Inflationary scalar and tensor hair

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Classical scalar and tensor hair

Visualizing small differences in the number of e-folds

Quantum inflationary scalar and tensor hair

Stochastic slow-roll inflation

Application: equilibrium state of a scalar field in the de Sitter background

Conclusions

The term 'cosmic no-hair theorem' should not be understood literally!

All inflationary models, in spite of locally approaching the de Sitter space-time inside the Hubble radius, have scalar and tensor 'hair' - spatial inhomogeneity - outside it. This inhomogeneity does not disappear with time. Just the opposite, its amplitude at a given comoving scale typically remains constant not only during inflation, but long time after its end up to the moment of the second Hubble radius crossing of this scale. Moreover, a part of scalar inflationary hair have been already observed through measurements of CMB angular temperature anisotropy and polarization, and we expect the discovery of tensor hair (primordial gravitational wave background from inflation) in future. In terms of 'no-hair' theorems, this situation is similar, but just opposite to that in General Relativity (GR), where we have the no-hair property of black holes outside their event horizons, but not inside them.

GR with a cosmological constant

De Sitter is *not* a generic late-time attractor!

Generic late-time asymptote of classical solutions of GR with a cosmological constant Λ both without and with hydrodynamic matter (A. A. Starobinsky, JETP Lett. 37, 55 (1983), also called the Fefferman-Graham expansion):

$$ds^2 = dt^2 - \gamma_{ik} dx^i dx^k$$

$$\gamma_{ik} = e^{2H_0 t} a_{ik} + b_{ik} + e^{-H_0 t} c_{ik} + \ldots$$

where $H_0^2 = \Lambda/3$ and the matrices a_{ik}, b_{ik}, c_{ik} are functions of spatial coordinates. a_{ik} contains two independent physical functions (after 3 spatial rotations and 1 shift in time + spatial dilatation) and can be made unimodular, in particular. b_{ik} is unambiguously defined through the 3-D Ricci tensor constructed from a_{ik}. c_{ik} contains a number of arbitrary physical functions (two - in the vacuum case, or with radiation) - classical tensor hair.

Power-law inflation - scalar hair

A similar but more complicated construction with an additional dependence of H on spatial coordinates in the case of power-law inflation driven by a scalar field with an exponential potential:

$$L = -\frac{R}{2} + \frac{1}{2} g^{ik}\Phi_{,i}\Phi_{,k} - V_0 e^{-\mu\Phi}, \; a(t) \propto t^q, q = \frac{2}{\mu^2} \gg 1$$

$(8\pi G = c = 1)$.
Generic late-time asymptote with both classical scalar and tensor hair (V. Müller, H.-J. Schmidt and A. A. Starobinsky, Class. Quant. Grav. 7, 1163 (1990)):

$$ds^2 = dt^2 - t^{2q} h_{\alpha\beta} dx^\alpha dx^\beta \,,$$

$$h_{\alpha\beta} = a_{\alpha\beta} + \sum_n b^{(n)}_{\alpha\beta} t^{-n} \,,$$

$$\Phi = \frac{2}{\mu} \ln t - \sum_n \Phi^{(n)} t^{-n} \,,$$

where $n \in kn_1 + ln_2, k = 3q - 1, l = 2(q - 1)$, n_1, n_2 integers ≥ 0, but at least one of then has to be positive.

Similar solution for the conformally related $L = f(R) \propto R^M, 1 < M < 2$ gravity where $q = \frac{(M-1)(2M-1)}{2-M}$.

For viable inflationary cosmological models, in which the duration of an inflationary stage is finite for any observer inside our past light cone, inflation is an intermediate attractor.

Theorem. In inflationary models in GR and $f(R)$ gravity, there exists an open set of classical solutions with a non-zero measure in the space of initial conditions at curvatures much exceeding those during inflation which have a metastable inflationary stage with a given number of e-folds.

For the $R + R^2$ model, this was proved in A. A. Starobinsky and H.-J. Schmidt, Class. Quant. Grav. 4, 695 (1987).

Outcome of inflation

In the super-Hubble regime ($k \ll aH$) in the coordinate representation:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dx^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\mathcal{R}(\mathbf{r})\delta_{lm} + \sum_{a=1}^{2} g^{(a)}(\mathbf{r})\, e_{lm}^{(a)}$$

$$e_l^{l(a)} = 0, \;\; g^{(a)}_{,l}\, e_m^{l(a)} = 0, \;\; e_{lm}^{(a)}\, e^{lm(a)} = 1$$

$\mathcal{R}$ describes primordial scalar perturbations, g – primordial tensor perturbations (primordial gravitational waves (GW)).

The most important quantities:

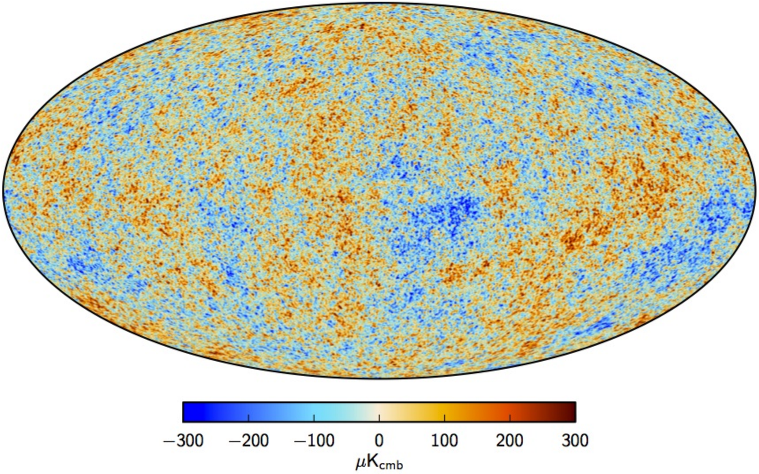
$$n_s(k) - 1 \equiv \frac{d \ln P_{\mathcal{R}}(k)}{d \ln k}, \quad r(k) \equiv \frac{P_g}{P_{\mathcal{R}}}$$

Existence of constant modes

For FLRW models filled by ideal fluids, it was known already to Lifshitz (1946). For a wide class of modified scalar-tensor gravity theories, it was proved in A. A. Starobinsky, S. Tsujikawa and J. Yokoyama, Nucl. Phys. B 610, 383 (2001). However, their existence is much more general. From the mathematical point of view, constant modes appear simply due to the existence of non-degenerate solutions of the same gravity models in the isotropic and spatially flat FLRW space-time. By construction, these solutions always have 3 non-physical (gauge) arbitrary constants of integration due to the possibility of arbitrary and independent rescaling of all spatial coordinates. Making these constants slightly inhomogeneous converts them to the leading terms of physical constant modes (one scalar and two tensor ones). Moreover, it straightforwardly follows from this that these constants (now functions of spatial coordinates) need not be small, they can be arbitrarily large.

CMB temperature anisotropy

Planck-2015: P. A. R. Ade et al., arXiv:1502.01589



New cosmological parameters relevant to inflation

Now we have numbers: N. Agranim et al., arXiv:1807.06209

The primordial spectrum of scalar perturbations has been measured and its deviation from the flat spectrum $n_s = 1$ in the first order in $|n_s - 1| \sim N_H^{-1}$ has been discovered (using the multipole range $\ell > 40$):

$$< \mathcal{R}^2(\mathbf{r}) >= \int \frac{P_{\mathcal{R}}(k)}{k}\, dk, \;\; P_{\mathcal{R}}(k) = (2.10 \pm 0.03) \cdot 10^{-9} \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$$k_0 = 0.05 \ \mathrm{Mpc}^{-1}, \quad n_s - 1 = -0.035 \pm 0.004$$

Two fundamental observational constants of cosmology in addition to the three known ones (baryon-to-photon ratio, baryon-to-matter density and the cosmological constant). Existing inflationary models can predict (and predicted, in fact) one of them, namely $n_s - 1$, relating it finally to $N_H = \ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$. (note that $(1 - n_s) N_H \sim 2$).

The most recent upper limit on r

BICEP/Keck Collaboration: P. A. R. Ade et al., Phys. Rev. Lett. 127, 151301 (2021); arXiv:2110.00483:

$r_{0.05} < 0.036$ at the 95% C.L.

For comparison, in the chaotic inflationary model $V(\varphi) \propto |\varphi|^n$, $r = \frac{4n}{N}$, $1 - n_s = \frac{n+2}{2N}$. The r upper bound gives $n \lesssim 0.5$ for $N_{0.05} = (55 - 60)$, but then $1 - n_s \leq 0.023$. Thus, this model is disfavoured by observational data.

Kinematic origin of scalar perturbations

Local duration of inflation in terms of $N_{tot} = \ln\left(\frac{a(t_{fin})}{a(t_{in})}\right)$ is different in different points of space: $N_{tot} = N_{tot}(\mathbf{r})$. Then

$$\mathcal{R}(\mathbf{r}) = \delta N_{tot}(\mathbf{r})$$

Correct generalization to the non-linear case: the space-time metric after the end of inflation at super-Hubble scales

$$ds^2 = dt^2 - a^2(t)e^{2N_{tot}(\mathbf{r})}(dx^2 + dy^2 + dz^2)$$

First derived in A. A. Starobinsky, Phys. Lett. B 117, 175 (1982) in the case of one-field inflation.

Visualizing small differences in the number of e-folds

Duration of inflation in terms of e-folds was finite for all points inside our past light cone. For $\ell \lesssim 50$, neglecting the Silk and Doppler effects, as well as the ISW effect due the presence of dark energy,

$$\frac{\Delta T(\theta,\phi)}{T_\gamma} = -\frac{1}{5}\mathcal{R}(r_{LSS},\theta,\phi) = -\frac{1}{5}\delta N_{tot}(r_{LSS},\theta,\phi)$$

For $n_s = 1$,

$$\ell(\ell+1)C_{\ell,s} = \frac{2\pi}{25}P_\zeta$$

For $\frac{\Delta T}{T} \sim 10^{-5}$, $\delta N \sim 5\times 10^{-5}$, and for $H \sim 10^{14}\,\mathrm{GeV}$, like in the minimal (one-parametric) inflationary models, $\delta t \sim 5t_{Pl}$!

Planck time intervals are seen by the naked eye!

Quantum inflationary scalar and tensor hair

Successive construction of viable slow-roll inflationary models is based on two independent assumptions.

1. Existence of a metastable quasi-de Sitter stage in our remote part which preceded the hot Big Bang. During it, the expansion of the Universe was accelerated and close to the exponential one, $|\dot{H}| \ll H^2$.

2. The origin of all inhomogeneities in the present Universe is the effect of gravitational creation of pairs of particles - antiparticles and field fluctuations during inflation from the adiabatic vacuum (no-particle) state for Fourier modes covering all observable range of scales (and possibly somewhat beyond).

In fact, metric perturbations h_{lm} are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in $\mathcal{R},\ g$).

In particular:

$$\hat{\mathcal{R}}_k = \mathcal{R}_k\, i(\hat{a}_{\mathbf{k}} - \hat{a}^{\dagger}_{\mathbf{k}}) + \mathcal{O}\left((\hat{a}_{\mathbf{k}} - \hat{a}^{\dagger}_{\mathbf{k}})^2\right) + ... + \mathcal{O}(10^{-100})(\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}) + ,,,$$

The last term is time dependent, it is affected by physical decoherence and may become larger, but not as large as the second term.

Remaining quantum coherence: deterministic correlation between $\mathbf{k}$ and $-\mathbf{k}$ modes - shows itself in the appearance of acoustic oscillations (primordial oscillations in case of GW).

All these predictions are beyond semiclassical gravity!

Semiclassical gravity: space-time metric g_{ik} is not quantized and

$$\frac{1}{8\pi G}\left(R_{\mu}^{\nu}-\frac{1}{2}\,\delta_{\mu}^{\nu}R\right)(g_{ik})=\left\langle\,\hat{T}_{\mu}^{\nu}\right\rangle$$

Instead,

$$\frac{1}{8\pi G}\left(\hat{R}_{\mu}^{\nu}-\frac{1}{2}\,\delta_{\mu}^{\nu}\hat{R}\right)(\hat{g}_{ik})=\hat{T}_{\mu}^{\nu}$$

is used.

$\langle\mathcal{R}\rangle=0$ does not mean the absence of perturbations.

Perturbative anomalous growth of light scalar fields in the de Sitter space-time

Background - fixed - de Sitter or, more interestingly, quasi-de Sitter space-time (slow roll inflation).
Occurs for $0 \le m^2 \ll H^2$ where $H \equiv \frac{\dot{a}}{a}$, $a(t)$ is a LFRW scale factor. The simplest and textbook example:
$m = 0$, $H = H_0 = const$ for $t \ge t_0$ and the initial quantum state of the scalar field at $t = t_0$ is the adiabatic vacuum for modes with $k/a(t_0) \gg H_0$ and some infrared finite state otherwise:

$$< \phi^2 >= \frac{H_0^2 N}{4\pi^2} + const$$

Here $N = \ln \frac{a}{a(t_0)} \gg 1$ is the number of e-folds from the beginning of inflation and the constant depends on the initial quantum state (Linde, 1982; AS, 1982; Vilenkin and Ford, 1982).
Straightforward generalization to the slow-roll case $|\dot{H}| \ll H^2$.

For $0 < m^2 \ll H^2$, the Bunch-Davies equilibrium value

$$< \phi^2 >= \frac{3H_0^4}{8\pi^2 m^2} \gg H_0^2$$

is reached after a large number of e-folds $N \gg \frac{H_0^2}{m^2}$.
Purely infrared effect - creation of real field fluctuations; renormalization is not important and does not affect it.

For the de Sitter inflation (gravitons only) (AS, 1979):

$$P_g(k) = \frac{16GH_0^2}{\pi}; \quad < h_{ik}h^{ik} >= \frac{16GH_0^2N}{\pi} .$$

The assumption of small perturbations breaks down for $N \gtrsim 1/GH_0^2$. Still ongoing discussion on the final outcome of this effect. My opinion - no screening of the background cosmological constant, instead - stochastic drift through an infinite number of locally de Sitter, but globally non-equivalent vacua.

But scalar perturbations are always larger than tensor ones in slow-roll inflationary models, and they become non-linear at very large scales earlier!

Stochastic approach to inflation ("stochastic inflation"):

$$\hat{R}_\mu^\nu - \frac{1}{2}\delta_\mu^\nu \hat{R} = 8\pi G\, \hat{T}_\mu^\nu(\hat{g}_{\alpha\beta})$$

- not as a function of $< \hat{g}_{\alpha\beta} >$!

Leads to QFT in a stochastic background.

Stochastic inflation:

1) can deal with an arbitrary large (though sufficiently smooth) global inhomogeneity;

2) takes backreaction of created fluctuations into account;

3) goes beyond any finite order of loop corrections.

Fully developed in Starobinsky (1984,1986) though the first simplified application (but beyond the one-loop approximation) was already in Starobinsky (1982).

1. A. A. Starobinsky, Phys. Lett. B 117, 175 (1982).

2. A. A. Starobinsky, Lect. Notes in Physics 246, 107 (1986).

Langevin equation for the large-scale field

The first main idea: splitting of the inflaton field ϕ into a large-scale and a small-scale parts with respect to H. More exactly, the border is assumed to lie at $k = \epsilon a H$ with $\exp\left(-\frac{H^2}{|\dot{H}|}\right) \ll \epsilon \ll 1$.

The second main idea: a non-commutative part of the large-scale field is very small (it is composed from decaying modes), so we may neglect it. Then the remaining part is equivalent (not equal!) to a stochastic c-number (classical) field with some distribution function.

$$\frac{d\phi}{d\tau^{(n)}} = -\frac{1}{3H^{n+1}}\frac{dV}{d\phi} + f ~,$$

$$< f(\tau_1^{(n)})\, f(\tau_2^{(n)}) >= \frac{H^{3-n}}{4\pi^2}\, \delta(\tau_1^{(n)} - \tau_2^{(n)})\,.$$

The time-like variables $\tau^{(n)} = \int H^n(t, \mathbf{r})\, dt$, where $H^2 = 8\pi G V(\phi)/3$.

This is not a time reparametrization $t \to f(t)$ in GR. Different $\tau^{(n)}$ describe different stochastic processes and even have different dimensionality. Different "clocks" are needed to measure them:

1) $n = 0$: phase of a wave function of a massive particle ($m \gg H$);

2) $n = 1$: scalar metric perturbations (δN formalism);

3) $n = 3$: dispersion of a light scalar field generated during inflation

$$< \chi^2 >= \frac{1}{4\pi^2} < \int H^3 \, dt >= \frac{< \tau^{(3)} >}{4\pi^2} \, .$$

See F. Finelli *et al.*, Phys. Rev. D **79**, 044007 (2009) for more details.

The Gaussian white noise f describes the flow of small-scale linear field modes through the border $k = \epsilon a H$ to the large-scale region in the course of the universe expansion.

Applicability conditions – the standard slow-roll ones:

$$V'^2 \ll 48\pi G V^2, \quad |V''| \ll 8\pi G V/3$$

Einstein-Smoluhovsky (Fokker-Planck) equation

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial \phi}\left(\frac{V'}{3H^{n+1}}\rho\right) + \frac{1}{8\pi^2}\frac{\partial^2}{\partial \phi^2}\left(H^{3-n}\rho\right) .$$

Probability conservation: $\int \rho \, d\phi = 1$.

Remarks

- More generally, the last term can be written the form

$$\frac{1}{8\pi^2}\frac{\partial}{\partial \phi}\left(H^{(3-n)\alpha}\frac{\partial}{\partial \phi}\left(H^{(3-n)(1-\alpha)}\rho\right)\right)$$

  with $0 \leq \alpha \leq 1$.
  $\alpha = 0$ – Ito calculus.
  $\alpha = 1/2$ – Stratonovich calculus.
  However, keeping terms explicitly depending on α exceeds the accuracy of the stochastic approach. Thus, α may put 0.

- All results are independent of the form of a cutoff in the momentum space as far as it occurs for $k \ll aH$ $(\epsilon \ll 1)$.
- Backreaction is taken into account: $\delta T_\mu^\nu = (V - V_{clas})\,\delta_\mu^\nu$.
- No necessity in any infrared cutoff. Problems with the so called "volume weighting" arise because quantities like $a^3\rho$ are considered which are not normalizable, thus, they may not be considered as probabilities of anything from the mathematical point of view ("unitarity breaking"). Their physical justification is also flawed since it based on the wrong assumption that all Hubble physical volumes ("observers") emerging from expansion of a previous inflationary patch are clones of each other while it is not so.

Transition to predictions for the post-inflationary evolution

From $\rho(\phi,\tau)$ during inflation to the distribution $w(\tau)$ over the total local duration of inflation:

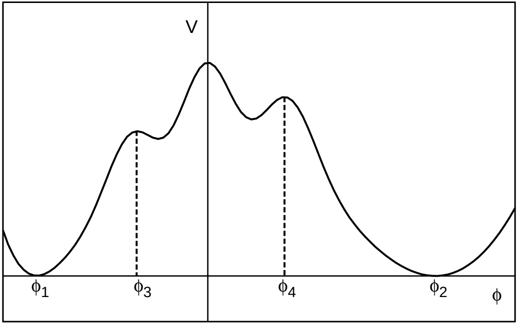
$$w(\tau) = \lim_{\phi\to\phi_{end}} j = \lim_{\phi\to\phi_{end}} \frac{|V'|}{3H^{n+1}}\,\rho(\phi,\tau)\,.$$

For the graceful exit to a post-inflationary epoch, the stochastic force should be much less than the classical one during last e-folds of inflation.

The same way to obtain the joint distribution $w(0,\tau_1;\ |\mathbf{r}|,\tau_2)$ from the 2-point joint probability distribution $\rho(\phi_1,0,\tau_1;\phi_2,|\mathbf{r}|,\tau_2)$ during inflation.

Probabilities to go to different vacua after inflation

Let inflation may end in two vacua: $\phi = \phi_1$ and $\phi = \phi_2$ with $V(\phi_1) = V(\phi_2) = 0$ (to consider a larger number of post-inflationary vacua, ϕ should have more than one-dimensional internal space).



Boundary conditions at the end of inflation:
$\rho(\phi_1, \tau) = \rho(\phi_2, \tau) = 0$.

Method of calculation (Starobinsky (1984,1986), see also V. Vennin and A. A. Starobinsky, Eur. Phys. J. C 75, 413 (2015); arXiv:1506.04732 for more details): consider the quantities

$$Q_m(\phi) = \int_0^\infty \tau^m \rho(\phi, \tau)\, d\tau$$

where $\tau = 0$ corresponds to the local beginning of inflation.
$Q_m(\phi_1) = Q_m(\phi_2) = 0$.

By integrating the Fokker-Planck equation over τ, we get for $m = 0$:

$$Q_0(\phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} d\psi \, \exp\left(-\frac{\pi}{GH^2(\psi)}\right) \times$$

$$\left(C_0 - \int_{\phi_1}^{\psi} \rho_0(\psi_1)\, d\psi_1\right) ,$$

$$C_0 = \frac{\int_{\phi_1}^{\phi_2} d\phi \, \exp\left(-\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} \rho_0(\psi)\, d\psi}{\int_{\phi_1}^{\phi_2} d\phi \, \exp\left(-\frac{\pi}{GH^2(\phi)}\right)} .$$

$P_1 = C_0$ – the absolute probability to go to the vacuum $\phi = \phi_1$;
$P_2 = 1 - C_0$ – the absolute probability to go to the vacuum $\phi = \phi_2$.
No n dependence in C !

Local duration of inflation

$$Q_1(\phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} d\psi \exp\left(-\frac{\pi}{GH^2(\psi)}\right) \times$$

$$\left(C_1 - \int_{\phi_1}^{\psi} Q_0(\psi_1)\, d\psi_1\right) ,$$

$$C_1 = \frac{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} Q_0(\psi)\, d\psi}{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right)} .$$

$$< \tau_1 >= \frac{C_1}{C_0}, \quad < \tau_2 >= \frac{\tilde{C}_1}{1 - C_0},$$

$$< \tau >_{tot} = C_0 < \tau_1 > + (1 - C_0) < \tau_2 > = \int_{\phi_1}^{\phi_2} Q_0(\phi)\, d\phi .$$

$\tilde{C}_1$ is C_1 with ϕ_1 and ϕ_2 interchanged.

Correlations and PDF

Following A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994).

In the leading approximations, all Green functions and joint n-point probability distributions of the inflaton field can be expressed through solutions of the same Fokker-Planck equation with different initial conditions only. In particular, in the case $H \approx H_0$ during inflation (for simplicity only), the general two-point PDF for points lying outside each other's light cones in the stochastic approach is:

$$\rho_2[\phi_1(\mathbf{r_1}, t_1), \phi_2(\mathbf{r_2}, t_2)] =$$

$$\int \Pi[\phi_1(\mathbf{r_1}, t_1)|\phi_r(\mathbf{r_1}, t_r)]\Pi[\phi_2(\mathbf{r_2}, t_2)|\phi_r(\mathbf{r_2}, t_r)]\rho_1(\phi_r, t_r)\, d\phi_r$$

where t_r is the time in the past when both points were inside one Hubble volume and $\Pi[\phi_1(\mathbf{r}, t_1)|\phi_2(\mathbf{r}, t_2)]$ satisfies the Fokker-Planck equation with respect to both its time arguments with the initial condition

$$\Pi[\phi_1(\mathbf{r}, t_1)|\phi_2(\mathbf{r}, t_1)] = \delta(\phi_1 - \phi_2)$$

Through the N-formalism - joint probability distributions of a space-time metric after inflation.

For more details including formulas for higher moments in single-field, slow-roll inflation, see V. Vennin and A. A. Starobinsky, Eur. Phys. J. C 75, 413 (2015); arXiv:1506.04732.

QFT of a self-interacting scalar field in the de Sitter background

A.A. Starobinsky and J. Yokoyama, Phys. Rev. D 50, 6357 (1994).

The equilibrium (static) solution for the 1-point distribution:

$$\rho_{eq}(\phi) = \mathrm{const}\, e^{-2v}\,, \quad v = \frac{4\pi^2 V(\phi)}{3H_0^4}\ .$$

Arbitrary Green functions and n-point distributions can be constructed, too, using solutions of the same Fokker-Planck equation.

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 , \quad 0 < \lambda \ll 1, \; H_0^2 = \frac{8\pi G V_0}{3} .$$

Three regimes:

1. Perturbative regime $\sqrt{\lambda} H_0^2 \ll m^2 \ll H_0^2$.

$$< \phi^2 > = \frac{3H_0^4}{8\pi^2 m^2} \left(1 - 3\beta + 24\beta^2 + ...\right) , \quad \beta = \frac{3\lambda H_0^4}{8\pi^2 m^4} .$$

Recently this result was exactly reproduced using the standard two-loop calculation in A. Yu. Kamenshchik, A. A. Starobinsky and T. Vardanyan, arXiv:2109.05625.
Compare to the same result in the one-loop (Gaussian) approximation:

$$< \phi_G^2 > = \frac{3H_0^4}{8\pi^2 m^2} \left(1 - 3\beta + 18\beta^2 + ...\right) .$$

2. Massless self-interacting regime $|m^2| \ll \sqrt{\lambda} H_0^2$.

$$< \phi^2 >= \sqrt{\frac{3}{2\pi^2}} \frac{\Gamma(0.75)}{\Gamma(0.25)} \frac{H_0^2}{\sqrt{\lambda}} \approx 0.132 \frac{H_0^2}{\sqrt{\lambda}}$$

$$< \phi_G^2 >= \frac{1}{\pi\sqrt{8}} \frac{H_0^2}{\sqrt{\lambda}} \approx 0.113 \frac{H_0^2}{\sqrt{\lambda}}$$

3. Symmetry breaking regime $m^2 < 0, \quad \sqrt{\lambda} H_0^2 \ll |m^2| \ll H_0^2$.

$$< \phi^2 >= \frac{|m^2|}{\lambda} + \frac{3 H_0^4}{16\pi^2 |m^2|} + \mathcal{O}\left(e^{-1/(4\beta)}\right)$$

The (modulus of) exponent is the action for the Hawking-Moss instanton.
See also F. Finelli *et al.*, Phys. Rev. D 82, 064020 (2010).

Conclusions

- Both classical and quantum inflationary hair do exist, and scalar hair are even partly observable already.
- They are approximately time-independent in the super-Hubble regime during and after inflation.
- Their Gaussian statistics is in the agreement with the assumed mechanism of their creation from the adiabatic vacuum state during inflation for a finite range of scales covering all observable ones.
- Using the formalism of stochastic slow-roll inflation, the behaviour of scalar hair can be quantitatively described in the strongly non-linear regime.
- No quantum instability of a self-interacting scalar field in the de Sitter background.

CONGRATULATIONS, BEST WISHES

AND NEXT SUCCESSES TO THIBAULT !