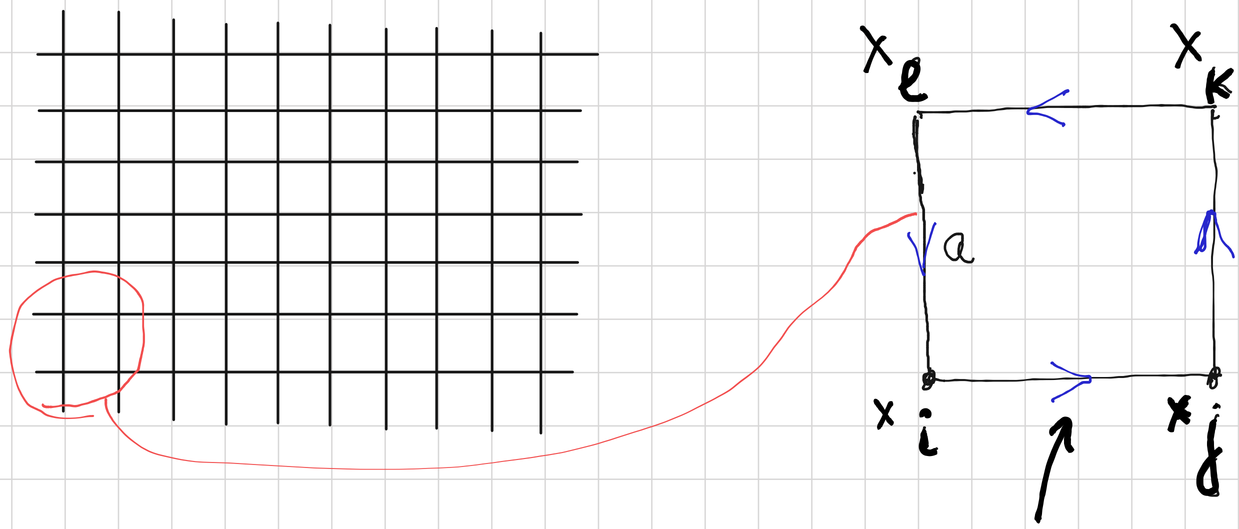
Discrete Gravity

A. Chamseddine, V. Mukhanov

Lattice gauge theory



$$U_{ij} = \exp\left(i a A_{\mu}(x)\right)$$

$$x = \frac{x_i + x_j}{2}$$

$$\hat{\mu} = \frac{x_j - x_i}{a}$$

gauge field

$$S(P) = \mathrm{Re}\,\mathrm{tr}\, U_{ij} U_{jk} U_{kl} U_{li}$$

$S(P)$ is invariant under local $U(n)$ transformations

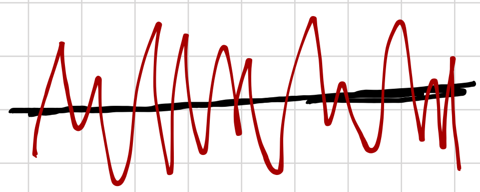
$$U_{ij} \to V_i^{\dagger} U_{ij} V_j$$

Lattice gravity ??

- Motivation

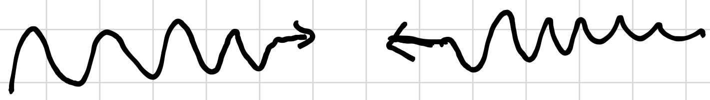
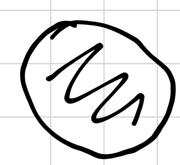
l_{pl}

One cannot check scales smaller than Planck scales

- 

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$|h_{\mu\nu}| > 1 \text{ at } l < l_{pl}$$

-  $\Longrightarrow$ 

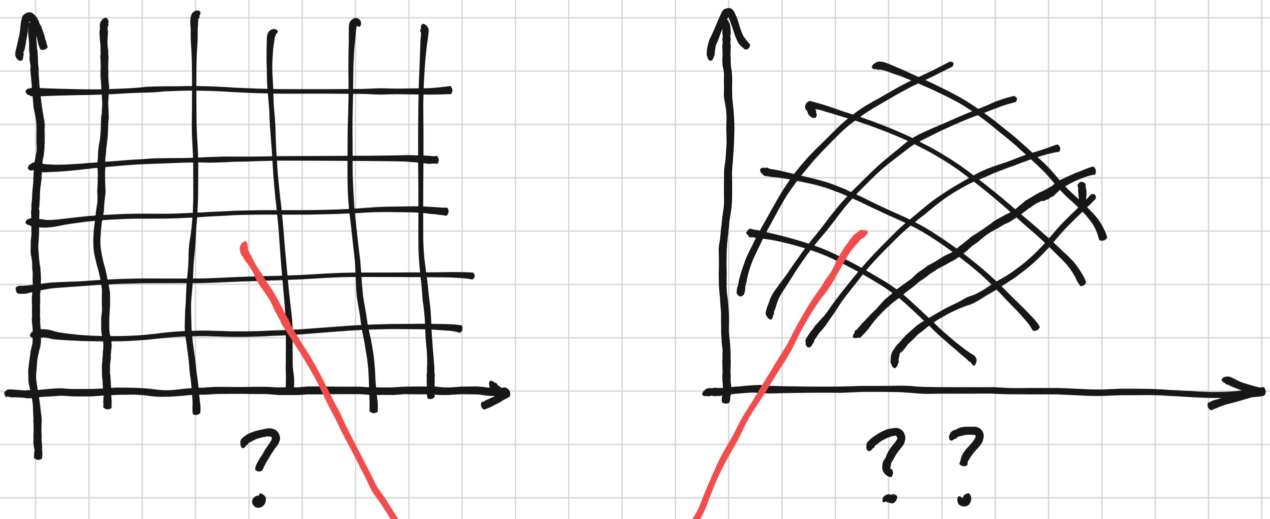
$E > m_{pl}$

B.H with

$r_g \sim E > m_{pl}$

- In QFT there is infinite number degrees of freedom in sub-Plackian scales, but it would be more logical to have finite number d. of freedom per Planck volume $\Rightarrow$ manifold is discrete and consist of elementary cells where "points" are indistinguish.

- **Obstacles**

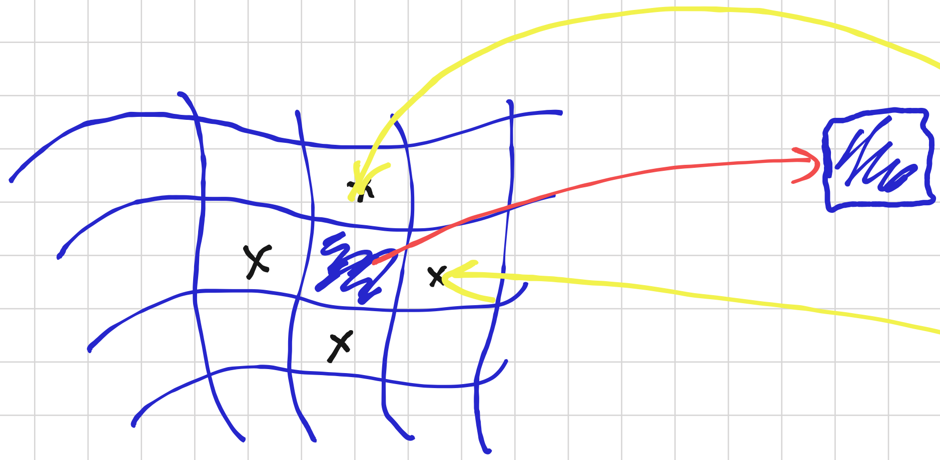


Planck volume

Dimension - ?

- In continuous case : d-dim manifold is defined as topological space for which each points has a neighborhood homeomorphic to R^d

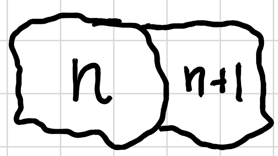
- In discrete case we define the dimension d assuming that each cell has 2d neighboring cells that share a common boundary with each individual cell



$$\mathbf{n} \equiv (n^1, n^2, \dots n^d) \equiv n^\alpha$$

In 2d (n^1, n^2) and (n^1+1, n^2), (n^1, n^2+1)

(n^1-1, n^2), (n^1, n^2-1) are neighboring cells



$f(n)$, $f(n+1)$, ...

with each cell we associate finite number degrees of freedom (for instance f can be a scalar function) $\equiv$ compare with QM

- Shift operators

$$\mathbf{E}_\beta(n) f(n^\alpha) \equiv f(n^\alpha + \delta^\alpha_\beta)$$

- Shift operators form a basis in a linear d-dim space in each cell

$$(a\,\mathbf{E}_\alpha + b\,\mathbf{E}_\beta)\, f(n^\gamma) = a f(n^\gamma + \delta^\gamma_\alpha) + b f(n^\gamma + \delta^\gamma_\beta)$$

$$\mathbf{E}^{-1}_\alpha(n) f(n) \equiv f(n - 1_\alpha)$$

- Tangent operators

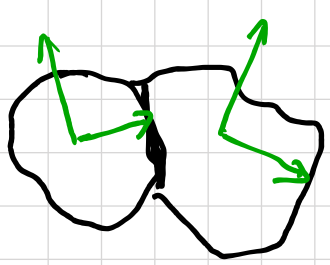
$$\mathbf{e}_\alpha(n) \equiv \frac{1}{2}\left(\mathbf{E}_\alpha(n) - \mathbf{E}_\alpha^{-1}(n)\right)$$

- 2d example $\quad \mathbf{e}_2(n^1,n^2)\, f(n^1,n^2) = \dfrac{f(n^1,n^2+1) - f(n^1,n^2-1)}{2}$

- Scalar product

$$\mathbf{V}_a \bullet \mathbf{V}_b = \delta_{ab}$$

d - vielbeins



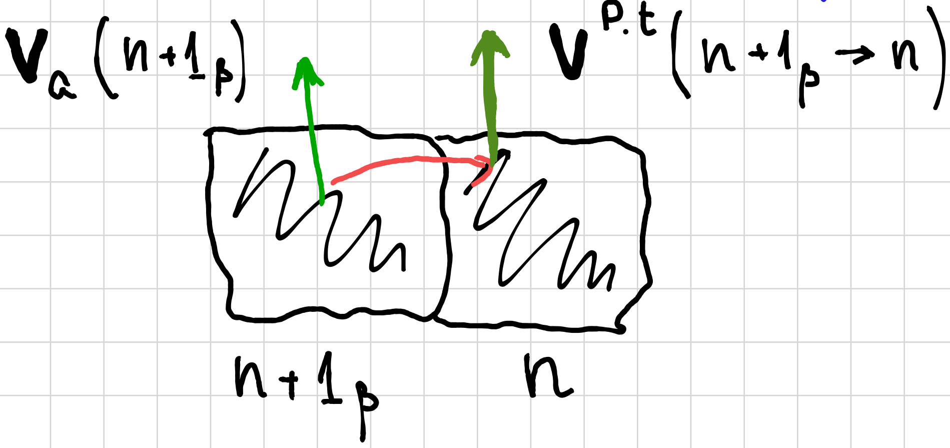
The local symmetry group is the group of rotations $SO(d)$

$$\mathbf{V}_a(n) \to \tilde{\mathbf{V}}_a(n) = R_a^b(n)\,\mathbf{V}_b$$

$$\mathbf{e}_\alpha(n) = e_\alpha^b(n)\,\mathbf{V}_b(n)$$

soldering form (16 in 4d, 9 in 3d)

- Parallel transport



$$\mathbf{v}_a^{p.t.}(n+1_\beta \to n) = (\Omega_\beta^{-1}(n))_a^b \mathbf{v}_b(n),$$

where $(\Omega_\beta^{-1}(n))_a^b$ is inverse of

spin connection group elements

$$\Omega_\beta(n) = \exp\left(\omega_\beta^{cd}(n) J_{cd}\right).$$

and J_{cd} are the generators of rotation group in the vector representation

$$\mathbf{e}_\alpha^{p.t.}(n+1_\beta \to n) = \mathbf{e}_\alpha(n) + \Gamma_{\alpha\beta}^\gamma(n)\mathbf{e}_\gamma(n)$$

- $\mathbf{e} \cdot \mathbf{v}$ is scalar $\Longrightarrow$

$$\Gamma_{\alpha\beta}^\gamma(n) e_{a\gamma}(n) = (\Omega_\beta(n))_a^b e_{b\alpha}(n+1_\beta) - e_{a\alpha}(n),$$

that is $\Gamma(n)$ can be entirelly expressed in terms of spin connections $\omega(n)$ and soldering form in cell n and nearby cells $n \pm 1$

- Torsion and Curvature

$$T^{\gamma}_{\alpha\beta} \equiv \Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha} = 0 \Longrightarrow$$

$$(\Omega_\beta(n))_a^{\ b} e_{b\alpha}(n+1_\beta) - e_{a\alpha}(n) = (\Omega_\alpha(n))_a^{\ b} e_{b\beta}(n+1_\alpha) - e_{a\beta}(n)$$

$$\Downarrow$$

ω can be entirely expressed in terms of soldering forms in n and $n+1$ cells

- under local $SO(d)$ transformations

$$\Omega_\beta(n) \to \widetilde{\Omega}_\beta(n) = \mathcal{R}(n)\Omega_\beta(n)\mathcal{R}^{-1}(n+1_\beta)$$

But

$$\Upsilon_\alpha(n) \equiv \Omega_\alpha(n) E_\alpha(n)$$

transforms covariantly

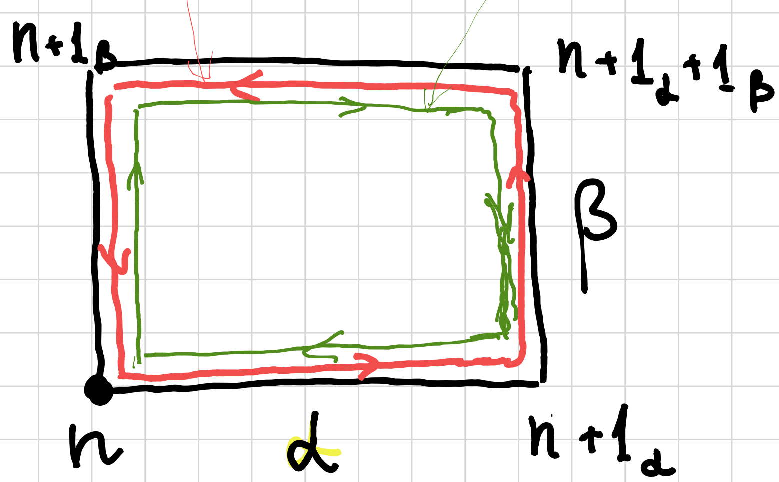
$$\Upsilon_\alpha(n) \to \widetilde{\Upsilon}_\alpha(n) = \mathcal{R}(n)\Upsilon_\alpha(n)\mathcal{R}^{-1}(n).$$

$$\Downarrow$$

The spin connection curvature

$$
\begin{aligned}
R_{\alpha\beta}(n) &= \frac{1}{2}\Big(\Upsilon_\alpha(n)\Upsilon_\beta(n)\Upsilon_\alpha^{-1}(n)\Upsilon_\beta^{-1}(n) - \Upsilon_\beta(n)\Upsilon_\alpha(n)\Upsilon_\beta^{-1}(n)\Upsilon_\alpha^{-1}(n)\Big) \\
&= \frac{1}{2}\Big(\Omega_\alpha(n)\Omega_\beta(n+1_\alpha)\Omega_\alpha^{-1}(n+1_\beta)\Omega_\beta^{-1}(n) - (\alpha \leftrightarrow \beta)\Big)
\end{aligned}
$$

is therefore obviously covariant with respect to $SO(d)$ local transformations



For $SO(2)$, $SO(3)$ and $SO(4)$

$$R_{\alpha\beta} = R_{\alpha\beta}{}^{ab} J_{ab}$$

$$R(n) = R_{\alpha\beta}{}^{ab}\, e_a^\alpha e_b^\beta$$

- Action

$$S = \sum_n v(n)\, R(n)$$

$$v(n)\bar{e}^\alpha(n)\,\Omega_\alpha(n) = v(n+1_\alpha)\Omega_\alpha(n)\bar{e}^\alpha(n+1_\alpha)$$

Example : 2d - case - $n=(n^1,n^2)$

$$\Omega_\alpha(n^1,n^2) = \cos\frac{1}{2}\omega_\alpha(n^1,n^2) + \tau\sin\frac{1}{2}\omega_\alpha(n^1,n^2), \tag{51}$$

where $\omega_\alpha(n^1,n^2) \equiv \omega_\alpha^{12}(n^1,n^2)$. Using freedom in choice of gauge and partitioning the manifold into cells (which in continuous limit corresponds to freedom in choice of coordinates) we can set

$$e_1^1 = e_2^2 = e\left(n^1,n^2\right), \quad e_2^1 = e_1^2 = 0. \tag{52}$$

at each cell $n=(n^1,n^2)$. In this case, the torsion-free conditions (27) simplify to

$$\begin{aligned} &e(n^1+1,n^2)\sin\omega_1(n^1,n^2) - e(n^1,n^2+1)\cos\omega_2(n^1,n^2) + e(n^1,n^2) = 0,\\ &e(n^1+1,n^2)\cos\omega_1(n^1,n^2) + e(n^1,n^2+1)\sin\omega_2(n^1,n^2) - e(n^1,n^2) = 0. \end{aligned} \tag{53}$$

Solving these equations gives the following explicit expressions for $\omega_\alpha(n^1,n^2)$ in terms of soldering forms in three adjacent cells

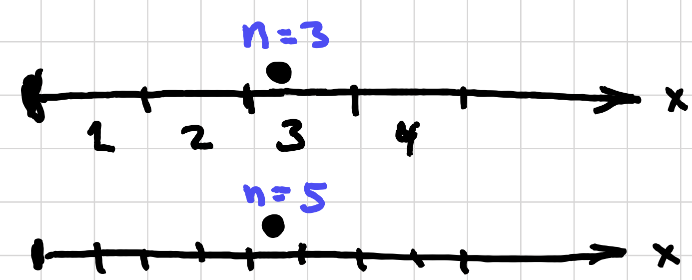
$$\begin{aligned} \omega_1(n^1,n^2) &= \frac{\pi}{4} - \arcsin\left(\frac{e^2(n^1+1,n^2) - e^2(n^1,n^2+1) + 2e^2(n^1,n^2)}{2\sqrt{2}e(n^1+1,n^2)e(n^1,n^2)}\right),\\ \omega_2(n^1,n^2) &= \frac{\pi}{4} - \arccos\left(\frac{e^2(n^1,n^2+1) - e^2(n^1+1,n^2) + 2e^2(n^1,n^2)}{2\sqrt{2}e(n^1,n^2+1)e(n^1,n^2)}\right), \end{aligned} \tag{54}$$

Substituting (49)-(51) into (38) and (39) we find that the only nonvanishing independent component of the curvature is

$$R_{12}{}^{12}(n) = 2\sin\left[\frac{1}{2}\left(\omega_2(n^1+1,n^2) - \omega_1(n^1,n^2+1) + \omega_1(n^1,n^2) - \omega_2(n^1,n^2)\right)\right], \tag{55}$$

- **Continuous limit**

$$x^\alpha = \varepsilon n^\alpha$$
$$\varepsilon \to 0, \quad n^\alpha \to \infty$$



$$E_\alpha(x) f(x) = f(x+\varepsilon_\alpha)$$

$$\mathbf{e}_\alpha(x) = \frac{1}{2\epsilon_\alpha}\left(\mathbf{E}_\alpha(x) - \mathbf{E}_\alpha^{-1}(x)\right) f(x) = \frac{f(x+\epsilon_\alpha) - f(x-\epsilon_\alpha)}{2\epsilon_\alpha}.$$

It follows that in the limit $\epsilon \to 0$,

$$\mathbf{e}_\alpha = \frac{\partial}{\partial x^\alpha},$$

$$\mathbf{v}_a^{p.t.}(x+\epsilon_\beta \to x) = \left(\Omega_\beta^{-1}(x)\right)_a^b \mathbf{v}_b(x),$$

where

$$\Omega_\beta(x) = \exp\left(\epsilon_\beta \omega_\beta^{cd}(x) J_{cd}\right)$$

Curvature

$$R_{\alpha\beta}(x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon_\alpha\epsilon_\beta}\left(\Omega_\alpha(x)\Omega_\beta(x+\epsilon_\alpha)\Omega_\alpha^{-1}(x+\epsilon_\beta)\Omega_\beta^{-1}(x) - (\alpha\leftrightarrow\beta)\right) = R_{\alpha\beta}{}^{ab} J_{ab}$$

and

$$R_{\alpha\beta}{}^{cd}(x) = \partial_\alpha \omega_\beta^{cd} - \partial_\beta \omega_\alpha^{cd} + \omega_\alpha^{cl}\omega_{\beta l}{}^{d} - \omega_\beta^{cl}\omega_{\alpha l}{}^{d},$$

$$S = \sum_n \mathcal{V}(n) R(n) \Longrightarrow S = \int \det(e^a_\alpha)\, R(x)\, dx^1 \dots dx^d$$

$$\sum_n \to \int dx^1 \dots dx^d \qquad \mathcal{V}(n) \to \det(e^a_\alpha) \qquad R(n) \to R(x)$$