STABILITY OF KERR FOR SMALL ANGULAR MOMENTUM

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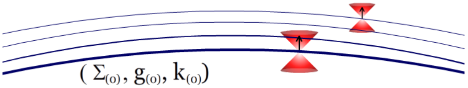
HAPPY BIRTHDAY THIBAULT !

RICCI FLAT (EV) MANIFOLDS

- EINSTEIN VACUUM(EVE) $\mathrm{Ric} = 0.$
- INITIAL DATA SETS $(\Sigma^3_{(0)}, g_{(0)}, k_{(0)})$ + Constraints
- MFGHD Bruhat(1953)-Geroch

Ric(g)=0



- STATIONARY SOLUTS. Kerr family $Kerr(a, m)$, $|a| < m$.

POSITIVE COSMOLOGICAL CONSTANT. Friedrich, Hintz-Vasy

WHAT IS MATHEMATICAL GR ?

1. ELUCIDATE THE MATHEMATICAL STRUCTURE OF CLASSICAL GR

2. FORMULATE AND ADDRESS ITS CENTRAL PROBLEMS.
   - PHYSICALLY RELEVANT
   - SATISFIES OUR MATHEMATICAL SENSIBILITIES: BEAUTY, RIGOR, NOVEL MATHEMATICAL CHALLENGES.

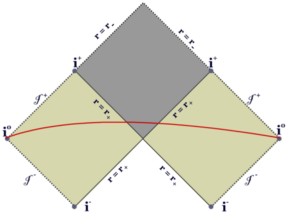
3. ESTABLISH CONECTIONS TO OTHER PROBLEMS PDE, GEOMETRY ...

MATHEMATICAL ENTANGLEMENT

WIGNER: " *Mathematical concepts introduced for solving specific problems have unexpected, mysterious, consequences in seemingly unrelated areas*"

STABILITY OF KERR

CONJECTURE. Small perturbations of a given, subextremal, Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.

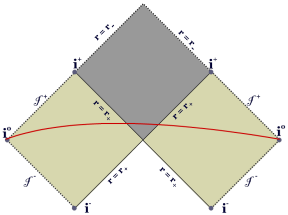


THEOREM[K-Szeftel(2021)] True if $|a|/m \ll 1$.

STABILITY OF KERR

CONJECTURE[Stability of (external) Kerr].

Small perturbations of a given exterior Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.



THEOREM[K-Szeftel(2021)] "True" if $|a|/m \ll 1$.

- STABILITY OF MINKOWSKI (1990-1993)
- STABILITY OF SCHWARZSCHILD (2018-2020) Polarized
- STABILITY OF SCHWARZSCHILD (2021) Codim. 3

THEOREM[K-Szeftel(2021)] "True" if $|a|/m \ll 1$.

1. Kl-Szeftel 2019. Construction of GCM spheres in perturbations of Kerr, arXiv:1911.00697

2. Kl-Szeftel 2019. Effective results on uniformization and intrinsic GCM spheres in perturbations of Kerr, arXiv:1912.12195

3. Giorgi-Kl-Szeftel 2019. A General Formalism for the stability of Kerr, arXiv:2002.0274, 2020.

4. Kl-Szeftel 2021. Kerr stability for small angular momentum, ArXiv:2104.11857v1, April 2021.

GENERAL STABILITY PROBLEM $\mathcal{N}[\phi] = 0.$

NONLINEAR EQUATIONS. $\mathcal{N}[\phi_0 + \psi] = 0, \; \mathcal{N}[\phi_0] = 0.$

1. ORBITAL STABILITY(OS). $\psi = O(1)$ as $t \to \infty$
2. ASYMPT STABILITY(AS). $\psi \longrightarrow 0$ as $t \to \infty.$
3. ORBITAL ASYMPT STABILITY(OAS). $\boxed{\phi_0 + \psi \to \phi_f}.$

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LINEARIZED EQUATIONS. $\mathcal{N}'[\phi_0]\psi = 0.$

1. MODE STABILITY (MS). No growing modes.
2. BOUNDEDNESS.
3. QUANTITATIVE DECAY.
4. ZERO MODES. $\Rightarrow$ AOS
5. LINEARIZED EV. Huge family of zero modes.
   - Due to 2-parameter family $Kerr(a, m)$.
   - Due to general covariance $\mathbf{g} \equiv \Phi_* \mathbf{g}, \quad \Phi : \mathcal{M} \longrightarrow \mathcal{M}.$

LINEAR STABILITY. $G'(\mathbf{g}_0)\delta\mathbf{g} = 0$

FORMAL MODE ANALYSIS

- Metric perturbations. Regge-Wheeler, Vishveshara-Zerilli.
- Newmann-Penrose. Bardeen-Press, Teukolsky, Press-Teukolsky.
- Chandrasekhar transformation. Chandrasekhar, Wald
- Full mode stability. Whiting.

LINEAR STABILITY. $G'(\mathbf{g}_0)\delta\mathbf{g} = 0$

VECTORFIELD METHOD. MORAWETZ-ENERGY ESTIMATES

- Mink. methods. Morawetz, Kl, Christodoulou-Kl
- Scalar wave- Schw. Soffer, Blue-S, Dafermos-Rodnianski, Marzuola-Metcalf-Tataru-Tohaneanu
- Scalar wave- $a \ll m$. D-R, T-T, Andersson-Blue
- Scalar wave- $a < m$. D-R-Shlapentokh Rothman
- Lin stability- $a = 0$. Dafermos-Holzegel- Rodnianski
- Lin Teukosky - $a \ll m$. Ma, Dafermos-Holzegel-Rodnianski
- Lin Teukosky - $a < m$. S T- Teixeira da Costa
- Lin stability- $a \ll m$. A-Bäckdahl-Blue-Ma, Hintz-Vassy

NONLINEAR STABILITY PROBLEM

- NEED: Dynamically defined **gauge condition** and mechanism to track the **final state**

- NEED: Robust mechanism for deriving **sufficient decay** for the main linearized quantities
  –with respect to the gauge

- EXAMPLE: Stability of $\psi \equiv 0$, $\quad \Box\psi = F(\psi, \partial\psi)$.
  - VECTORFIELD METHOD
  - NULL CONDITION

- NEED: A version of the null condition
  –with respect to the gauge

- NEED: A strategy to disentangle the nonlinear interdependence of the above.

STABILITY OF MINKOWSKI Christodoulou-K(1993).

GEOMETRIC FRAMEWORK I

1. Null Pair $(e_3, e_4), \quad \mathbf{g}(e_3, e_4) = -2.$

2. Horizontal structures $\mathcal{H} := \{e_3, e_4\}^{\perp}.$

3. Null Frames $e_3, e_4, (e_a)_{a=1,2}.$

4. Connection coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \omega, \underline{\omega}.$

$$\begin{aligned}
\chi_{ab} &= \mathbf{g}(\nabla_a e_4, e_a), \qquad \underline{\chi}_{ab} = \mathbf{g}(\nabla_a e_3, e_b) \\
\chi_{ab} &= \hat{\chi}_{ab} + \frac{1}{2}\mathrm{tr}\,\chi\delta_{ab} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\chi \in_{ab} \\
\underline{\chi}_{ab} &= \underline{\hat{\chi}}_{ab} + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\delta_{ab} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\underline{\chi} \in_{ab}
\end{aligned}$$

${}^{(a)}\mathrm{tr}\,\chi = {}^{(a)}\mathrm{tr}\,\underline{\chi} = 0 \Rightarrow$ Integrability.

GEOMETRIC FRAMEWORK II

5. Curvature coefficients α, $\underline{\alpha}$, β, $\underline{\beta}$, ρ, ${}^*\rho$

$$\alpha_{ab} = \mathbf{R}(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} = \mathbf{R}(e_a, e_3, e_b, e_3)$$

6. Complexification

$$X = \chi + i\,{}^*\chi, \quad \underline{X} = \underline{\chi} + i\,{}^*\underline{\chi}, \quad A = \alpha + i\,{}^*\alpha, \ldots$$

7. Complete set of null decompositions
   - Connection $\Gamma = \left\{X, \underline{X}, \Xi, \underline{\Xi}, H, \underline{H}, Z, \omega, \underline{\omega}, \right\}$
   - Curvature $R = \{A, B, P, \underline{B}\,, \underline{A}\}$

8. Cartan-Bianchi Equations

$$\begin{aligned} d\Gamma + [\Gamma, \Gamma] &= R \\ dR + [R, \Gamma] &= 0. \end{aligned}$$

9. Comparison to Christodoulou-K and Newmann-Penrose (NP).

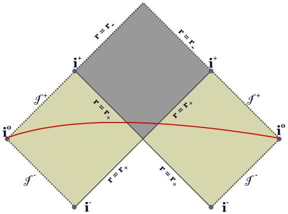
KERR FAMILY $\mathcal{K}(a, m)$

BOYER-LINDQUIST (t, r, θ, φ).

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr \\ q = r + ia\cos\theta, \\ \Sigma^2 = (r^2+a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

STATIONARY, AXISYMMETRIC. $\mathbf{T} = \partial_t$, $\mathbf{Z} = \partial_\varphi$



KERR FAMILY $\mathcal{K}(a, m)$

PRINCIPAL NULL FRAME.

$$e_4, e_3 = \frac{r^2+a^2}{q\sqrt{\Delta}}\partial_t \pm \frac{\sqrt{\Delta}}{q}\partial_r + \frac{a}{q\sqrt{\Delta}}\partial_\varphi, \qquad \mathbf{D}_4 e_4 = 0.$$

$$e_1 = \frac{1}{|q|}\partial_\theta, \quad e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi.$$

CRUCIAL FACT.

1. In Kerr, relative to a principal null frame,

$$A,\ \underline{A},\ B,\ \underline{B} = 0, \quad P = -\frac{2m}{q^3}, \qquad \hat{\chi},\ \underline{\hat{\chi}},\ \xi,\ \underline{\xi} = 0.$$

2. In Schwarzschild, in addition
   - $\{e_3, e_4\}^\perp$ is integrable, $\qquad \Im(X) = \Im(\underline{X}) = 0.$
   - ${}^*\rho = \Im(P) = 0$

3. In Minkowski, in addition $\qquad \rho = \Re(P) = 0.$

$O(\epsilon)$ - PERTURBATIONS

ASSUME. There exists a perturbed geometric structure on $\mathcal{M}$

$$\check{\Gamma} := \Gamma - \Gamma_{Kerr}, \ \check{R} = R - R_{Kerr} =: O(\epsilon)$$

FRAME DEPENDENCE. $(f, \underline{f})_{a=1,2}, \ \lambda = 1 + O(\epsilon)$

$$\begin{aligned} e_4' &= \lambda\Big(e_4 + f_a e_a + O(\epsilon^2)\Big) \\ e_3' &= \lambda^{-1}\Big(e_3 + \underline{f}_a e_a + O(\epsilon^2)\Big) \\ e_a' &= e_a + \frac{1}{2}\underline{f}_a e_4 + \frac{1}{2} f_a e_3 + O(\epsilon^2) \end{aligned}$$

REMARKABLE FACT!

- Curvature components $A, \underline{A}$ are $O(\epsilon^2)$ invariant.
- In linear theory $A, \underline{A}$ verify decoupled wave equations!

TEUKOLSKY AND gRW EQTS

TEUKOLSKY.

$$\dot{\square}_{\mathbf{g}} A + M \cdot \nabla_3 A + \underline{M} \cdot \nabla_4 A = \mathrm{Err}(\check{\Gamma}, \check{R})$$

CHANDRASEKHAR TRANSF. $A \to \mathfrak{q}$

$$\mathfrak{q} \quad = \quad q\overline{q}^3 \left(\nabla_3 \nabla_3 A + C \cdot \nabla_3 A + D \cdot A \right), \qquad q = r + ia\cos\theta.$$

gRW EQUATION.

$$\dot{\square}_2 \mathfrak{q} - i \frac{4a\cos\theta}{|q|^2} \nabla_{\mathbf{T}} \mathfrak{q} - V\mathfrak{q} = a\, L_{\mathfrak{q}}[A] + \mathrm{Err}_{\mathfrak{q}}(\check{\Gamma}, \check{R}),$$

$$L_{\mathfrak{q}}[A] \quad - \quad \text{linear in } \nabla_3 \nabla A,\ \nabla_3 A,\ \nabla A,\ A.$$

$$\mathrm{Err}_{\mathfrak{q}}(\check{\Gamma}, \check{R}) \quad - \quad \text{quadratic in } \check{\Gamma}, \check{R}$$

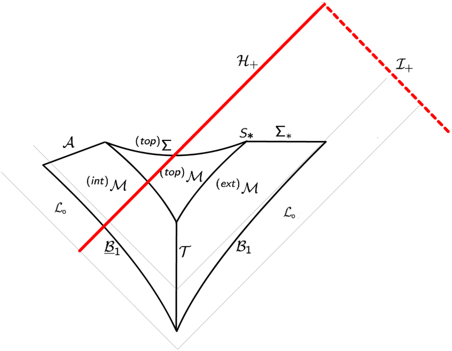
FACT. To control all other terms we need to fix a gauge! .

Adapted null frame + functions (r, θ).

STRATEGY

1. Expect to be able to treat A using gRW
2. To control Err we need to control all $\check{\Gamma}, \check{R}$.
3. To control all other components of $\check{\Gamma}, \check{R}$ we need an appropriate gauge condition
4. Set up a continuation argument, based on
   - Family of finite **GCM-admissible spacetimes**
   - Boot-strap assumptions for $\check{\Gamma}, \check{R}$.

GCM ADMISSIBLE $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$



- S_* GCM surface, **effective uniformization** coordinates. Determines the axis and angular momentum.
- Σ_* GCM hypersurface
- ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$ Determined by transport equations form Σ_*
- Spacetime, including the axis, mass and angular momentum, are constantly upgraded.

MAIN RESULT

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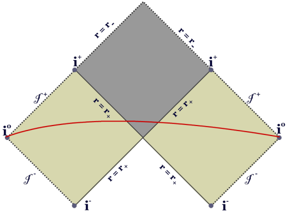
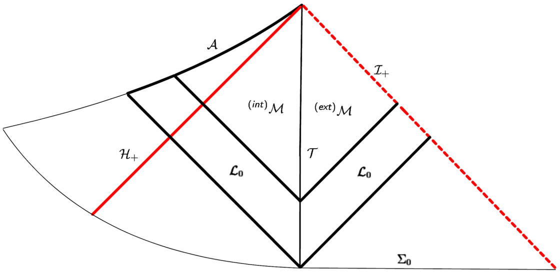


THEOREM[K-Szeftel(2021)] True if $|a|/m \ll 1$.

MAIN RESULT

- INITIAL LAYER $\mathcal{L}_0$ (K-Nicolo)
- FOLIATED BY RETARDED AND ADVANCED TIMES $u, \underline{u}$
- FUTURE NULL INFINITY $\mathcal{I}_0^+$
- EVENT HORIZON $\mathcal{M}_\infty$ limit of finite **GCM admissible** spacetimes



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