Higher-Order Post-Newtonian ADM Dynamics of Compact Binary Systems: a DJS "Tour de Force"

Piotr Jaranowski

Faculty of Physics, University of Białystok, Poland

Damour Fest

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The "DJS" Collaboration: Post-Newtonian Two-Body Problem, Effective-One-Body Approach

- Thibault Damour,
  Institut des Hautes Études Scientifiques,
  Bures-sur-Yvette, France
- Gerhard Schäfer,
  Institute of Theoretical Physics,
  Friedrich Schiller University Jena, Germany
- Piotr Jaranowski,
  Faculty of Physics,
  University of Białystok, Poland

T. Damour

P. Jaranowski

G. Schäfer

1. Post-Newtonian Two-Body Problem:
   EOM + Gravitational-Wave Luminosities
2. 2.5PN-accurate Two-Body EOM: a History of Derivation
3. A DJS "Tour de Force": 3PN and 4PN Conservative
   2-Point-Mass ADM Hamiltonians
4. 3PN and 4PN Conservative Two-Body EOM:
   a History of Derivation
5. Other Applications of the ADM formalism
   for the PN Two-Body Problem
6. The Future of the PN Two-Body Problem

Post-Newtonian Two-Body Problem (Without Spins)

There are two sub-problems, usually analyzed separately:

- problem of deriving equations of motion (EOM),
- problem of computing gravitational-wave luminosities.

EOM	N	1PN	2PN	2.5PN	3PN	3.5PN	4PN	4.5PN	5PN*	5.5PN	6PN	6.5PN	7PN
Luminosity	—	—	—	N	—	1PN	1.5PN	2PN	2.5PN	3PN	3.5PN	4PN	4.5PN

Red color = worked out completely;
orange color = worked out almost completely (as far as I know).

*5PN will be completely done soon, due to the upcoming work of the DESY group (J. Blümlein, private communication).

EOM at orders N, 1PN, 2PN, and 3PN are purely conservative,
EOM at orders 2.5PN and 3.5PN are purely dissipative.

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2.5PN-Accurate Two-Body Equations of Motion (Without Spins)

- 0PN (Newtonian): Newton 1687.
- 1PN ($\propto v^2/c^2$): Lorentz & Droste 1917 (extended-body derivations),
  Einstein, Infeld, & Hoffmann 1938 (surface-integral method),
  Fock 1939 & Petrova 1940 (PhD thesis, published 1949 only)
  (extended-body derivations),
  Robertson 1938 (1PN periastron advance).
- 2PN ($\propto v^4/c^4$):
  Ohta, Okamura, Kimura, & Hiida 1974 (incomplete),
  Damour & Deruelle 1981 (not complete), Damour 1982,
  Damour & Schäfer 1985 (ADM-gauge Lagrangian),
  Kopeikin 1985 (extended-body derivation),
  Damour & Schäfer 1987 (2PN periastron advance via ADM Hamiltonian).
- 2.5PN ($\propto v^5/c^5$):
  Damour & Deruelle 1981 (not complete), Damour 1982,
  Grishchuk & Kopeikin 1983 (extended-body derivation),
  Schäfer 1985 (ADM-Hamiltonian-based derivation).

- Damour 1983—deriving the rate of the decay $\dot{P}$ of the orbital period of the two-body system directly from 2.5PN-accurate EOM;
  this work ended—at least to a large extent—the quadrupole formula controversy (which was vivid in the 70th/80th of the XX century).
  [See, e.g., D. Kennefick, *Traveling at the Speed of Thought*, Ch. 11.]

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Constraint Equations for 2-Point-Mass Systems

- Source terms for the constraint equations are derived from the 2-point-mass energy-momentum tensor

$$T^{\alpha\beta}(x^\mu) := \sum_{a=1}^{2} m_a \int_{-\infty}^{+\infty} \frac{u_a^\alpha u_a^\beta}{\sqrt{-\det(g_{\mu\nu})}} \delta^{d+1}\big(x^\mu - \xi_a^\mu(\tau_a)\big)\, \mathrm{d}\tau_a,$$

  τ_a is the proper time along the world line $x^\mu = \xi_a^\mu(\tau_a)$ of the ath particle, and $u_a^\alpha := \mathrm{d}\xi_a^\alpha/\mathrm{d}\tau_a$.

- The constraint equations:

$$\sqrt{\gamma}\, R - \frac{1}{\sqrt{\gamma}} \left(\gamma_{ik}\,\gamma_{j\ell}\,\pi^{ij}\,\pi^{k\ell} - \frac{(\gamma_{ij}\,\pi^{ij})^2}{d-1} \right) = \sum_{a=1}^{2} \sqrt{\boxed{\gamma_a^{ij}}\, p_{ai} p_{aj} + m_a^2}\, \delta^d(\mathbf{x} - \mathbf{x}_a),$$

$$-2D_j\pi^{ij} = \sum_{a=1}^{2} \boxed{\gamma_a^{ij}}\, p_{aj}\, \delta^d(\mathbf{x} - \mathbf{x}_a),$$

  $\gamma_{ij} := g_{ij}$, $\pi^{ij} := \sqrt{\gamma}(K^{ij} - \gamma^{ij}\gamma^{kl}K_{kl})$, K_{ij} is the extrinsic curvature and R is the spatial scalar curvature of the hypersurface $t = const$, D_j is the spatial d-dimensional covariant derivative (acting on a tensor density), $\gamma_a^{ij} := \gamma^{ij}(\mathbf{x}_a)$ is perturbatively unambigously defined and finite (at least up to the 4PN order).

Fixing the Gauge

- The ADM Transverse-Traceless (TT) gauge:

$$\gamma_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)} \delta_{ij} + h_{ij}^{\mathrm{TT}}, \quad \pi^{ii} = 0.$$

- Splitting of the field momentum:

$$\pi^{ij} = \tilde{\pi}^{ij}(V^k) + \pi^{ij}_{\mathrm{TT}},$$
$$\tilde{\pi}^{ij}(V^k) = \partial_i V^j + \partial_j V^i - \frac{2}{d}\,\delta^{ij}\,\partial_k V^k.$$

The super/subscript TT denotes the application of the d-dimensional (spatially nonlocal) TT-projection operator:

$$f_{ij}^{\mathrm{TT}} := \delta_{ij}^{\mathrm{TT}kl} f_{kl},$$

where
$$\delta_{ij}^{\mathrm{TT}kl} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{d-1}\delta_{ij}\delta_{kl} - \frac{1}{2}(\delta_{ik}\partial_j\partial_l + \delta_{jl}\partial_i\partial_k + \delta_{il}\partial_j\partial_k + \delta_{jk}\partial_i\partial_l)\Delta^{-1} + \frac{1}{d-1}(\delta_{ij}\partial_k\partial_l + \delta_{kl}\partial_i\partial_j)\Delta^{-1} + \frac{d-2}{d-1}\partial_i\partial_j\partial_k\partial_l\Delta^{-2}.$$

Reduced Matter+Field ADM Hamiltonian

- ϕ and V^i are expressed in terms of $(\boldsymbol{x}_a, \boldsymbol{p}_a, h_{ij}^{\mathrm{TT}}, \pi^{ij}_{\mathrm{TT}})$ by a perturbative solving of the constraint equations—this is done by PN expansion of ϕ and V^i (the numbers in parentheses denote the formal order in $1/c$):

$$\phi = \phi_{(2)} + \phi_{(4)} + \cdots , \quad V^i = V^i_{(3)} + V^i_{(5)} + \cdots .$$

- The constraints yield a system of elliptic equations for ϕ and V^i, which has the structure (h_{ij}^{TT} and π^{ij}_{TT} enter the ellipsis)

$$\Delta\phi = -\sum_a m_a(1 + \cdots)\,\delta^d(\boldsymbol{x} - \boldsymbol{x}_a) + \cdots ,$$

$$\Delta V^i + \left(1 - \frac{2}{d}\right)\partial_i\partial_j V^j = -\frac{1}{2}\sum_a (p_{ai} + \cdots)\,\delta^d(\boldsymbol{x} - \boldsymbol{x}_a) + \cdots .$$

- The reduced form of the total matter+field ADM Hamiltonian:

$$H_{\mathrm{red}}\left[\boldsymbol{x}_a, \boldsymbol{p}_a, h_{ij}^{\mathrm{TT}}, \pi^{ij}_{\mathrm{TT}}\right] = -\sum_{n=2}^{\infty}\int \mathrm{d}^d\boldsymbol{x}\,\Delta\phi_{(n)}\left[\boldsymbol{x}_a, \boldsymbol{p}_a, h_{ij}^{\mathrm{TT}}, \pi^{ij}_{\mathrm{TT}}\right].$$

The equations of motion for the particles:

$$\dot{\boldsymbol{p}}_a = -\frac{\delta H_{\mathrm{red}}}{\delta \boldsymbol{x}_a}, \quad \dot{\boldsymbol{x}}_a = \frac{\delta H_{\mathrm{red}}}{\delta \boldsymbol{p}_a} \quad (a = 1, 2).$$

Evolution equations for the field degrees of freedom:

$$\frac{\partial}{\partial t} h_{ij}^{\mathrm{TT}} = \delta_{ij}^{\mathrm{TT}kl}\frac{\delta H_{\mathrm{red}}}{\delta \pi^{kl}_{\mathrm{TT}}}, \quad \frac{\partial}{\partial t}\pi^{ij}_{\mathrm{TT}} = -\delta_{kl}^{\mathrm{TT}ij}\frac{\delta H_{\mathrm{red}}}{\delta h_{kl}^{\mathrm{TT}}}.$$

Conservative Matter Hamiltonian

- Legendre transformation with respect to the field variables leads to the Routhian,

$$R\left[\boldsymbol{x}_a, \boldsymbol{p}_a, h_{ij}^{\mathrm{TT}}, \dot{h}_{ij}^{\mathrm{TT}}\right] := H_{\text{red}}\left[\boldsymbol{x}_a, \boldsymbol{p}_a, h_{ij}^{\mathrm{TT}}, \dot{h}_{ij}^{\mathrm{TT}}\right] - \int \mathrm{d}^d x\, \pi_{\mathrm{TT}}^{ij} \dot{h}_{ij}^{\mathrm{TT}}.$$

- Elimination of the field variables h_{ij}^{TT}, $\dot{h}_{ij}^{\mathrm{TT}}$: they are "integrating out", i.e., replaced by time-symmetric solutions (as a functional of the particle variables) of their field equations,

$$H_{\text{cons}}[\boldsymbol{x}_a, \boldsymbol{p}_a] := R\left[\boldsymbol{x}_a, \boldsymbol{p}_a, h^{\mathrm{TT}}_{\text{sym}\,ij}(\boldsymbol{x}; \boldsymbol{x}_a, \boldsymbol{p}_a), \dot{h}^{\mathrm{TT}}_{\text{sym}\,ij}(\boldsymbol{x}; \boldsymbol{x}_a, \boldsymbol{p}_a)\right],$$

where time derivatives of $\boldsymbol{x}_a$ and $\boldsymbol{p}_a$ are eliminated through the use of lower-order equations of motion.

- Work of Blanchet & Damour (1988):
  - — starting at the 4PN level it is impossible to express (in any gauge) the near-zone metric as a functional of the instantaneous state of the source: 4PN metric is the sum of an instantaneous functional of the source variables and of a nonlocal-in-time tail contribution.

  One has to add the time-symmetric part of the tail contribution to the near-zone conservative Hamiltonian, so the total Hamiltonian reads

$$H_{\text{cons}}[\boldsymbol{x}_a, \boldsymbol{p}_a] = H_{\text{cons}}^{\text{near-zone}}(\boldsymbol{x}_a, \boldsymbol{p}_a) + H^{\text{tail sym}}[\boldsymbol{x}_a, \boldsymbol{p}_a].$$

Conservative Matter Hamiltonian: Near-Zone Contribution

- Equations for the field degrees of freedom can be combined to get

$$\Box h_{ij}^{\mathrm{TT}} = S_{ij}^{\mathrm{TT}}, \quad \Box := -c^{-2}\partial_t^2 + \Delta,$$

$$S_{ij}(\mathbf{x},t) = S_{ij}\big(\mathbf{x}-\mathbf{x}_a(t), \mathbf{p}_a(t), h_{ij}^{\mathrm{TT}}(\mathbf{x},t), \pi^{ij}_{\mathrm{TT}}(\mathbf{x},t)\big).$$

- Time-symmetric and near-zone solution of the field equation,

$$h_{\mathrm{sym}\,ij}^{\mathrm{TT\,loc}} = \left(\Box_{\mathrm{sym}}^{-1} S_{ij}\right)^{\mathrm{TT}} = \frac{1}{2}\left((\Box_{\mathrm{adv}}^{-1} + \Box_{\mathrm{ret}}^{-1}) S_{ij}\right)^{\mathrm{TT}}$$

$$= \left((\Delta^{-1} + c^{-2}\Delta^{-2}\partial_t^2 + c^{-4}\Delta^{-3}\partial_t^4 + \cdots) S_{ij}\right)^{\mathrm{TT}}.$$

- After making the PN expansion of the source terms, $S_{ij} = S_{(4)ij} + S_{(6)ij} + \cdots$, one gets $h_{\mathrm{sym}\,ij}^{\mathrm{TT\,loc}}(\mathbf{x},t) = h_{(4)ij}^{\mathrm{TT}}(\mathbf{x},t) + h_{(6)ij}^{\mathrm{TT}}(\mathbf{x},t) + \cdots$, $\quad \Delta h_{(4)ij}^{\mathrm{TT}} = S_{(4)ij}^{\mathrm{TT}}$, $\quad \Delta h_{(6)ij}^{\mathrm{TT}} = S_{(6)ij}^{\mathrm{TT}} + \ddot{h}_{(4)ij}^{\mathrm{TT}}$.
The functions $h_{(4)ij}^{\mathrm{TT}}$ and $h_{(6)ij}^{\mathrm{TT}}$ are enough to compute 4PN-accurate conservative Hamiltonian.

- After replacing h_{ij}^{TT} by $h_{\mathrm{sym}\,ij}^{\mathrm{TT\,loc}}$ one gets the near-zone conservative Hamiltonian, which is local in time:

$$H_{\mathrm{cons}}^{\text{near-zone}}(\mathbf{x}_a, \mathbf{p}_a) := R\left[\mathbf{x}_a, \mathbf{p}_a, h_{\mathrm{sym}\,ij}^{\mathrm{TT\,loc}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \dot{h}_{\mathrm{sym}\,ij}^{\mathrm{TT\,loc}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a)\right].$$

Conservative Matter Hamiltonian: Time-Symmetric Tail Contribution

- The time-symmetric part of the 4PN tail metric (Blanchet & Damour 1988) contributes to the two-body EOM through a nonlocal-in-time Hamiltonian:

$$\text{Reg}\big\{H_{\text{4PN}}^{\text{tail sym}}\big\}[\boldsymbol{x}_a, \boldsymbol{p}_a; s] = -\frac{1}{5}\,\frac{G^2 M}{c^8}\,\dddot{I}_{ij}\,\mathrm{Pf}_{2s/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\,\dddot{I}_{ij}(t+v),$$

where Pf_T denotes a Hadamard partie finie with time scale $T := 2s/c$,

$$\mathrm{Pf}_T\int_0^{+\infty}\frac{\mathrm{d}v}{v}\,g(v) := \int_0^{T}\frac{\mathrm{d}v}{v}\,\big(g(v)-g(0)\big) + \int_T^{+\infty}\frac{\mathrm{d}v}{v}\,g(v),$$

and I_{ij} is the Newtonian quadrupole moment of the binary system:

$$I_{ij} := \sum_a m_a\left(x_a^i\,x_a^j - \frac{1}{3}\,\delta^{ij}\,\boldsymbol{x}_a^2\right).$$

- The regularized tail Hamiltonian depends on an arbitrary length scale s, which plays the role of an intermediate scale between the scale of the system r_{12} and the wavelength $\lambda/(2\pi)$,

$$r_{12} \ll s \ll \lambda/(2\pi).$$

4PN-Accurate Conservative Hamiltonian: UV/IR Divergences

$$H^{\text{cons}}_{\leq 4\text{PN}}[\boldsymbol{x}_a, \boldsymbol{p}_a] = H_{\text{N}}(\boldsymbol{x}_a, \boldsymbol{p}_a) + H_{1\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a) + H_{2\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a) + H_{3\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a) + H_{4\text{PN}}[\boldsymbol{x}_a, \boldsymbol{p}_a];$$

$$H_{\text{N}}(\boldsymbol{x}_a, \boldsymbol{p}_a) = \text{Reg}_{\text{UV}}\left\{ H^{\text{near-zone}}_{\text{N}}(\boldsymbol{x}_a, \boldsymbol{p}_a)\right\},$$

$$H_{1\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a) = \text{Reg}_{\text{UV}}\left\{ H^{\text{near-zone}}_{1\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a)\right\},$$

$$H_{2\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a) = \text{Reg}_{\text{UV}}\left\{ H^{\text{near-zone}}_{2\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a)\right\},$$

$$H_{3\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a) = \text{Reg}_{\text{UV}}\left\{ H^{\text{near-zone}}_{3\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a)\right\},$$

$$H_{4\text{PN}}[\boldsymbol{x}_a, \boldsymbol{p}_a] = \text{Reg}_{\text{UV}}\left\{ H^{\text{near-zone}}_{4\text{PN, IR conv}}(\boldsymbol{x}_a, \boldsymbol{p}_a)\right\} + \text{Reg}_{\text{IR}}\left\{ H^{\text{near-zone}}_{4\text{PN, IR div}}(\boldsymbol{x}_a, \boldsymbol{p}_a)\right\} + \text{Reg}\left\{ H^{\text{tail sym}}_{4\text{PN}}[\boldsymbol{x}_a, \boldsymbol{p}_a]\right\}.$$

Structure of the Hamiltonian Density: Contact and Field-Like Terms

- The near-zone Hamiltonian can be written as

$$H = \int \mathcal{H}(\boldsymbol{x})\,\mathrm{d}^d\boldsymbol{x}\,, \quad \mathcal{H}(\boldsymbol{x}) = \mathcal{H}_{\mathrm{c}}^{(D)}(\boldsymbol{x}) + \mathcal{H}_{\mathrm{f}}^{(D)}(\boldsymbol{x}) + \partial_i D^i(\boldsymbol{x}),$$

$\partial_i D^i$ gives no contribution to the H.

Contact Terms: Hadamard's "Partie Finie"

- Structure of contact $\mathcal{H}_{\mathrm{c}}^{(D)}$ terms:

$$\mathcal{H}_{\mathrm{c}}^{(D)} = S_1(\boldsymbol{x})\,\delta^d(\boldsymbol{x} - \boldsymbol{x}_1) + (1 \leftrightarrow 2).$$

- "Partie finie", i.e., the finite part (FP) of a singular function:

$$S(\boldsymbol{x}_a + r_a\boldsymbol{n}_a) = \sum_{m=-m_{\max}}^{\infty} a_m(\boldsymbol{x}_a)\,r_a^m, \quad \mathrm{FP}_a\,S := \frac{1}{\Omega_{d-1}} \oint \mathrm{d}\Omega_{d-1}\,a_0(\boldsymbol{n}_a),$$

Ω_{d-1} is the volume of the $(d-1)$-dimensional unit sphere.

- Regularization of contact terms:

$$S(\boldsymbol{x})\,\delta^d(\boldsymbol{x} - \boldsymbol{x}_a) = (\mathrm{FP}_a\,S)\delta^d(\boldsymbol{x} - \boldsymbol{x}_a),$$

$$\left(\int \mathrm{d}^d\boldsymbol{x}\,S(\boldsymbol{x})\,\delta^d(\boldsymbol{x} - \boldsymbol{x}_a)\right)_{\mathrm{reg}} := \mathrm{FP}_a\,S.$$

- Important feature (in general not valid in $d = 3$ dimensions):

$$\mathrm{FP}_a\big(f_1 f_2 \cdots\big) = \big(\mathrm{FP}_a\,f_1\big)\big(\mathrm{FP}_a\,f_2\big)\cdots.$$

Field-Like Terms

- Structure of field-like $\mathcal{H}_{\mathrm{f}}^{(D)}$ terms in $d = 3$ dimensions (at least up to 4PN order):

$$\mathcal{H}_{\mathrm{f}}^{(D)} = \sum_{\boldsymbol{\ell}} c_{\boldsymbol{\ell}} \, (\boldsymbol{n}_1 \cdot \boldsymbol{p}_1)^{\ell_1} \, (\boldsymbol{n}_2 \cdot \boldsymbol{p}_1)^{\ell_2} \, (\boldsymbol{n}_1 \cdot \boldsymbol{p}_2)^{\ell_3} \, (\boldsymbol{n}_2 \cdot \boldsymbol{p}_2)^{\ell_4} \times r_1^{\ell_5} r_2^{\ell_6} (r_1 + r_2 + r_{12})^{\ell_7};$$

using prolate spheroidal coordinates one can reduce field-like integrands to

$$c_{\boldsymbol{\ell}} \, r_1^{\ell_1} r_2^{\ell_2} (r_1 + r_2 + r_{12})^{\ell_3}.$$

- In $d = 3$ dimensions (JS 1998):

$$\int r_1^{\alpha} r_2^{\beta} (r_1 + r_2 + r_{12})^{\gamma} \, \mathrm{d}^3\boldsymbol{x} = 2\pi \, r_{12}^{\alpha+\beta+\gamma+3} \times \Big(B(\alpha+2, \beta+2) \, B_{1/2}(-\alpha-\beta-\gamma-4, \alpha+\beta+4) - B(-\alpha-\beta-4, \beta+2) \, B_{1/2}(-\alpha-\gamma-2, \alpha+2) - B(-\alpha-\beta-4, \alpha+2) \, B_{1/2}(-\beta-\gamma-2, \beta+2) \Big),$$

where B is the beta function and $B_{1/2}$ is the incomplete beta function:

$$B_{1/2}(\alpha, \beta) = \frac{1}{\alpha \, 2^{\alpha}} \, {}_2F_1\Big(1-\beta, \alpha; \alpha+1; \frac{1}{2}\Big),$$

${}_2F_1$ is the Gauss hypergeometric function.

Field-like terms: Riesz-Implemented Hadamard Regularization

- Let the integrand $i(\boldsymbol{x})$ develops only local poles, then its RH-regularized value reads

$$I^{\mathrm{RH}}(3;\epsilon_1,\epsilon_2) := \int_{\mathbb{R}^3} i(\boldsymbol{x}) \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \left(\frac{r_2}{s_2}\right)^{\epsilon_2} \mathrm{d}^3\boldsymbol{x}$$

$$= A + c_1\left(\frac{1}{\epsilon_1} + \ln\frac{r_{12}}{s_1}\right) + c_2\left(\frac{1}{\epsilon_2} + \ln\frac{r_{12}}{s_2}\right) + \mathcal{O}(\epsilon_1,\epsilon_2),$$

  s_1 and s_2 are arbitrary UV regularization scales.

- The pole, say $\propto 1/\epsilon_1$, comes from the part of the integrand $i(\boldsymbol{x})$ which develops logarithmic singularities (i.e. locally behaves like $1/r_1^3$),

$$i(\boldsymbol{x}) = \cdots + \tilde{c}_1(\boldsymbol{n}_1)\, r_1^{-3} + \cdots, \quad \text{when } \boldsymbol{x} \to \boldsymbol{x}_1.$$

  The pole part can be recovered by RH regularization of the integral of $\tilde{c}_1(\boldsymbol{n}_1)\, r_1^{-3}$ over the ball $B(\boldsymbol{x}_1,\ell_1)$:

$$I_1^{\mathrm{RH}}(3;\epsilon_1) := \int_{B(\boldsymbol{x}_1,\ell_1)} \tilde{c}_1(\boldsymbol{n}_1)\, r_1^{-3} \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \mathrm{d}^3\boldsymbol{r}_1 = c_1\left(\frac{1}{\epsilon_1} + \ln\frac{\ell_1}{s_1}\right) + \mathcal{O}(\epsilon_1).$$

Field-Like Terms: Implementation of DR

- It is enough to replace $I_1^{\text{RH}}(3;\epsilon_1)$ and $I_2^{\text{RH}}(3;\epsilon_2)$ by their d-dimensional versions $I_1(d,\ell_0)$ and $I_2(d,\ell_0)$, where ℓ_0 relates the Newtonian G_N and the D-dimensional $(D=d+1)$ G_D gravitational constants,

$$G_D = G_\text{N}\,\ell_0^{\varepsilon}, \qquad \varepsilon := d-3.$$

- One considers d-dimensional version of the expansion of $i(\boldsymbol{x})$,

$$i(\boldsymbol{x}) = \cdots + \tilde{c}_1(d,\ell_0;\boldsymbol{n}_1)\,r_1^{6-3d} + \cdots\,, \quad \text{when } \boldsymbol{x}\to\boldsymbol{x}_1,$$

and defines

$$I_1(d,\ell_0) := \int_{B(\boldsymbol{x}_1,\ell_1)} \tilde{c}_1(d,\ell_0;\boldsymbol{n}_1)\,r_1^{6-3d}\,\mathrm{d}^d\boldsymbol{r}_1 = c_1\Big(-\frac{1}{2\varepsilon}+\ln\frac{\ell_1}{\ell_0}\Big) + B_1 + \mathcal{O}(\varepsilon).$$

- The DR correction to the RH-regularized integral $I^{\text{RH}}(3;\epsilon_1,\epsilon_2)$:

$$\begin{aligned} I^{\text{RH}}(3;\epsilon_1,\epsilon_2) - I_1^{\text{RH}}(3;\epsilon_1) - I_2^{\text{RH}}(3;\epsilon_2) + I_1(d,\ell_0) + I_2(d,\ell_0) \\ = A + \Delta A - \frac{c_1+c_2}{2\varepsilon} + B\ln\frac{r_{12}}{\ell_0} + \mathcal{O}(\varepsilon). \end{aligned}$$

The result is as if all computations were fully done in d dimensions.

UV Regularization of 3PN/4PN Hamiltonians

- Regularization of the 3PN Hamiltonian:

$$\mathrm{Reg}_{\mathrm{UV}}\{H_{\mathrm{3PN}}(\mathbf{x}_a,\mathbf{p}_a)\} := \lim_{\varepsilon\to 0}\left\{H^{\mathrm{RH}}_{\mathrm{3PN}}(\mathbf{x}_a,\mathbf{p}_a) + \Delta H^{\mathrm{DR\ correction}}_{\mathrm{3PN}}(\mathbf{x}_a,\mathbf{p}_a;\varepsilon)\right\},$$

$$\Delta H^{\mathrm{DR\ correction}}_{\mathrm{3PN}}(\mathbf{x}_a,\mathbf{p}_a;\varepsilon) = C(\mathbf{x}_a,\mathbf{p}_a) + \mathcal{O}(\varepsilon) \quad (\text{no pole part!}).$$

- Regularization of the near-zone and IR-convergent part of the 4PN Hamiltonian:

$$\begin{aligned}\mathrm{Reg}_{\mathrm{UV}}\{H^{\text{near-zone}}_{\mathrm{4PN,\ IR\ conv}}(\mathbf{x}_a,\mathbf{p}_a)\} := \lim_{\varepsilon\to 0}\Big\{&H^{\mathrm{RH}}_{\mathrm{4PN,\ IR\ conv}}(\mathbf{x}_a,\mathbf{p}_a)\\ &+\Delta H^{\mathrm{DR\ correction}}_{\mathrm{4PN}}(\mathbf{x}_a,\mathbf{p}_a;\varepsilon)\\ &+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{D_1(\mathbf{x}_a,\mathbf{p}_a)}{\varepsilon} + D_2(\mathbf{x}_a,\mathbf{p}_a)\ln\frac{r_{12}}{\ell_0}\right)\Big\},\end{aligned}$$

$$\Delta H^{\mathrm{DR\ correction}}_{\mathrm{4PN}}(\mathbf{x}_a,\mathbf{p}_a;\varepsilon) = C_1(\mathbf{x}_a,\mathbf{p}_a) + \frac{C_2(\mathbf{x}_a,\mathbf{p}_a)}{\varepsilon} + C_3(\mathbf{x}_a,\mathbf{p}_a)\ln\frac{r_{12}}{\ell_0} + \mathcal{O}(\varepsilon).$$

The Regularized 4PN Hamiltonian

- DJS (2014–15) used two different methods to perform IR regularization:
  (i) modifying behavior of the $h^{\mathrm{TT}}_{(6)ij}$ at spatial infinity;
  (ii) d-dimensional version of the RH regularization.
  In both methods one introduces a new IR regularization length scale $\bar{s}$.
- Both methods yield the same result modulo a time derivative and a change in the constant C, which enters the Hamiltonian through the term $\ln(r_{12}/\bar{s}) + C$.
- One identifies the length scale used in the near-zone IR regularization with the length scale employed to define the tail contribution: $s = \bar{s}$.
- The dependence on s cancels between the local- and nonlocal-in-time contributions, and the total 4PN Hamiltonian reads

$$
\begin{aligned}
H_{\mathrm{4PN}}[\boldsymbol{x}_a, \boldsymbol{p}_a; C] &= \mathrm{Reg}_{\mathrm{UV}}\{H^{\text{near-zone}}_{\text{4PN, IR conv}}\}(\boldsymbol{x}_a, \boldsymbol{p}_a) \\
&\quad + \mathrm{Reg}_{\mathrm{IR}}\{H^{\text{near-zone}}_{\text{4PN, IR div}}\}(\boldsymbol{x}_a, \boldsymbol{p}_a; s, C) + \mathrm{Reg}\{H^{\text{tail sym}}_{\mathrm{4PN}}\}[\boldsymbol{x}_a, \boldsymbol{p}_a; s] \\
&= \chi(\boldsymbol{x}_a, \boldsymbol{p}_a) + \frac{2}{5}\frac{G^2 M}{c^8}\left(\dddot{I}_{ij}\right)^2 C \\
&\quad - \frac{1}{5}\frac{G^2 M}{c^8}\,\dddot{I}_{ij}\,\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\,\dddot{I}_{ij}(t+v).
\end{aligned}
$$

- Determination of the value of the constant C:
  using beyond-near-zone information taken from Bini & Damour (2013)
  [it was enough to consider the linear in $\nu := m_1 m_2/(m_1+m_2)^2$ part of the 4PN-accurate gauge-invariant link $E_{\leq \mathrm{4PN}}(j;\nu)$ valid for circular orbits].

3PN Conservative Two-Body EOM (Without Spins): the First Four Independent and Mutually Compatible Derivations

The three derivations used δ-sources and DR:

- Damour, Jaranowski, & Schäfer (1998–2001),
  ADM-Hamiltonian-based derivation,
  initial purely 3-dim. derivation plagued by two UV-divergence-related ambiguity parameters (one parameter fixed by the requirement of Poincaré invariance),
  final 2001 non-ambiguous derivation used DR;
- Blanchet, Damour, Esposito-Farèse, & Faye (2000–2004),
  harmonic-coordinate-based derivation,
  initial purely 3-dimensional Lorentz-invariant derivation plagued by one UV-divergence-related ambiguity parameter,
  final 2004 non-ambiguous derivation used DR;
- Foffa & Sturani (2011),
  effective-field-theory approach, non-ambiguous derivation using DR.

There exists only one pure 3-dimensional derivation using an extended body model together with the strong-field point-particle limit and a surface-integral approach (in harmonic coordinates):

- Itoh & Futamase (2003–2004).

4PN Conservative Two-Body EOM (Without Spins): the First Four Independent and Mutually Compatible Derivations

All four derivations used δ-sources and DR:

- Damour, Jaranowski, & Schäfer (2012–2015), ADM-Hamiltonian-based derivation, final 2014 derivation used DR and beyond-near-zone information taken from Bini & Damour (2013);
- Bernard, Blanchet, Bohé, Faye, Marchand, & Marsat (2016–2017), harmonic-coordinate-based derivation, final 2017 derivation used DR and beyond-near-zone information taken from Bini & Damour (2013); then a new treating of IR divergences by means of DR by Bernard, Blanchet, Bohé, Faye, Marchand, & Marsat (2017–2018);
- Foffa, Porto, Rothstein, & Sturani (2019), effective-field-theory approach, non-ambiguous derivation using DR;
- Blümlein, Maier, Marquard, & Schäfer (2020), effective-field-theory approach, non-ambiguous derivation using DR.

Other Applications of the ADM Hamiltonian Approach

- Spin-dependent effects in dynamics of compact binaries (DJS 2008—a novel derivation of the NLO SO corrections).
- Conservative Hamiltonians are one of the key ingredients of the effective-one-body formalism (DJS 2000—adding orbital 3PN dynamics, DJS 2008—adding NLO SO interaction, DJS 2015—adding orbital 4PN dynamics).
- Implementation in phase space of the Poincaré invariance of the two-body dynamics described by the ADM Hamiltonian (initiated by DJS in 2000)—a powerfull check of the correctness of the ADM Hamiltonian.
- Radiation-reaction effects in EOM and gravitational-wave luminosities. 2.5PN and 3.5PN dissipative ADM Hamiltonians for spinless many-body point-mass systems were derived by JS (1997),

  $$H^{\rm diss}_{\leq 3.5\rm PN}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) = H_{2.5\rm PN}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) + H_{3.5\rm PN}(\boldsymbol{x}_a, \boldsymbol{p}_a, t).$$

  These Hamiltonians were applied to derive, at the LO (the quadrupole formula) and the NLO, gravitational-wave luminosity $\mathcal{L}$ of the two-body system in quasi-elliptical motion. The luminosity is computed from the formula

  $$\mathcal{L} = -\left\langle \frac{\partial}{\partial t} H^{\rm diss}_{\leq 3.5\rm PN}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) \right\rangle,$$

  where $\langle \cdots \rangle$ denotes time averaging over one period of the motion. Also the LO spin-orbit and spin(1)-spin(2) dissipative Hamiltonians were derived.

The Future of the PN Two-Body Problem: Results to be Achieved

- Computation of gravitational-wave luminosity of two-point-mass system at the 4PN/4.5PN orders, both for circular and for generic quasi-elliptical orbits, and construction of the 4.5PN-accurate templates for inspiralling compact binaries.
- Completion of computations of 5PN, 5.5PN, 6PN, ... EOM of two-point-mass systems together with computation of gravitational-wave luminosities at 5PN, 5.5PN, 6PN, ... orders, and construction of $\geq$5PN-accurate templates for inspiralling compact binaries.
- Computation, within the PN framework, higher-order tidal corrections to dynamics of binaries containing neutron stars.
- Higher-order perturbative solutions of two-body problem are complicated, both from computational and from conceptual point of view. Therefore it is highly desired to have more than one independent derivation of any analytical result:
  - — making independent derivations (e.g. within the ADM Hamiltonian approach) of gravitational-wave luminosities of two-point-mass system at the 2PN, 3PN, ... order;
  - — making rederivation of 4PN two-body equations of motion using an extended body model.

The Future of the PN Two-Body Problem: New/Refreshed Tools

- Looking for a new treatment of regularization issues related to usage of δ-sources, which would simplify higher-order PN computations.
  - Replacing δ-sources by sources described by some sequence of classical functions ("the δ sequence"); then there is no need for using distributional derivatives of singular homogeneous functions.
    At the 3PN level DJS (2008) successfully recomputed all UV logarithmically divergent terms using d-dimensional Riesz kernels to model point masses.
  - Looking for some extension/modification of Schwartz distribution theory that would be suitable for purely 3-dimensional regularization.
    Such an attempt was made by Blanchet & Faye (2000), but their "extended Hadamard regularization" can not be combined with DR.
- A new project (just starting): increasing the level of algorithmization and automatization of computation of 2-point-mass ADM Hamiltonians, starting from PN iterations of constraint equations up to UV/IR regularization, performed using a mixture of 3-dimensional RH regularization and DR.
  - Rederivation of the 4PN two-point-mass ADM Hamiltonian, without usage of gravitational self-force results and without introducing any ambiguity parameter (work in progress).
  - 2-point-mass Hamiltonians at orders at least 4.5PN, 5PN, and 5.5PN seem to be within reach.

Conservative 4PN-Accurate 2-Point-Mass Hamiltonian

$$H_{\leq 4\mathrm{PN}}[\boldsymbol{x}_a, \boldsymbol{p}_a] = \frac{\boldsymbol{p}_1^2}{2m_1} - \frac{Gm_1m_2}{2r_{12}} + \frac{1}{c^2}\Bigg\{ -\frac{(\boldsymbol{p}_1^2)^2}{8m_1^3} + \frac{Gm_1m_2}{4r_{12}}\bigg(-6\,\frac{\boldsymbol{p}_1^2}{m_1^2} + 7\,\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1m_2} + \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1m_2}\bigg) + \frac{G^2m_1^2m_2}{2r_{12}^2}\Bigg\} + \frac{1}{c^4}\Bigg\{\frac{(\boldsymbol{p}_1^2)^3}{16m_1^5} + \frac{1}{8}\,\frac{Gm_1m_2}{r_{12}}\bigg(5\,\frac{(\boldsymbol{p}_1^2)^2}{m_1^4} - \frac{11}{2}\,\frac{\boldsymbol{p}_1^2\,\boldsymbol{p}_2^2}{m_1^2m_2^2} - \frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2}$$

$$+ 5\,\frac{\boldsymbol{p}_1^2\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2} - 6\,\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\,\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2}\bigg) + \frac{G^2m_1m_2}{4r_{12}^2}\bigg(m_2\Big(10\,\frac{\boldsymbol{p}_1^2}{m_1^2} + 19\,\frac{\boldsymbol{p}_2^2}{m_2^2}\Big) - m_1\,\frac{27\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2) + 6\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1m_2}\bigg) - \frac{G^3m_1^2m_2(2m_1+5m_2)}{8r_{12}^3}\Bigg\}$$

$$+ \frac{1}{c^6}\Bigg\{ -\frac{5(\boldsymbol{p}_1^2)^4}{128m_1^7} + \frac{Gm_1m_2}{32r_{12}}\bigg(-14\,\frac{(\boldsymbol{p}_1^2)^3}{m_1^6} + 4\,\frac{\big((\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2 + 4\,\boldsymbol{p}_1^2\,\boldsymbol{p}_2^2\big)\boldsymbol{p}_1^2}{m_1^4m_2^2} + 6\,\frac{\boldsymbol{p}_1^2\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^4m_2^2} - 10\,\frac{\big(\boldsymbol{p}_1^2\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2 + \boldsymbol{p}_2^2\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\big)\boldsymbol{p}_1^2}{m_1^4m_2^2} + 24\,\frac{\boldsymbol{p}_1^2\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1^4m_2^2}$$

$$+ 2\,\frac{\boldsymbol{p}_1^2\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^3m_2^3} + \frac{\big(7\,\boldsymbol{p}_1^2\,\boldsymbol{p}_2^2 - 10\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2\big)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1^3m_2^3} + \frac{\big(\boldsymbol{p}_1^2\,\boldsymbol{p}_2^2 - 2\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2\big)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1^3m_2^3} + 15\,\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^3m_2^3} - 18\,\frac{\boldsymbol{p}_1^2\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3}{m_1^3m_2^3} + 5\,\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3}{m_1^3m_2^3}\bigg)$$

$$+ \frac{G^2m_1m_2}{r_{12}^2}\bigg(\frac{1}{16}(m_1 - 27m_2)\frac{(\boldsymbol{p}_1^2)^2}{m_1^4} - \frac{115}{16}m_1\frac{\boldsymbol{p}_1^2\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2 + 371\,\boldsymbol{p}_1^2\,\boldsymbol{p}_2^2}{m_1^2m_2^2} + \frac{17}{16}\frac{\boldsymbol{p}_1^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4}{m_1^3} - \frac{3}{2}m_1\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1^3m_2} - \frac{1}{8}m_1\frac{\big(15\,\boldsymbol{p}_1^2\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2) + 11\,(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)\big)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)}{m_1^3m_2}$$

$$+ \frac{125}{12}m_2\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\,(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2} - \frac{1}{48}(220m_1 + 193m_2)\frac{\boldsymbol{p}_1^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2}\bigg) + \frac{G^3m_1m_2}{r_{12}^3}\bigg(-\frac{1}{48}\Big(425\,m_1^2 + \Big(473 - \frac{3}{4}\pi^2\Big)m_1m_2 + 150\,m_2^2\Big)\frac{\boldsymbol{p}_1^2}{m_1^2}$$

$$+ \frac{1}{16}\Big(77(m_1^2 + m_2^2) + \Big(143 - \frac{1}{4}\pi^2\Big)m_1m_2\Big)\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1m_2} + \frac{1}{16}\Big(20\,m_1^2 - \Big(43 + \frac{3}{4}\pi^2\Big)m_1m_2\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2}{m_1^2} + \frac{1}{16}\Big(21(m_1^2 + m_2^2) + \Big(119 + \frac{3}{4}\pi^2\Big)m_1m_2\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1m_2}\bigg) + \frac{G^4m_1m_2^3}{8r_{12}^4}\bigg(\Big(\frac{227}{3} - \frac{21}{4}\pi^2\Big)m_1 + m_2\bigg)\Bigg\}$$

$$+ \frac{1}{c^8}\Bigg\{\frac{7(\boldsymbol{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}}\bigg(\frac{45(\boldsymbol{p}_1^2)^4}{128m_1^8} - \frac{9(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1^2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{16m_1^6m_2^2} - \frac{3(\boldsymbol{p}_1^2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1^2)^2\boldsymbol{p}_2^2}{64m_1^6m_2^2} - \frac{21(\boldsymbol{p}_1^2)^3\boldsymbol{p}_2^2}{64m_1^6m_2^2} - \frac{35(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^5(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3}{256m_1^5m_2^3}$$

$$+ \frac{25(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3\boldsymbol{p}_1^2}{128m_1^5m_2^3} + \frac{33(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3(\boldsymbol{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{256m_1^5m_2^3} - \frac{45(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{128m_1^5m_2^3} - \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1^2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{256m_1^5m_2^3} + \frac{25(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{64m_1^5m_2^3}$$

$$+ \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{64m_1^5m_2^3} - \frac{3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^3}{64m_1^5m_2^3} + \frac{3\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^5(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{256m_1^5m_2^3} - \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{128m_1^5m_2^3} - \frac{25(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1^2)^2\boldsymbol{p}_2^2}{256m_1^5m_2^3} - \frac{23(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{256m_1^5m_2^3}$$

$$+ \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{128m_1^5m_2^3} - \frac{7(\boldsymbol{p}_1^2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{256m_1^5m_2^3} - \frac{5(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^4\boldsymbol{p}_1^2}{64m_1^4m_2^4} + \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^4(\boldsymbol{p}_1^2)^2}{64m_1^4m_2^4} - \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{4m_1^4m_2^4} + \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{16m_1^4m_2^4} - \frac{5(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_2^2}{64m_1^4m_2^4}$$

$$+ \frac{21(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{64m_1^4m_2^4} - \frac{3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1^2)^2\boldsymbol{p}_2^2}{32m_1^4m_2^4} - \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{4m_1^4m_2^4} + \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{16m_1^4m_2^4} + \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_2^2}{16m_1^4m_2^4} - \frac{\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_2^2}{32m_1^4m_2^4} + \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4(\boldsymbol{p}_2^2)^2}{64m_1^4m_2^4}$$

$$- \frac{3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_1^2(\boldsymbol{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\boldsymbol{p}_1^2)^2(\boldsymbol{p}_2^2)^2}{128m_1^4m_2^4}\bigg) + \frac{G^2m_1m_2}{r_{12}^2}m_1\bigg(\frac{369(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^6}{160m_1^6} - \frac{889(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4\boldsymbol{p}_1^2}{192m_1^6} + \frac{49(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1^2)^2}{16m_1^6} - \frac{63(\boldsymbol{p}_1^2)^3}{64m_1^6} - \frac{549(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^5(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{128m_1^5m_2} + \frac{67(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2}{16m_1^5m_2}$$

$$- \frac{167(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{256m_1^5m_2} - \frac{851(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{128m_1^5m_2} + \frac{1099(\boldsymbol{p}_1^2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{256m_1^5m_2} + \frac{3263(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1^2)^2}{3840m_1^4m_2^2}$$

$$- \frac{3571(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{320m_1^4m_2^2} + \frac{3073(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{480m_1^4m_2^2} + \frac{4349(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{1280m_1^4m_2^2} - \frac{3461\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4\boldsymbol{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\boldsymbol{p}_1^2)^2\boldsymbol{p}_2^2}{3840m_1^4m_2^2}$$

$$- \frac{13(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3}{8m_1^3m_2^3} + \frac{191(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3\boldsymbol{p}_1^2}{192m_1^3m_2^3} - \frac{19(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{384m_1^3m_2^3} - \frac{5(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{384m_1^3m_2^3} + \frac{11(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{96m_1^3m_2^3}$$

$$- \frac{47(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{32m_1^3m_2^3} + \frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{384m_1^3m_2^3} - \frac{185\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{384m_1^3m_2^3} - \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^4\boldsymbol{p}_1^2}{4m_1^2m_2^4} - \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^3(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{2m_1^2m_2^4} + \frac{21(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_2^2}{6m_1^2m_2^4}$$

$$+ \frac{49(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{48m_1^2m_2^4} - \frac{133(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{p}_2^2}{24m_1^2m_2^4} - \frac{77(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_2^2}{96m_1^2m_2^4} + \frac{197(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\boldsymbol{p}_1^2(\boldsymbol{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\boldsymbol{p}_2^2)^3}{8m_2^6}\bigg) + \frac{G^3m_1m_2}{r_{12}^3}\bigg(m_1^2\bigg(\frac{5027(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4}{384m_1^4} - \frac{22993(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_1^2}{960m_1^4} - \frac{6695(\boldsymbol{p}_1^2)^2}{1152m_1^4}$$

$$- \frac{3191(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{640m_1^3m_2} + \frac{28561(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2}{1920m_1^3m_2} + \frac{8777(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{384m_1^3m_2} + \frac{752969\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{28800m_1^3m_2} - \frac{16481(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{2400m_1^2m_2^2}$$

$$+ \frac{791(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\boldsymbol{p}_2^2)^2}{32m_2^4}\bigg) + m_1m_2\bigg(\Big(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\Big)\frac{(\boldsymbol{p}_1^2)^2}{m_1^4} + \Big(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_1^2}{m_1^4} + \Big(\frac{375\pi^2}{8192} - \frac{23533}{1280}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^4}{m_1^4}$$

$$+ \Big(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\Big)\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2} + \Big(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\Big)\frac{\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{m_1^2m_2^2} + \Big(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2\boldsymbol{p}_1^2}{m_1^2m_2^2} + \Big(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1^2m_2^2}$$

$$- \Big(\frac{30383}{960} + \frac{36405\pi^2}{16384}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2} + \Big(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\Big)\frac{\boldsymbol{p}_1^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1^3m_2} + \Big(\frac{2369}{60} + \frac{35655\pi^2}{16384}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^3(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1^3m_2} + \Big(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)\boldsymbol{p}_1^2}{m_1^3m_2}$$

$$+ \Big(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1^3m_2}\bigg)\bigg) + \frac{G^4m_1m_2}{r_{12}^4}\bigg(m_1^3\Big(\frac{64861\boldsymbol{p}_1^2}{4800m_1^2} - \frac{91(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{8m_1m_2} + \frac{105\boldsymbol{p}_2^2}{32m_2^2} - \frac{9841(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2}{1600m_1^2} - \frac{7(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{2m_1m_2}\Big) + m_1^2m_2\bigg(\Big(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\Big)\frac{\boldsymbol{p}_1^2}{m_1^2}$$

$$+ \Big(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\Big)\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1m_2} + \Big(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\Big)\frac{\boldsymbol{p}_2^2}{m_2^2} + \Big(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)^2}{m_1^2} + \Big(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_1)(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)}{m_1m_2} + \Big(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\Big)\frac{(\boldsymbol{n}_{12}\cdot\boldsymbol{p}_2)^2}{m_2^2}\bigg)\bigg)$$

$$+ \frac{G^5m_1m_2}{r_{12}^5}\bigg(-\frac{m_1^4}{16} + \Big(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\Big)m_1^3m_2 + \Big(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\Big)m_1^2m_2^2\bigg) - \frac{G^2(m_1+m_2)}{10}\,\dddot{I}_{ij}\,\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\,\dddot{I}_{ij}(t+v)\Bigg\} + (1\leftrightarrow 2).$$