Ultrarelativistic limit of gravity, cosmological singularities and E_{10}

Marc Henneaux

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Introduction

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The purpose of this talk is to describe Carroll-invariant theories and review their connection with the BKL behaviour.

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More generally, dynamical Carroll-invariant field equations reduce to ordinary differential equations with respect to time.

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Generalization to p-forms, gravity, higher spins in arbitrary dimension : M. Henneaux and P. Salgado-Rebolledo, [arXiv :2109.06708 [hep-th]] (see also J. de Boer, J. Hartong, N. A. Obers, W. Sybesma and S. Vandoren [arXiv :2110.02319 [hep-th]]

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Note that both limits are compatible with gauge invariance.

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This implies straightforwardly

$$[E, B_k] = 0 \qquad [B_k, B_m] = 0$$

for $E = \int d^3x \mathscr{E}$ and $B_k = \int d^3x\, x_k \mathscr{E}$ (with $\mathscr{E} = \mathscr{E}^E$ or $\mathscr{E}^M$).

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$$\mathcal{H} = G_{ijkm} \pi^{ij} \pi^{mn} - c^6 R\sqrt{g}, \qquad \mathcal{H}_i = -2\pi_i{}^j{}_{|j}.$$

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and one sees therefore that $c \to 0$ (Carroll limit) is equivalent to $G \to \infty$ (strong coupling limit, Isham 1975) or $\epsilon = 0$ (zero Hamiltonian signature limit -Teitelboim 1978), keeping N^{resc} finite.

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There is also a magnetic limit (Henneaux and Salgado-Rebolledo 2021).

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which leads to a billiard description in hyperbolic space of the dynamics.

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While the BKL limit corresponds to a Carrollian description where one takes the electric limit for gravity, the appropriate limit for p-forms can be either electric or magnetic.

11-dimensional supergravity

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(Note : The Chern-Simons term is in fact compatible with Carroll invariance and survives in the Carroll limit.)

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a limit that exhibits therefore remarkable connections with hidden symmetries !

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THANK YOU !