E_{10} and the Wave Function of the Universe

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En l'honneur de Thibault Damour

The Man of a Thousand Talents

What brought us together

Conjecture: reduction of maximal supergravity to one dimension should exhibit E_{10} symmetry. [Julia(1982)] $\Rightarrow$

Is this the symmetry underlying M theory? $\Leftrightarrow$ **Infinite increase in hidden symmetries of maximal supergravity and unification of matter and gravitational degrees of freedom via a single duality symmetry.**

Around 2000 Thibault became interested in E_{10}, and with Marc Henneaux we started working on $\mathrm{E}_{10}/\mathrm{K}(\mathrm{E}_{10})$ sigma model – not worrying too much about what E_{10} really is. [Damour,Henneaux,HN(2001)]

This is something we still don't know (nor does anyone else)!

This investigation revealed (among other things):

- **Crucial link with BKL analysis of cosmological singularities.**
- **A concrete proposal of what M theory could/should be**
- **... and many further advances (with A. Kleinschmidt)**

Quantum BKL

[BKL = Belinski,Khalatnikov,Lifshitz(1972);Damour,Henneaux,HN,CQG20(2003)R145]

Metric ansatz with *diagonal* metric components

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \sum_{a=1}^{d} e^{-2\beta^a(t)} \mathrm{d}x_a^2 \qquad \textbf{(Kasner metric)}$$

Substitute ansatz into Einstein-Hilbert action →

$$\mathcal{L} = n^{-1} G_{ab} \dot{\beta}^a \dot{\beta}^b \equiv \sum_a (\dot{\beta}^a)^2 - \left(\sum_a \dot{\beta}^a \right)^2 \quad \Rightarrow \quad \mathcal{H} = G^{ab} \pi_a \pi_b \approx 0$$

Induced *DeWitt metric* G_{ab} in β-space coincides with the *Cartan-Killing metric* of some indefinite Kac-Moody algebra restricted to CSA (for maximal supergravity $d = 10$ and KMA $= \mathrm{E}_{10}$).

Project motion onto *unit hyperboloid* in β-space

$$\beta^a = \rho \omega^a \quad , \quad \omega^a G_{ab} \omega^b = -1 \quad , \quad \rho^2 = -\beta^a G_{ab} \beta^b$$

with $\omega^a = \omega^a(z)$ parametrizing unit hyperboloid.

⇒ BKL limit: singularity is reached for $\rho \to \infty$ (‘Zeno time’).

'Integrating out' matter, off-diagonal and spatial curvature degrees of freedom → Effective potential $V(\beta)$ with Toda walls. Drastic simplification in BKL limit: sharp walls partition β-space into *chambers* → metric oscillations towards singularity.

Quantize ⇒ Wheeler-DeWitt (WDW) operator $(d > 2)$

$$\mathcal{H} \equiv G^{ab}\partial_a\partial_b = -\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right) + \rho^{-2}\triangle_{\rm LB} \qquad \Rightarrow \ \rho = \textbf{'time'}$$

with Laplace-Beltrami $\triangle_{LB}$ on $(d-1)$-dimensional unit hyperboloid → wave function obeys WDW equation

$$\mathcal{H}\Psi(\rho, z) = 0$$

To solve separate as $\Psi(\rho, z) = R(\rho)F(z)$ with eigenfunctions $-\triangle_{\rm LB}F(z) = EF(z)$. Then [Kleinschmidt,Köhn,HN:PRD80(2009)061701]

$$R_\pm(\rho) = \rho^{-\frac{d-2}{2}}\exp\left(\pm i\sqrt{E - \frac{(d-2)^2}{4}}\cdot\ln\rho\right) \qquad (d > 2)$$

For Laplace-Beltrami use $(d-1)$-dimensional upper half plane $z = (v, \mathbf{u})$ with $v \in \mathbb{R}_>$ and $\mathbf{u} \in \mathbb{R}^{d-2}$ ⇒

$$\triangle_{\rm LB} = v^{d-1}\partial_v(v^{3-d}\partial_v) + v^2\partial_{\mathbf{u}}^2$$

With Dirichlet boundary conditions use inequality

[e.g.:Iwaniec:Spectral Methods of Automorphic Forms,AMS(2000) → KKN(2009)]

$$-(\triangle_{\rm LB}F, F) \geq \int \mathrm{d}v\, \mathrm{d}^{d-2}u\, v^{3-d}(\partial_v F)^2$$

and rewrite

$$(F, F) = \int \mathrm{d}v\, \mathrm{d}^{d-2}u\; v^{1-d}F^2 = \frac{2}{d-2}\int \mathrm{d}v\, \mathrm{d}^{d-2}u\, v^{2-d}F\partial_v F$$

Cauchy-Schwarz inequality (with $-\triangle_{\rm LB}F = EF$) $\Rightarrow$

$$E \geq \frac{(d-2)^2}{4}$$

$\Rightarrow$ wave function vanishes at singularity ($\rho\!=\!\infty$)!

This is DeWitt's proposed mechanism for resolving classical singularities in quantum gravity! [DeWitt(1964)]

Complex **wave function $\Psi(\rho, z)$ → creates *arrow of time***

... could also be relevant for resolution of BH information paradox (if information does not get crushed in singularity it cannot be lost) [Perry(2021):2106.03715]

Automorphic properties of wave function

Reflective walls imply boundary conditions:

$$\Psi(\rho, z) = \pm\Psi(\rho, w_I \cdot z)$$

for fundamental Weyl reflections w_I with $-(+)$ for Dirichlet (Neumann) boundary conditions. For E_{10} we take

$$z = \mathbf{u} + \mathrm{i}v\ ,\ \ \bar{z} = \bar{\mathbf{u}} - \mathrm{i}v \qquad (\mathbf{u} \in \mathbb{O}\,,\ v > 0\ ;\ \mathrm{i}\mathbf{u} = \bar{\mathbf{u}}\mathrm{i})$$

→ *octonionic upper half plane.* This leads to 'modular' action of fundamental Weyl reflections

$$w_{-1}(z) = \frac{1}{\bar{z}}\ \ ,\quad w_0(z) = -\theta\bar{z}\theta\quad ,\quad w_j(z) = -\varepsilon_j\bar{z}\varepsilon_j$$

where θ highest root of E_8 and ε_j simple roots of E_8.

NB: E_8 root lattice can be endowed with the structure of a *non-commutative* and *non-associative* ring ('octavians' O), such that E_8 roots are *units* of this ring!

Weyl group of E_{10} = '$\mathrm{PGL}_2(\mathbf{O})$' [Feingold,Kleinschmidt,HN:JAlg322(2009)1295].

Even subgroup of Weyl group = $\mathrm{PSL}_2(\mathbf{O})$ is the analog of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ ($\rightarrow$ Einstein gravity).
[see also L.Forte,CQG26(2009)045001]

However: a proper theory of of automorphic forms for $\mathrm{PSL}_2(\mathbf{O})$ remains to be developed!

Restriction of wave function to fundamental domain = division by modular group for string amplitudes.

These results can be supersymmetrized $\rightarrow$ fermionic Fock space has 2^{160} components even for this simplest of approximations. [Kleinschmidt,Köhn,HN(2009)]

More manageable fermionic Fock space via quantization of $N=1, D=4$ supergravity in Bianchi IX background [Damour,Spindel:PRD95(2017)126011] **$\rightarrow$ further evidence for AE_3 and $K(AE_3)$ in ordinary (Einstein) gravity.**

Geodesics on $\mathrm{E}_{10}/\mathrm{K}(\mathrm{E}_{10})$ are infinitely unstable

[Damour,HN:CQG22(2005)2849]

Recall geodesic deviation (Jacobi) equation

$$\delta\ddot{\xi}^{\mu} = \mathcal{R}^{\mu}{}_{\nu\lambda\rho}\,\dot{\xi}^{\nu}\,\delta\xi^{\lambda}\,\dot{\xi}^{\rho}$$

Choose CSA geodesic $\xi(t) \equiv \xi^a(t)H_a$ **(**$\Rightarrow \ddot{\xi}^a(t) = 0$**) with deviation in direction of** E^s_α ***i.e.*** $\delta\xi = \sum_{\alpha,s}\delta\xi^\alpha_s(E^s_\alpha + E^s_{-\alpha})$.
$\Rightarrow$ **sectional curvature with generic** $v \in$ **CSA:**

$$\mathcal{R}(v, E^+_\alpha, v, E^+_\alpha) = -(v^a\alpha_a)^2 < 0$$

is negative and decreases without bound: for every imaginary root α**, any multiple** $n\alpha$ **is also a root** $\forall n \in \mathbb{N}$
$\Rightarrow\ \delta\ddot{\xi}^{n\alpha}_s = n^2(v\cdot\alpha)^2\,\delta\xi^{n\alpha}_s \ \Rightarrow\ \delta\xi^{n\alpha}_s(t) \propto e^{n|v\cdot\alpha|t}$ **for** $E^+_{n\alpha}$.

This strongly suggests that analysis cannot be confined to diagonal metrics and BKL regime, but *must take into account all off-diagonal degrees of freedom.*

Extension to E_{10}?

[...ongoing work with Axel Kleinschmidt]

Classically, *bosonic* Hamiltonian of $D = 11$ supergravity (*alias* M theory) and E_{10} Casimir *agree* up to level $\ell = 3$ [Damour,Kleinschmidt,HN(2006)]. **Suggests: $\Box\Psi = 0$ with "$\Box$" = generalized WDW operator $\Leftrightarrow$**

$$\text{“}\Box\text{”} = \mathrm{E}_{10}\ \textbf{Casimir} = \frac{1}{2}H^a H_a + \varpi^a H_a + \sum_{\alpha>0} \sum_{s=1}^{\mathrm{mult}(\alpha)} E^s_{-\alpha} E^s_\alpha \qquad (*)$$

with all generators represented by (monstrous!) differential operators → no ordering ambiguities. However:

- **Mismatches for $\ell \geq 3$ (starting with 'dual graviton')**
- **Proposal does not incorporate fermions**
- **$\Box\Psi = 0$ → only trivial representation if Ψ is in highest weight irrep, otherwise (for non h.w. irreps) ill-defined Casimir??**

So this is definitely not the complete story!

Fermions & $\mathrm{K(E_{10})}$: *la clé au royaume*?

Fermions transform in spinorial representation of 'maximal compact' subgroup $\mathrm{K(E_{10})} \subset \mathrm{E_{10}}$. So far, only unfaithful *finite-dimensional* spinorial representations of $\mathrm{K(E_{10})}$ are known. [Damour,Kleinschmidt,HN;deBuyl,Henneaux,Paulot(2006)]

For affine case $\mathrm{K(E_9)} \subset \mathrm{E_9}$ we now know *infinitely many* such representations of ever increasing dimension [Kleinschmidt,Köhl,Lautenbacher,HN:2102.00870[math.RT]] $\rightarrow$ **can we 'exhaust' $\mathrm{K(E_{10})}$ with an infinite sequence of quotient groups??**

Question: does faithful incorporation of fermions require *'non-abelian' bosonization* $\rightarrow$ could enlarge bosonic coset space $\mathrm{E_{10}/K(E_{10})}$ back to full $\mathrm{E_{10}}$?

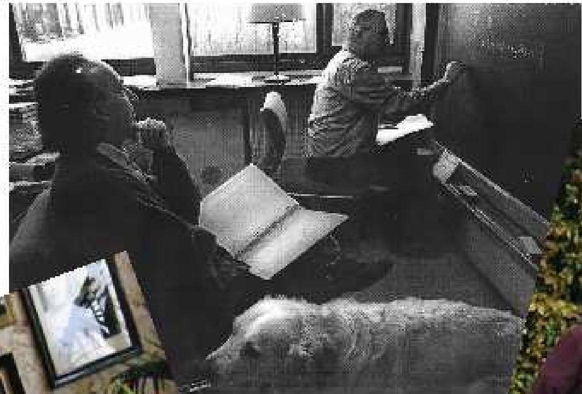
Hints from CFT [Goddard,Nahm,Olive(1985)]**: replace fermions in $\mathbf{128}_c$ by 120 bosons $\rightarrow$ $\mathbf{128}_s \oplus \mathbf{120}$ = adjoint of $\mathbf{E}_8$!**

... at the end of the day something like $(*)$ might still work for *full* theory *with* fermions...

Main Physics Questions

- **Can a sensible Wheeler-DeWitt operator be defined for E_{10}?**
- **Is Ψ a generalized 'modular form'?** [also:Brown,Ganor,Helfgott(2004)].
- **Does wave function Ψ *necessarily* involve *all* off-diagonal degrees of freedom close to singularity? $\Rightarrow$**
- **'Diffusion' into a huge phase space as singularity is approached? $\to$ information not lost, but may become unrecoverable.**
- **Does wave function continue to vanish at singularity? Exponential decay along off-diagonal directions is suggested by known results on automorphic functions** [Fleig,Gustavsson,Kleinschmidt, Persson: "Eisenstein Series and automorphic representations",C.U.P.(2018)]
- **Is life at the singularity infinitely simple or infinitely complex?** [cf. HN,CQG38(2021)187001]

$\to$ there is a vast *terra incognita* out there waiting to be discovered and explored...



JOYEUX ANNIVERSAIRE, THIBAULT!