$\begin{aligned} r &= a_r(1 - e_r \cos u)\,, \\ \frac{2\pi}{P} t = \ell &= u - e_t \sin u \\ \frac{2\pi}{\Phi}\phi &= V = 2 \arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2} \tan\frac{u}{2}\right]. \end{aligned}$$

- 1PN parametrization involves three kinds of eccentricities (e_r, e_t and e_ϕ). Parameters explicit functions of conserved energy and AM
- Motion is *doubly periodic*.
Radial motion $r(t)$ is periodic with period P; $n \equiv \frac{2\pi}{P}$
Angular motion $\phi(t)$ is periodic with a **different** period due to periastron precession.

$$\frac{2\pi}{\Phi} \equiv \frac{1}{K} \equiv \frac{1}{1+k}$$

3PN generalised Quasi-Keplerian reprn

3PN parametrization of the orbital motion of the binary
was constructed by Memmeshiemer, Gopakumar and Schäfer (2004)
in both ADM and modified harmonic coordinates.

$$
\begin{aligned}
r &= a_r \left(1 - e_r \cos u\right) , \\
l \equiv \frac{2\pi}{P}\,(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(V - u) \\
&\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right) \sin V + \frac{i_{6t}}{c^6} \sin 2V + \frac{h_{6t}}{c^6} \sin 3V , \\
\frac{2\pi}{\Phi}\,(\phi - \phi_0) &= V + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right) \sin 2V + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right) \sin 3V \\
&\quad + \frac{i_{6\phi}}{c^6} \sin 4V + \frac{h_{6\phi}}{c^6} \sin 5V , \\
\text{where,}\quad V &= 2 \arctan\left[\left(\frac{1 + e_\phi}{1 - e_\phi}\right)^{1/2} \tan\frac{u}{2}\right].
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Evolution of Orbital Elts under 5.5PN GRR

$$n = n(E, J)$$

- Differentiating with respect to t one obtains

$$\frac{dn}{dt} = \gamma_1(e_t, x, \nu)\,\frac{dE}{dt} + \gamma_2(e_t, x, \nu)\,\frac{d|\mathbf{J}|}{dt},$$

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  γ_1 and γ_2 PN expansions in powers of x.
- Use the balance equations,

$$\frac{dE}{dt} = -\frac{d\mathcal{E}}{dt},$$
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- Leads to secular evolution of orbital elements under 3PN gravitational radiation reaction (GRR) i.e. 5.5PN terms in the EOM ([Arun, Blanchet, BI, Sinha (2008)]).

Beyond Orbital Averages & Slow Secular Drifts

- To go beyond the adiabatic secular evolution one must go beyond the 'balance eqns' and examine the perturbed EOM
- If unperturbed EOM has sufficient integrals of motion to be integrable, the soln to the perturbed system is obtained by *variation of constants* in the generic soln of the unperturbed system. Method of Variation of Constants ([Damour 1983]) to deal with the three timescales (orbital period, periastron precession and radiation reaction) beyond the usual approximation of treating radiative time scale as an adiabatic process.

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$$l \equiv \int_{t_0}^{t} n\, dt + c_l(t)\,,$$

$$\lambda \equiv \int_{t_0}^{t} (1+k)\, n\, dt + c_\lambda(t)\,,$$

$$\frac{dc_\alpha}{dl} = G_\alpha(l; c_a)\ ;\ \alpha = 1, 2, l, \lambda\ ; a = 1\,,\,2\,.$$

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- Evolution of $c_\alpha(l)$ contains both a 'slow' (radiation-reaction time-scale) secular drift and 'fast' (orbital time-scale) periodic oscillations.

Method of variation of constants

- Introduce a two-scale decomposition for $c_\alpha(l)$

$$c_\alpha(l) = \bar{c}_\alpha(l) + \tilde{c}_\alpha(l)\,,$$

where $\bar{c}_\alpha(l)$ represents a slow drift (which can ultimately lead to *large* changes in the 'constants' c_α) and $\tilde{c}_\alpha(l)$ represents fast oscillations (which will stay always *small*, $\mathcal{O}(G_\alpha) = \mathcal{O}(c^{-5})$).

$$\frac{d\bar{c}_\alpha}{dl} = \bar{G}_\alpha(\bar{c}_a)\,,$$

$$\frac{d\tilde{c}_\alpha}{dl} = \tilde{G}_\alpha(l, \bar{c}_a)\,.$$

$$\bar{G}_\alpha(c_a) \equiv \frac{1}{2\pi}\int_0^{2\pi} dl\, G(l, c_a)\,,$$

$$\tilde{G}_\alpha(l; c_a) \equiv G_\alpha(l; c_a) - \bar{G}_\alpha(c_a).$$

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- Secular evolution of orbital elements generalizing Peters-Mathews
- Expressions for the orbital elements in terms of the conserved energy and angular momentum from GQKR and the assumption of energy and angular momentum balance equations to compute the secular evolution of the orbital elements like n, e_t, a_r and k under the gravitational radiation reaction in the quasi-elliptic case.

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- 3PN modes for compact binaries in eccentric orbits: Nonlinear Memory contributions: [Ebersold et al (2019)]

- Constructions of FD and TD templates for Detection, Parameter estimation of eccentric binaries
- Implications of eccentricity for detection, PE, TOG.
- After BBH[LVC, PRL,116, 061102 (2016)], BNS[LVC, PRL,119, 161101 (2017)], IMBH[LVC, PRL,125, 101102 (2020)], NSBH[LVK,ApJL,915, L5 (2021)], Asymmetric BBH (higher modes)[LVC,PRD 102, 043015 (2020), ApjL, 896, L44 (2020)]

Eccentric Binaries
could be the next category of GW event to be discovered
(known-unknown)

Some Current Eccentricity Research

- Hereditory, burst, eccentric behaviour [Loutrel, Yunes et al 2016, 17, 18]
- FD and TD eccentric inspiral [Tanay et al 2016]
- Solving PN accurate Kepler eqn (Mikkola) [Boetzel et al 2017]
- 3PN FD WF moderate and DA implications e [Moore, Yunes 2019 a,b]
- Modified gravity, eccentricity [Ma, Yunes 2019]
- Ready to use FD ICB moderate eccen [Tiwari et al 2019]
- Post circular, Pade for eccentric inspirals [Tiwari et al 2020]
- ENIGMA 2G, 3G [Chen et al 2021]
- PE correlns in low mass eccen GW151226, GW170608 [O'Shea e al 2021]
- Effective chip mass for inspiral freq evoln [Bose, Pai 2021]
- constraining orb eccen with aLIGO [Favata et al 2021]
- 2PN spinning, eccentric [Tanay et al 2021]
- Eccentric surrogate ..[islam et al 2021]
- Eccentric BNS for CE [Lenon et al 2021]
- Subsolar eccentric search [Nitz et al 2021]
- Eccen, spin-aligned EOB [Khalil et al 2021, Nagar et al 2021a,b]

Concluding Remarks

- Thibault's reviews on the problem of motion in the proceedings of Les Houches, Cargese, 300 years of Gravitation and those on the status of the equivalence principle have a lyrical clarity as he reminds us
  "*It is not sufficient to transplant in Einstein's Theory the technical steps of Newton's Theory but one needs to transmute within Einstein's conceptual framework the ideas that underlie the technical developments.*"

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- My initiation into GW research was the sabbatical with Thibault that allowed me to first chip in to model ICB with Luc and him and post 2009 to lend a shoulder via IndIGO to realize LIGO-India. Thanks Thibault for the initial opportunity and subsequent support that made it all possible later