

Optimal nonpermanent control problems on time scales

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Sampled-data days

8-10 septembre 2021

Laboratoire de Mathématiques de Bretagne Atlantique, Brest

- 1 Introduction
- 2 Time scale calculus
- 3 PMP for optimal (permanent) control problems on time scales
- 4 PMP for optimal nonpermanent control problems on time scales
- 5 Some general comments
- 6 Specific comments to optimal sampled-data control problems
- 7 PMP for state-constrained optimal nonpermanent control problems on time scales
- 8 A work in progress and challenges

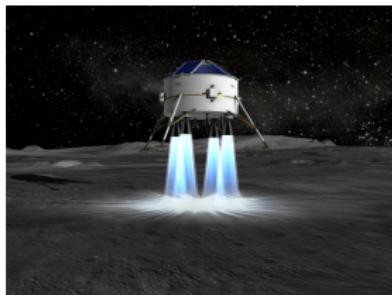
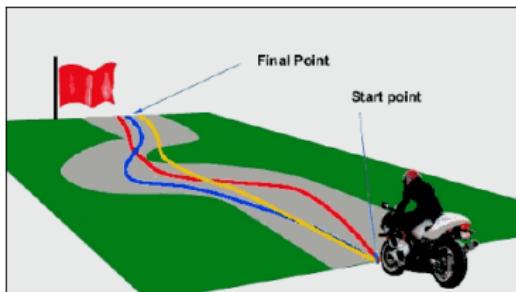
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Aim of the optimal control theory

To steer a controlled dynamical system

- from a given configuration to a desired target ;
- by minimizing a given criterion ;
- and by satisfying some constraints.

Applications in various domains : mechanics, economy, biology, etc.



Numerous mathematical models : ordinary differential equations, evolution partial differential equations, discrete equations, integral equations, stochastic equations, etc.

A typical continuous-time optimal control problem

$$\text{minimize} \quad M(x(T)) + \int_0^T L(x(\tau), u(\tau), \tau) d\tau,$$

subject to

$$\begin{cases} \text{state } x \in \text{AC}([0, T], \mathbb{R}^n), \quad \text{control } u \in L^\infty([0, T], \mathbb{R}^m), \\ \dot{x}(t) = f(x(t), u(t), t), & \text{a.e. } t \in [0, T], \\ x(0) = x_{\text{init}}, \\ u(t) \in U, & \text{a.e. } t \in [0, T], \end{cases}$$

where $U \subset \mathbb{R}^m$ nonempty.

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where $U \subset \mathbb{R}^m$ nonempty.

Hamiltonian : $H(x, u, p, t) := \langle p, f(x, u, t) \rangle - L(x, u, t)$.

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Pontryagin Maximum Principle (PMP)

~ 1950's

If (x^*, u^*) optimal, there exists an adjoint vector $p \in \text{AC}([0, T], \mathbb{R}^n)$ such that

$$\dot{x}^*(t) = \nabla_p H(x^*(t), u^*(t), p(t), t), \quad \dot{p}(t) = -\nabla_x H(x^*(t), u^*(t), p(t), t),$$

$$p(T) = -\nabla M(x(T)),$$

$$u^*(t) \in \arg \max_{v \in U} H(x^*(t), v, p(t), t).$$

Theoretical applications of the PMP

- ~ In some cases : solving explicitly the optimal control problem.
- ~ More generally : getting (partial) informations on the optimal solution.

- Indirect methods (based on PMP) : if the maximization condition writes

$$u^*(t) = \mathcal{F}(x^*(t), p(t), t),$$

then we provide a guess $p_{\text{init}} \in \mathbb{R}^n$ and compute numerically

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- **Direct methods** : a full discretization of the optimal control problem solved numerically with optimization algorithms.

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$$\begin{aligned} & \text{minimize} \quad M(x_N) + \sum_{k=0}^{N-1} L(x_k, u_k, k), \\ & \text{subject to} \quad \left\{ \begin{array}{ll} \text{state } x \in (\mathbb{R}^n)^{N+1}, & \text{control } u \in (\mathbb{R}^m)^N, \\ x_{k+1} - x_k = f(x_k, u_k, k), & k = 0, \dots, N-1, \\ x_0 = x_{\text{init}}, & \\ u_k \in U, & k = 0, \dots, N-1, \end{array} \right. \end{aligned}$$

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First questions

- Why is the Hamiltonian maximization weakened in the discrete setting ?
- Why is the convexity of U required in the discrete setting ?
- Why is there a shift on the adjoint vector in the discrete setting ?
- What happens for hybrid time structures ?

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A mathematical tool to answer these questions

Time scale calculus introduced by Hilger ~ 1990.

Two main objectives :

- **unification** of continuous and discrete analyses ;
- **extension** to more general time structures.

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Definition

A **time scale** \mathbb{T} is a nonempty closed subset of \mathbb{R} .

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Classical examples of time scales

- intervals $[a, b]$ continuous analysis
- finite sets $\{t_0, \dots, t_N\}$ discrete analysis
- mixes $[a, b] \cup \{c\} \cup [d, e]$ hybrid analysis
- $\{0\} \cup q^{\mathbb{N}}$ with $0 < q < 1$ q -differences equations
- Cantor sets fractal analysis

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Right-scattered and right-dense points

A point $r \in \mathbb{T}$ is said to be **right-scattered** if it is right-isolated.

A point $s \in \mathbb{T}$ is said to be **right-dense** if it is not.



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A notation

For any function $x : \mathbb{T} \rightarrow \mathbb{R}^n$, we denote by $x^\sigma := x \circ \sigma$.

Δ -derivative and Lebesgue Δ -integration

A function $x : \mathbb{T} \longrightarrow \mathbb{R}^n$ is said to be **Δ -differentiable** at $t \in \mathbb{T}$ if

$$x^\Delta(t) := \lim_{\substack{\tau \rightarrow t \\ \tau \in \mathbb{T}}} \frac{x^\sigma(t) - x(\tau)}{\sigma(t) - \tau} \in \mathbb{R}^n.$$

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$$\int_{\mathbb{T}} x(\tau) \Delta\tau.$$

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• In the continuous case $\mathbb{T} = [0, T]$:

$$x^\Delta(t) = \dot{x}(t) \quad \text{and} \quad \int_{\mathbb{T}} x(\tau) \Delta\tau = \int_0^T x(\tau) d\tau.$$

• In the discrete case $\mathbb{T} = \{0 = t_0 < \dots < t_N = T\}$:

$$x^\Delta(t_k) = \frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k} \quad \text{and} \quad \int_{\mathbb{T}} x(\tau) \Delta\tau = \sum_{k=0}^{N-1} (t_{k+1} - t_k) x(t_k).$$

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An optimal control problem on a general time scale \mathbb{T}

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PMP

2013

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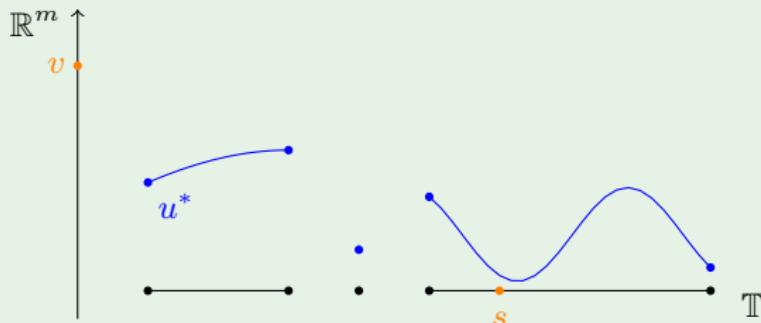
$$\text{if } s \in \text{RD : } u^*(s) \in \arg \max_{v \in \mathbf{U}} H(x^*(s), v, p(s), s),$$

$$\text{if } r \in \text{RS : } \nabla_u H(x^*(r), u^*(r), p^\sigma(r), r) \in \text{N}_{\mathbf{U}}[u^*(r)].$$

Needle perturbation of u^* at $s \in \mathbb{R}D$

$$\forall (s, v) \in \mathbb{R}D \times U, \quad \forall \alpha > 0, \quad u_\alpha(\cdot) := \begin{cases} v & \text{on } [s, s + \alpha) \cap \mathbb{T}, \\ u^*(\cdot) & \text{elsewhere.} \end{cases}$$

L^1 -type perturbation



Variation vector w

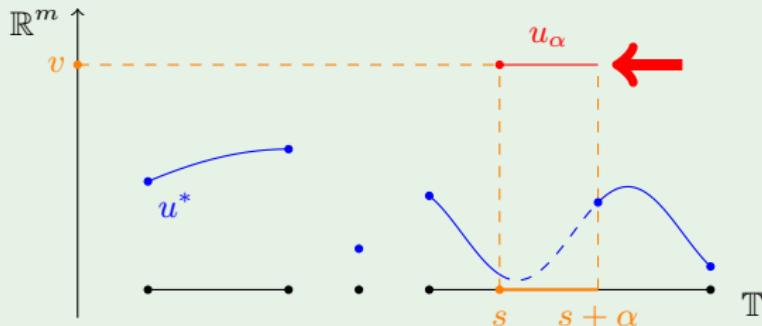
It holds that $x_\alpha = x^* + \alpha w + \text{rest}$ where

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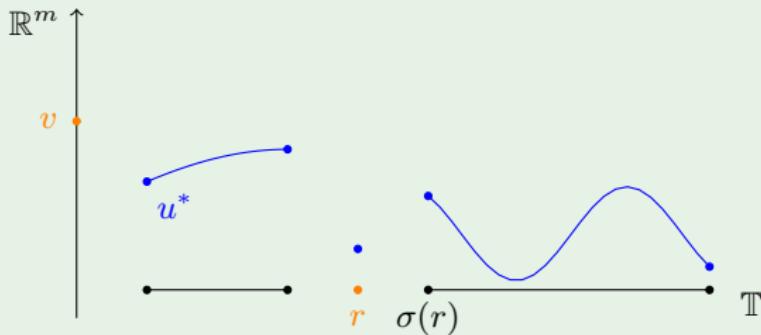
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Convex perturbation of u^* at $r \in \text{RS}$

$$\forall (r, v) \in \text{RS} \times \mathbb{U}, \quad \forall \alpha > 0, \quad u_\alpha(\cdot) := \begin{cases} u^*(r) + \alpha(v - u^*(r)) & \text{at } r, \\ u^*(\cdot) & \text{elsewhere.} \end{cases}$$

L^∞ -type perturbation



Variation vector w

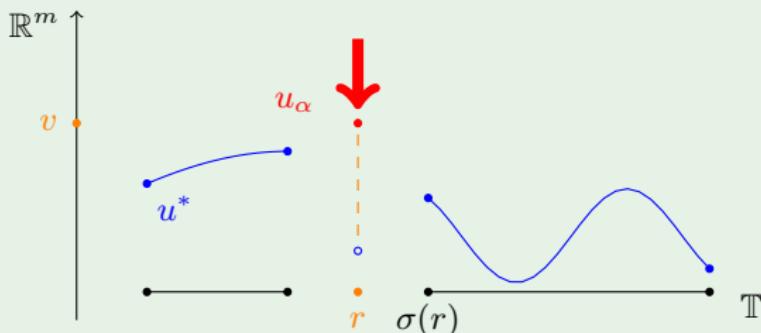
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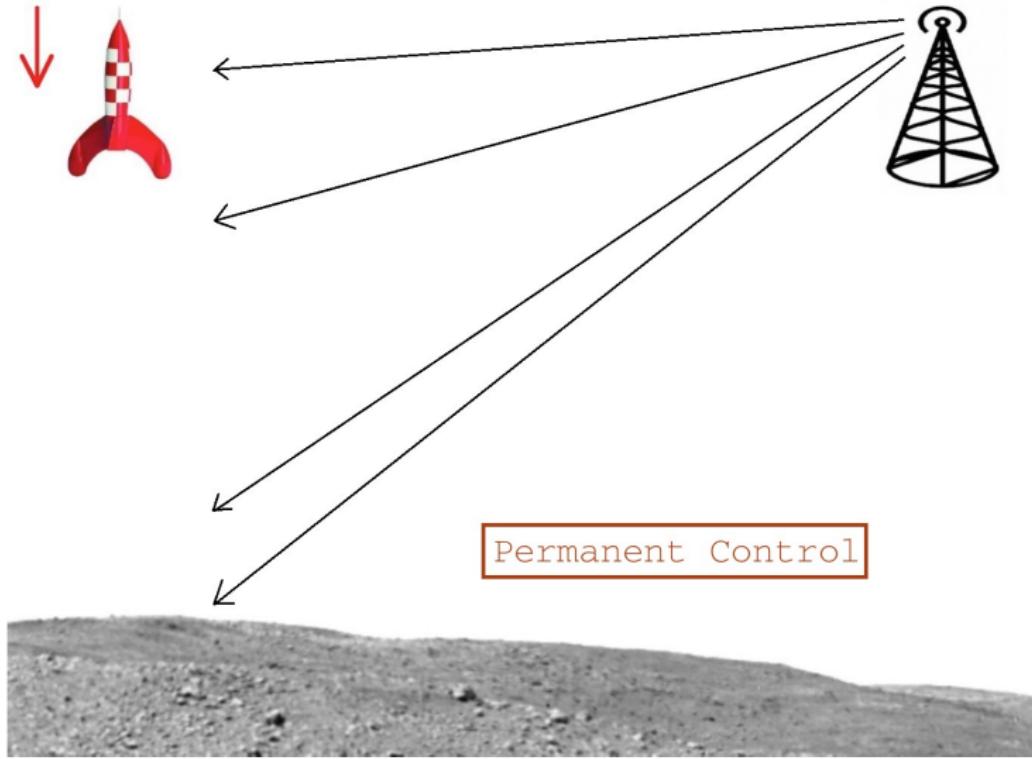
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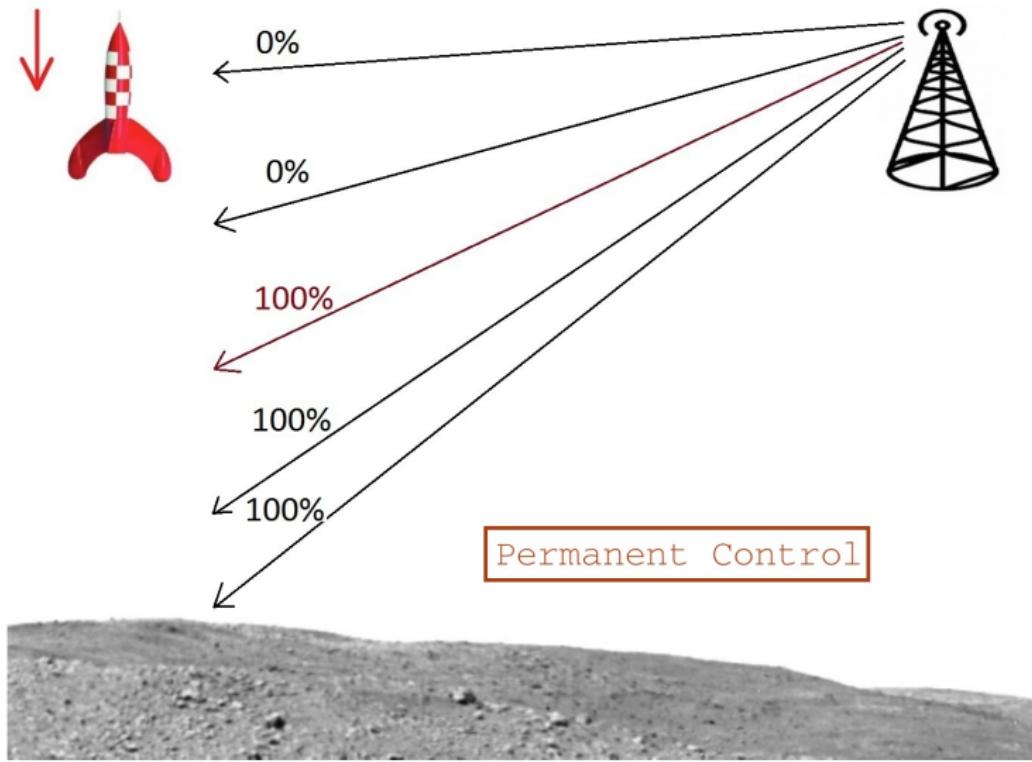
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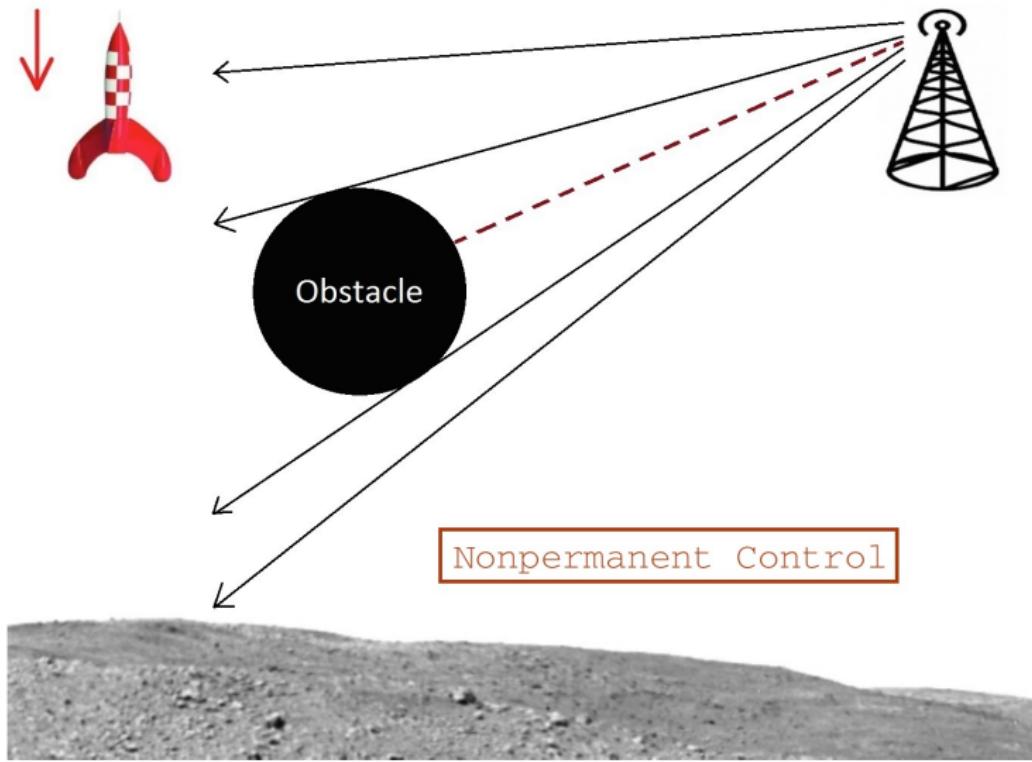
⚠ Classical theories deal with **permanent controls** ⚠



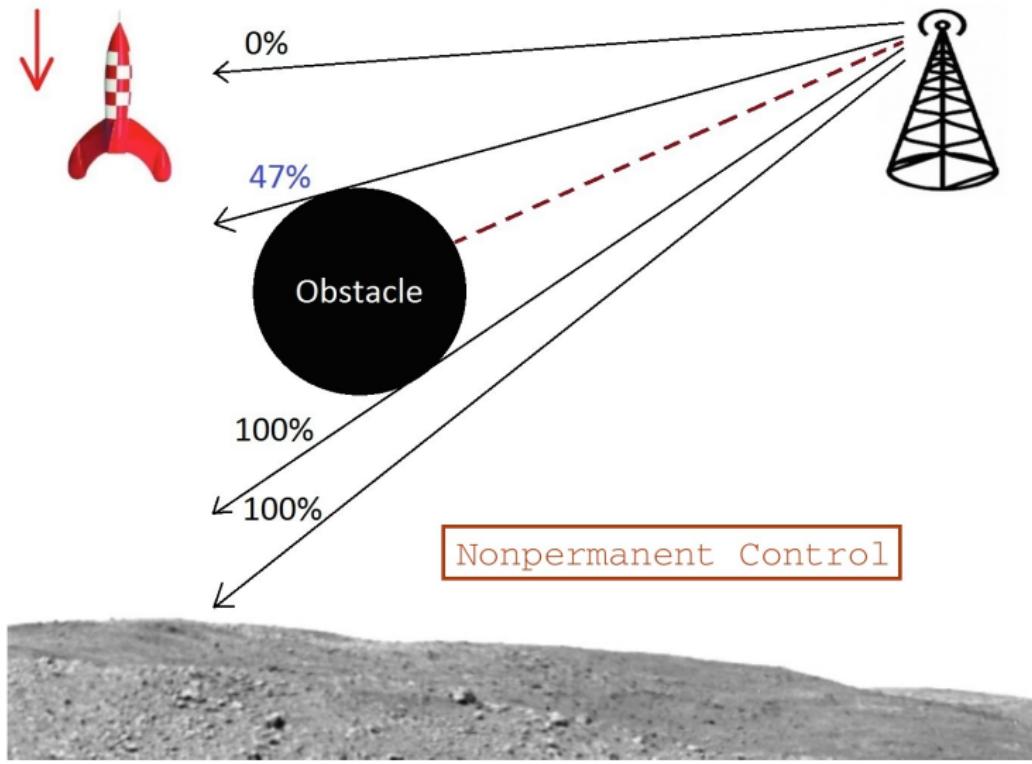
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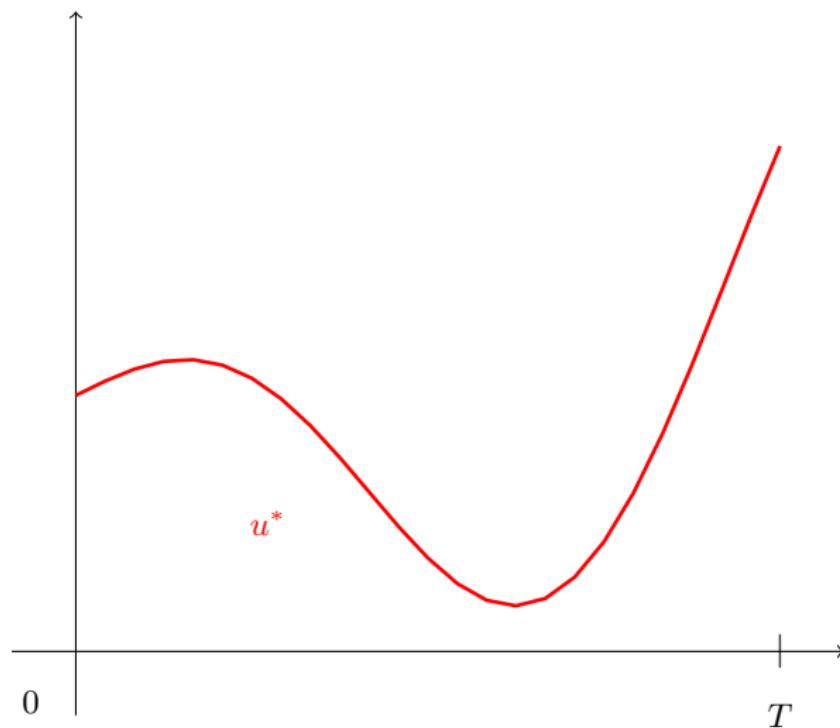
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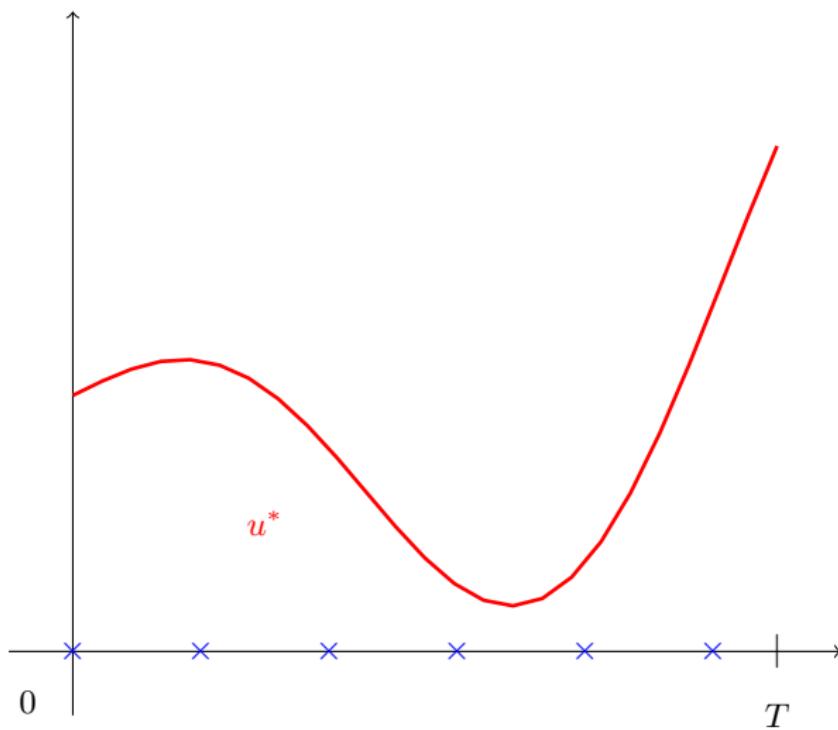
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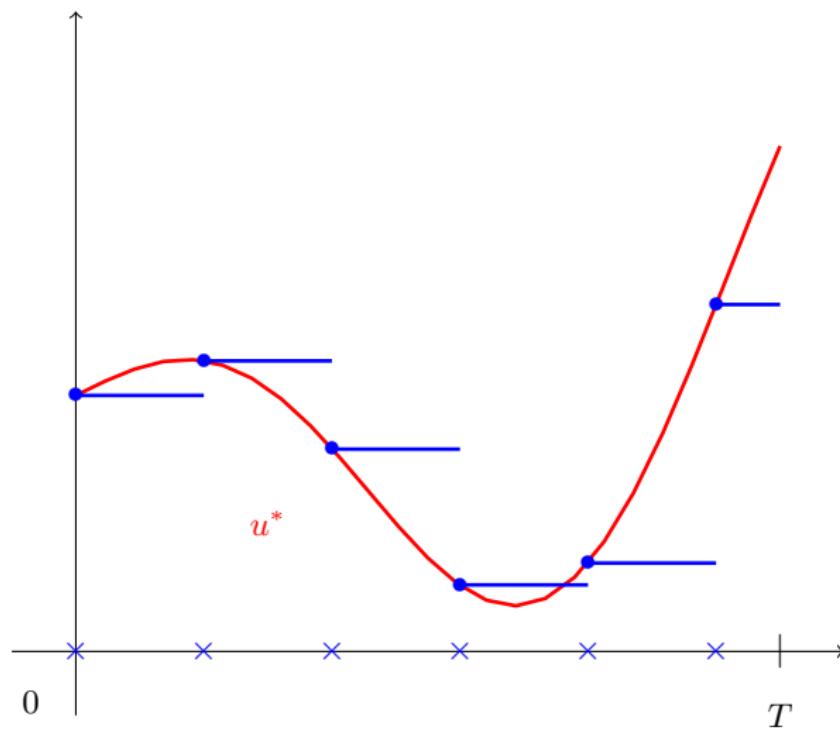
⚠ Optimal (permanent) controls may require ⚠
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⚠ While a permanent modification may be not feasible ⚠
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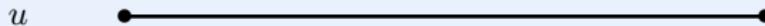
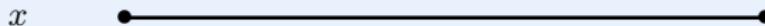


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Permanent controls

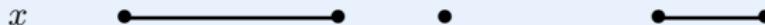
Continuous case :



Discrete case :

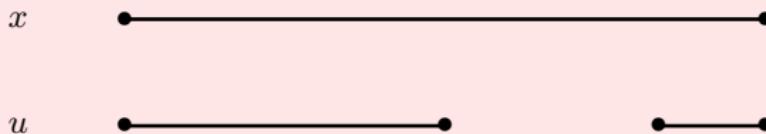


Time scale case :



Nonpermanent controls

Continuous case (with a non-control interval) :



Continuous case (with a sampled-data control) :



Discrete case :



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Nonpermanent controls and frozen values

Consider **two time scales** :

- the state x evolves on \mathbb{T} ;
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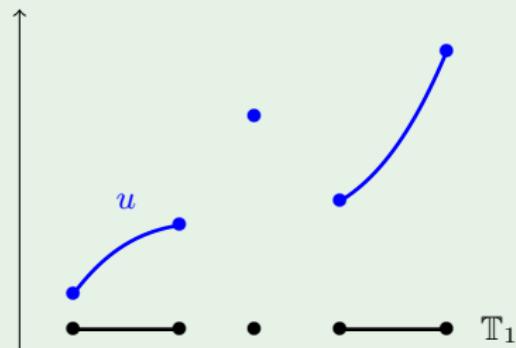
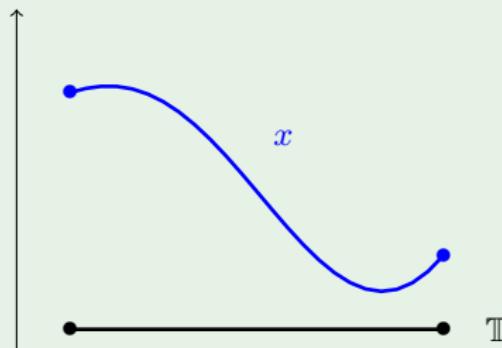
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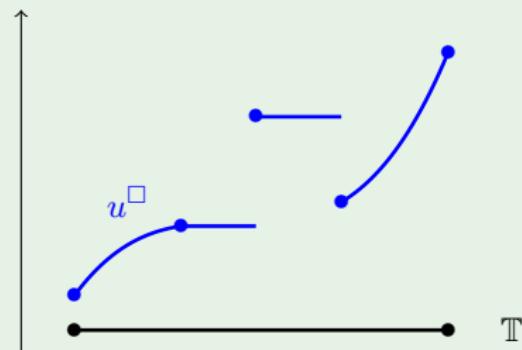
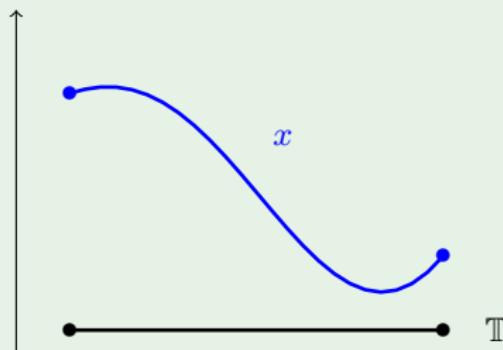
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An optimal nonpermanent control problem on general time scales $\mathbb{T}_1 \subset \mathbb{T}$

$$\text{minimize} \quad M(x(T)) + \int_{[0,T] \cap \mathbb{T}} L(x(\tau), \textcolor{orange}{u^\square(\tau)}, \tau) \Delta\tau,$$

subject to

$$\begin{cases} \text{state } x \in \text{AC}(\mathbb{T}, \mathbb{R}^n), & \text{control } u \in L^\infty(\mathbb{T}_1, \mathbb{R}^m), \\ x^\Delta(t) = f(x(t), \textcolor{orange}{u^\square(t)}, t), & \Delta\text{-a.e. } t \in [0, T] \cap \mathbb{T}, \\ x(0) = x_{\text{init}}, & \\ u(t) \in U, & \Delta_1\text{-a.e. } t \in [0, T] \cap \mathbb{T}_1, \end{cases}$$

where $U \subset \mathbb{R}^m$ nonempty convex.

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PMP

2016

If (x^*, u^*) optimal, there exists an adjoint vector $p \in \text{AC}(\mathbb{T}, \mathbb{R}^n)$ such that

$$x^{*\Delta}(t) = \nabla_p H(x^*(t), u^{*\square}(t), p^\sigma(t), t), \quad p^\Delta(t) = -\nabla_x H(x^*(t), u^{*\square}(t), p^\sigma(t), t),$$

$$p(T) = -\nabla M(x(T)),$$

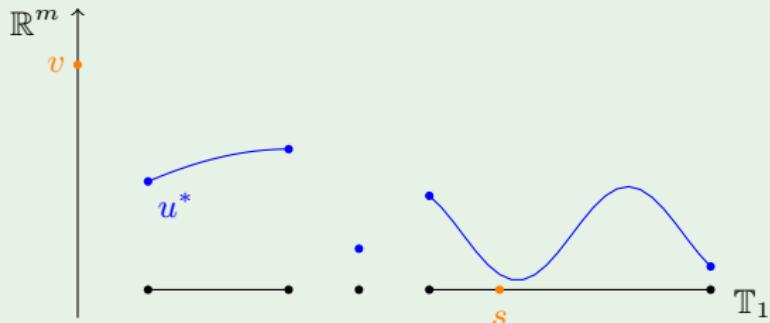
If $s \in \text{RD}_1$: $u^*(s) \in \arg \max_{v \in U} H(x^*(s), v, p(s), s),$

If $r \in \text{RS}_1$: $\int_{[r, \sigma_1(r)) \cap \mathbb{T}} \nabla_u H(x^*(\tau), u^*(r), p^\sigma(\tau), \tau) \Delta\tau \in N_U[u^*(r)].$

Needle perturbation of u^* at $s \in \text{RD}_1$

$$\forall (s, v) \in \text{RD}_1 \times U, \quad \forall \alpha > 0, \quad u_\alpha(\cdot) := \begin{cases} v & \text{on } [s, s + \alpha) \cap \mathbb{T}_1, \\ u^*(\cdot) & \text{elsewhere.} \end{cases}$$

L^1 -type perturbation



Variation vector w

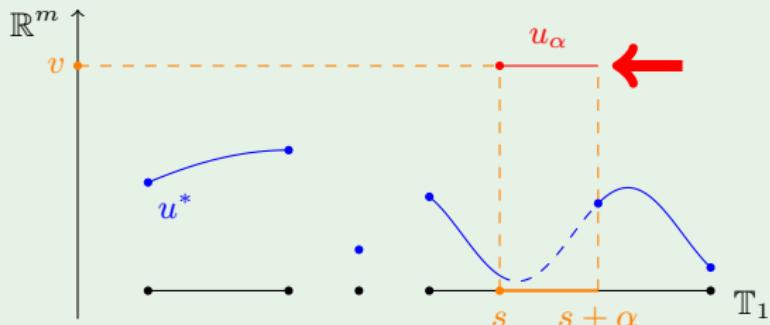
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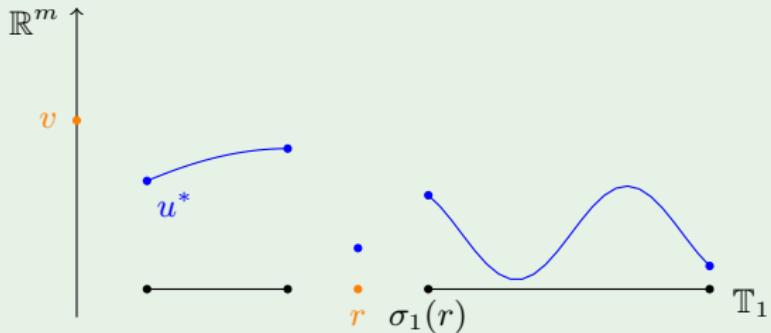
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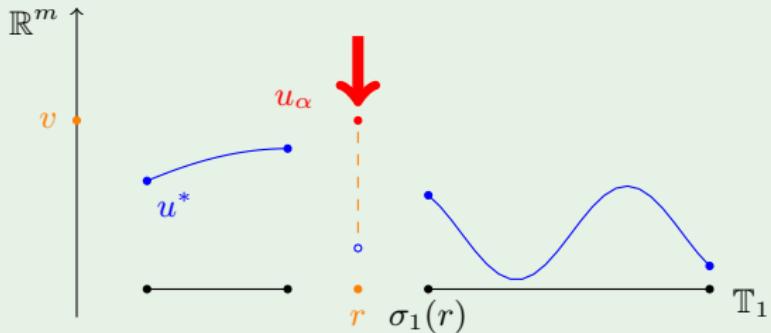
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Same PMP but ...

2013 & 2016

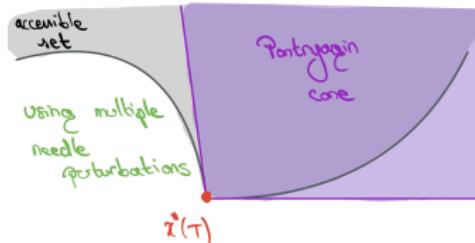
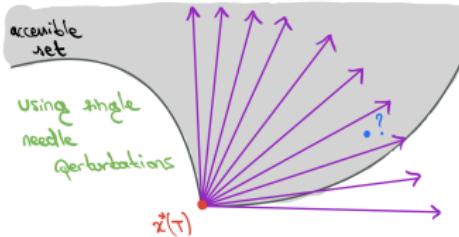
- The Hamiltonian has one variable $\lambda \in \{0, 1\}$ more :

$$H(x, u, p, \lambda, t) := \langle p, f(x, u, t) \rangle - \lambda L(x, u, t).$$

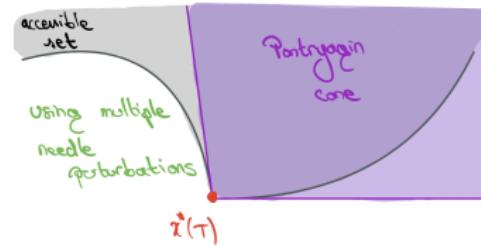
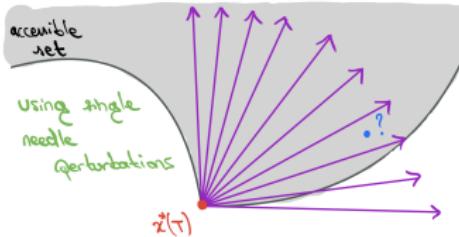
- The transversality condition writes :

$$\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} = \lambda \nabla M(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T))^\top \xi,$$

for some $\xi \in N_S[g(x^*(0), x^*(T))]$.



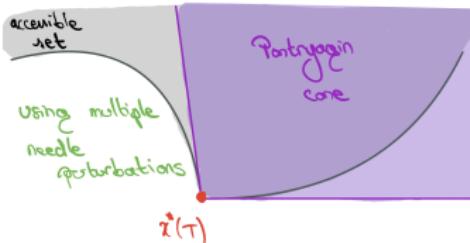
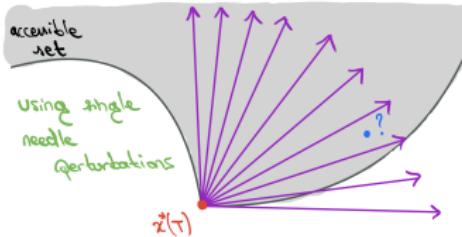
- ~~ Requires single needle perturbations (only).
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An obstruction for multiple needle perturbations on time scale



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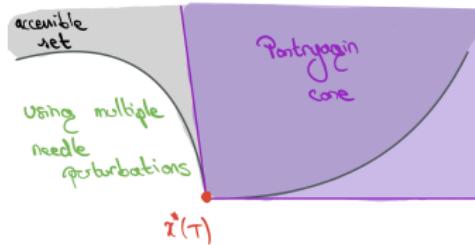
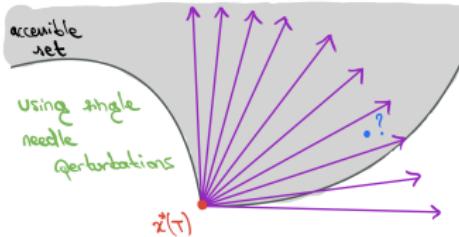


An obstruction for multiple needle perturbations on time scale



Ekeland variational principle

$$\begin{aligned} J_\varepsilon : \quad L^\infty(\mathbb{T}_1, U) \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (u, x_{\text{init}}) &\longmapsto \sqrt{(\mathbf{C}(u, x_{\text{init}}) - \mathbf{C}^* + \varepsilon)^+ + d_S^2(g(x_{\text{init}}, x_{(u, x_{\text{init}})}(T)))} \end{aligned}$$



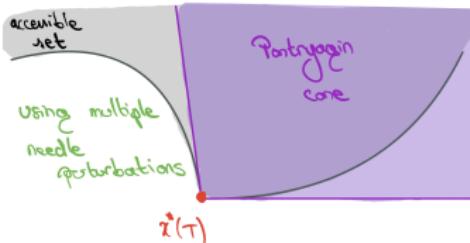
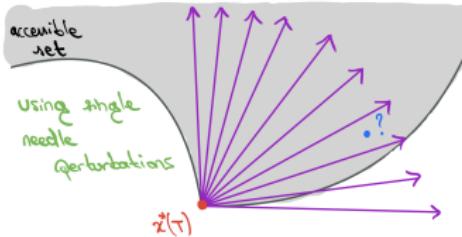
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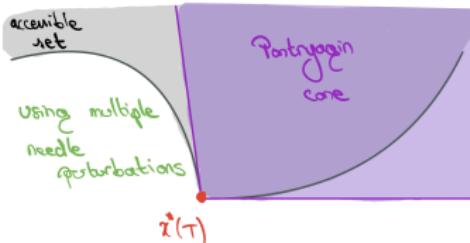
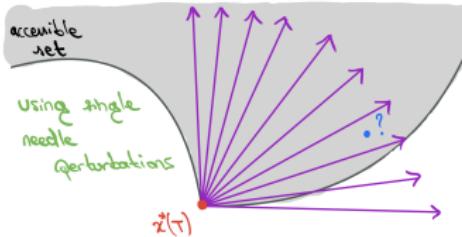
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Challenge n° 1 : derive a PMP in the “difficult” framework ...

... with no assumption on \mathbb{T}_1 and without assuming that U is closed.

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If the following properties are satisfied :

- The problem is feasible ;
- The admissible states are uniformly bounded ;
- U is compact ;
- For all $(x, t) \in \mathbb{R}^n \times \mathbb{T}$, the set of extended velocities

$$\left\{ (f(x, u, t), L(x, u, t) + \gamma) \mid (u, \gamma) \in U \times \mathbb{R}_+ \right\}$$

is convex ;

then there exists an optimal pair (x^*, u^*) .

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Some key points :

- To guarantee that the limit control can be associated to a nonpermanent control.
- To guarantee that it is Δ_1 -measurable (measurable selection theorem).

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Extension/nonextension of some classical properties

Cont-Permanent	Disc-Permanent	Cont-Nonpermanent
Hamiltonian maximization ✓	(✗)	(✗)
\mathcal{H} is continuous ✓	not applicable	(✗)
H is affine in $u \Rightarrow$ saturation of U ✓	✓	✗

Notation : maximized Hamiltonian $\mathcal{H}(t) := H(x^*(t), u^{*\square}(t), p^\sigma(t), \lambda, t)$.

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Challenge n° 2 : recovering a maximization condition

Under a convexity assumption (see Holtzmann & Halkin), one should find :

$$\text{If } s \in \text{RD}_1 : \quad \color{red}{u^*(s) \in \arg \max_{v \in U} H(x^*(s), v, p(s), s)},$$

$$\text{If } r \in \text{RS}_1 : \quad \color{red}{u^*(r) \in \arg \max_{v \in U} \int_{[r, \sigma_1(r)) \cap \mathbb{T}} H(x^*(\tau), v, p^\sigma(\tau), \lambda, \tau) \Delta \tau}.$$

Linear example : optimal consumption

Consider the optimal (permanent) control problem given by

$$\begin{aligned} & \text{minimize} \quad \int_0^{12} (u(\tau) - 1)x(\tau) d\tau, \\ & \text{subject to} \quad \left\{ \begin{array}{ll} \text{state } x \in AC([0, 12], \mathbb{R}), & \text{control } u \in L^\infty([0, 12], \mathbb{R}), \\ \dot{x}(t) = u(t)x(t), & \text{a.e. on } [0, 12], \\ x(0) = 1, & \\ u(t) \in [0, 1], & \text{a.e. on } [0, 12]. \end{array} \right. \end{aligned}$$

The PMP gives the optimal (permanent) control :

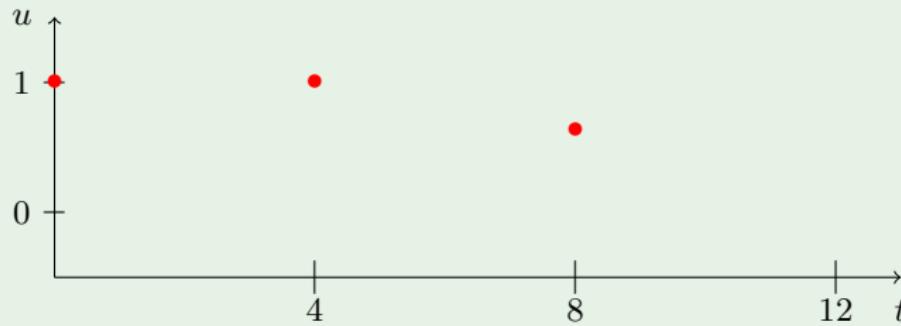


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Consider the optimal sampled-data control problem given by

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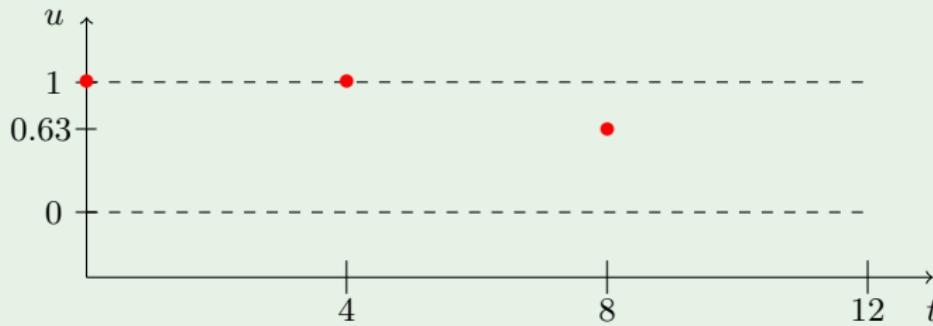


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Averaging due to the PMP conditions

- ~~ Take $\mathbb{T} = [0, T]$.
- ~~ With $\mathbb{T}_1 = [0, T]$, the optimal permanent control satisfies

$$\nabla_u H(x^*(t), u^*(t), p(t), \lambda, t) \in N_U[u^*(t)].$$

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$$\int_{t_k}^{t_{k+1}} \nabla_u H(x^*(\tau), u^*(t_k), p(\tau), \lambda, \tau) d\tau \in N_U[u^*(t_k)].$$

Example : the linear-quadratic (LQ) case

$$\begin{aligned} & \text{minimize} \quad \frac{1}{2} \int_0^T x(\tau)^\top Q(\tau) x(\tau) + u^\square(\tau)^\top R(\tau) u^\square(\tau) \, d\tau, \\ & \text{subject to} \quad \begin{cases} \text{state } x \in AC([0, T], \mathbb{R}^n), & \text{control } u \in L^\infty(\mathbb{T}_1, \mathbb{R}^m), \\ \dot{x}(t) = A(t)x(t) + B(t)u^\square(t), & \text{a.e. } t \in [0, T], \\ x(0) = x_{\text{init}}. & \end{cases} \end{aligned}$$

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$$u^*(t) = R(t)^{-1} B(t) p(t).$$

With $\mathbb{T}_1 = \{0 = t_0 < \dots < t_N = T\}$, the optimal sampled-data control satisfies

$$u^*(t_k) = \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} R(\tau) \, d\tau \right)^{-1} \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} B(\tau) p(\tau) \, d\tau \right).$$

- **Convergence result :** when the maximal distance δ between consecutive sampling times tends to 0, the (unique) optimal sampled-data control converges a.e. to the (unique) optimal permanent control.

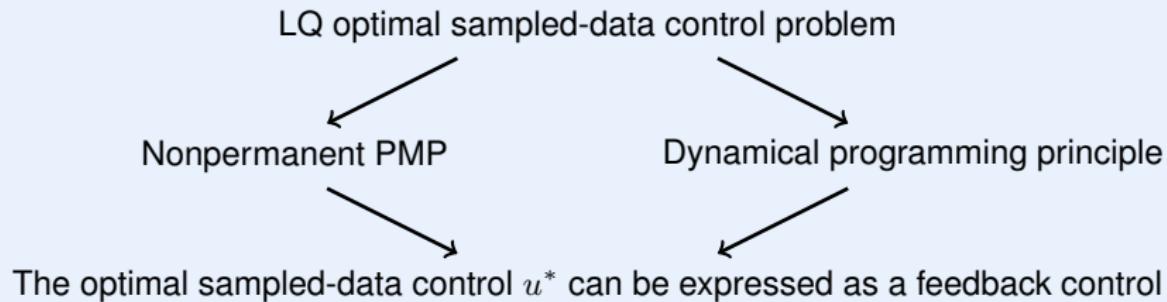
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LQ optimal sampled-data control problem

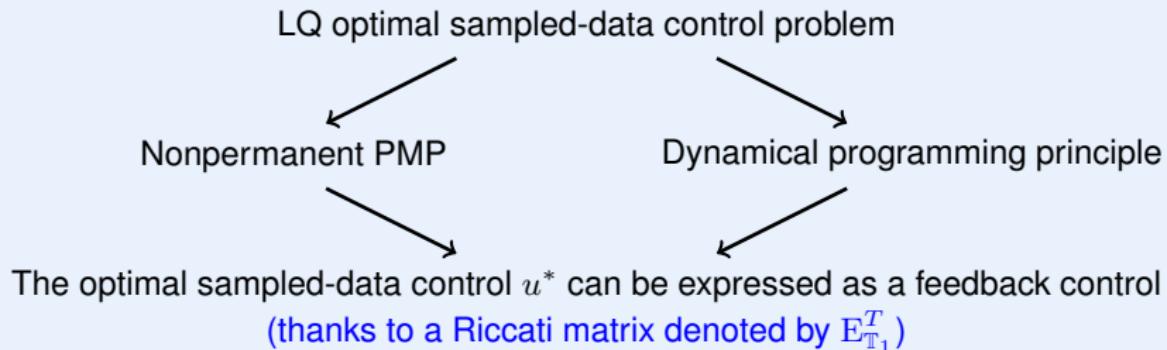
Dynamical programming principle

The optimal sampled-data control u^* can be expressed as a feedback control

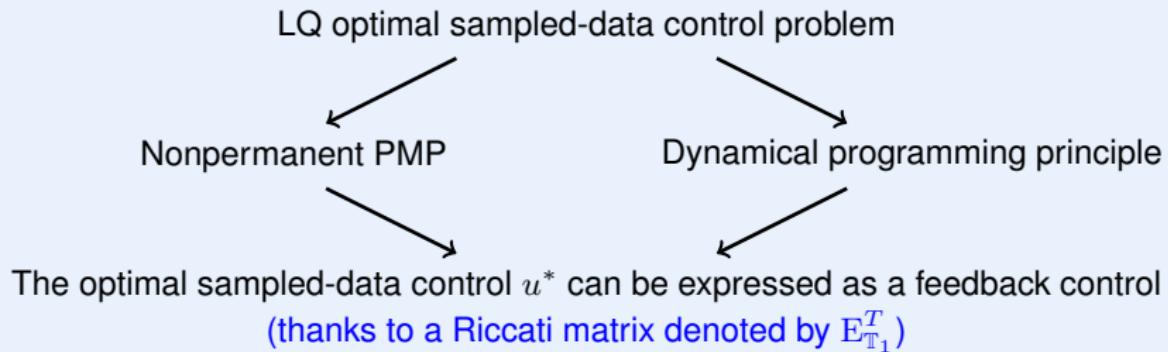
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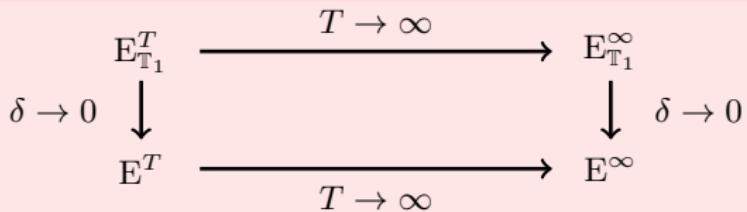


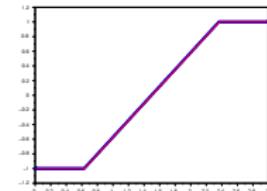
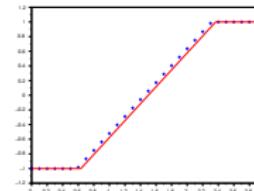
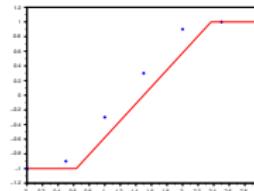
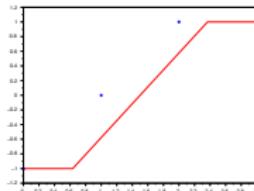
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Infinite horizon and convergence of Riccati matrices

2021





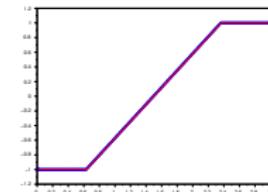
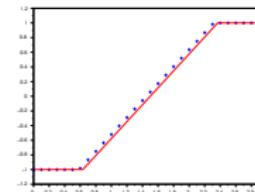
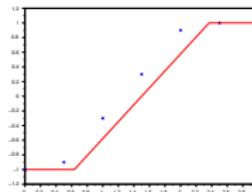
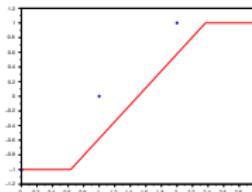
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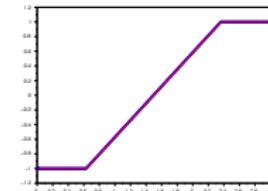
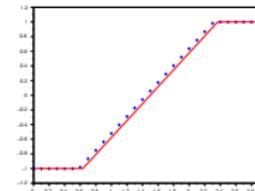
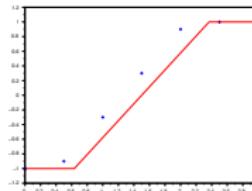
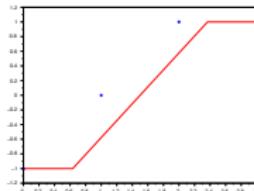
See Emmanuel's presentation.



Convergence in nonlinear problems with fixed endpoint ?

2021 & 2022

- Consider a nonlinear optimal (permanent) control problem with fixed endpoint x_f .
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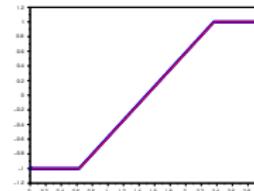
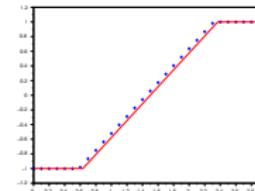
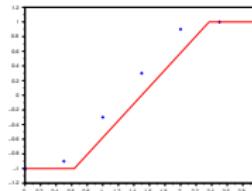
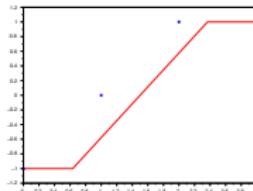


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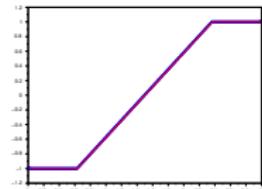
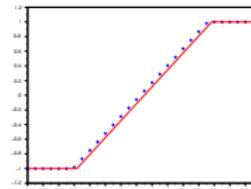
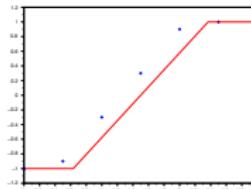
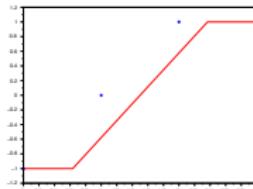
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An optimal sampled-data control problem with free sampling times

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Same PMP but ...

2019

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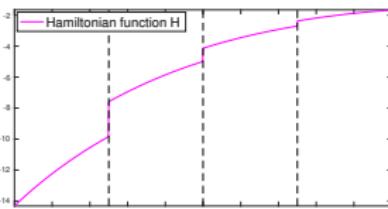
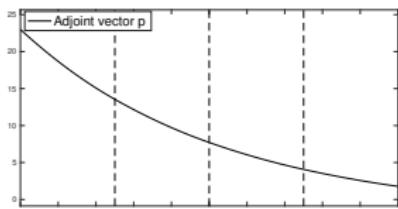
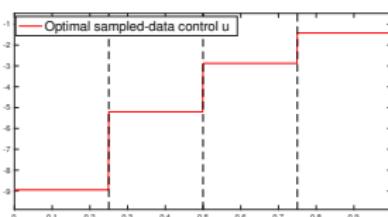
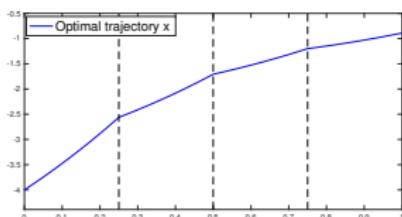
... with an additional necessary condition : \mathcal{H} is continuous !

Remark : the continuity of \mathcal{H} can be added in indirect numerical methods to compute the optimal sampling times.

Example : a LQ problem with fixed uniform sampling times

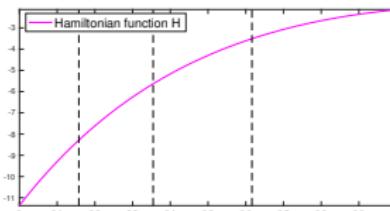
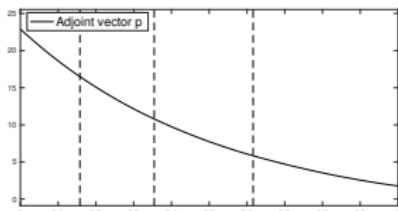
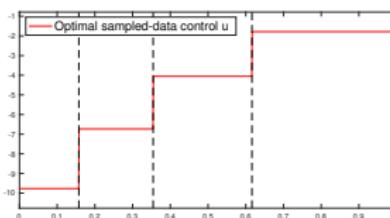
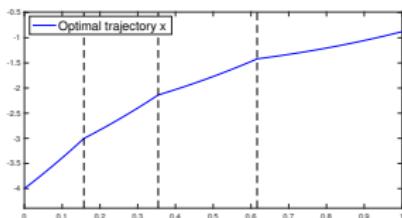
$$\begin{aligned} & \text{minimize} \quad x(1)^2 + \int_0^1 3x(\tau)^2 + u^\square(\tau)^2 d\tau, \\ & \text{subject to} \quad \left\{ \begin{array}{ll} \text{state } x \in AC([0, 1], \mathbb{R}), & \text{control } u \in L^\infty(\mathbb{T}_1, \mathbb{R}), \\ \dot{x}(t) = x(t) - u^\square(t) + t, & \text{a.e. } t \in [0, 1], \\ x(0) = -4, & \end{array} \right. \end{aligned}$$

where $\mathbb{T}_1 = \{0 = t_0 < \dots < t_N = 1\}$ is a fixed uniform partition.



Example : the same LQ problem but with free sampling times

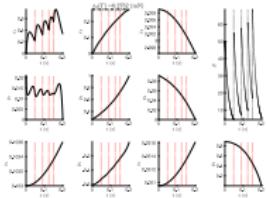
$$\begin{aligned}
 & \text{minimize} \quad x(1)^2 + \int_0^1 3x(\tau)^2 + u^\square(\tau)^2 d\tau, \\
 & \text{subject to} \quad \left\{ \begin{array}{ll} \text{state } x \in AC([0, 1], \mathbb{R}), & \text{control } u \in L^\infty(\mathbb{T}_1, \mathbb{R}), \\ \mathbb{T}_1 = \{0 = t_0 < \dots < t_N = 1\} \text{ free}, & \\ \dot{x}(t) = x(t) - u^\square(t) + t, & \text{a.e. } t \in [0, 1], \\ x(0) = -4. & \end{array} \right.
 \end{aligned}$$



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$$\begin{cases} \dot{C}(t) = -\frac{C(t)}{\tau_c} + \frac{1}{\tau_c} \sum_{k=0}^N R_k \exp\left(-\frac{t-t_k}{\tau_c} u_k \text{heavy}(t-t_k)\right), \\ \dot{F}(t) = -\left(\frac{1}{\tau_1 + \tau_2 \frac{C(t)}{K_m + C(t)}}\right) F(t) + A \frac{C(t)}{K_m + C(t)}. \end{cases}$$

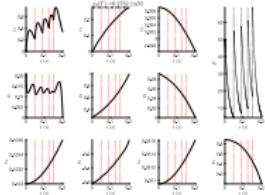
See Bernard's and Jérémie's presentations



In a first place, how to construct a satisfactory general framework ?

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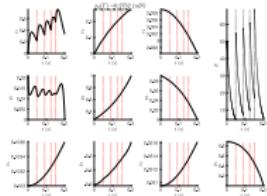


Challenge n°3 :

Extension of the PMP to dynamics/costs depending on the sampling times t_k ?

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A state-constrained optimal nonpermanent control problem

$$\begin{aligned} & \text{minimize} \quad M(x(0), x(T)) + \int_{[0, T] \cap \mathbb{T}} L(x(\tau), u^\square(\tau), \tau) \Delta\tau, \\ & \text{subject to} \quad \left\{ \begin{array}{ll} \text{state } x \in AC(\mathbb{T}, \mathbb{R}^n), & \text{control } u \in L^\infty(\mathbb{T}_1, \mathbb{R}^m), \\ x^\Delta(t) = f(x(t), u^\square(t), t), & \Delta\text{-a.e. } t \in [0, T] \cap \mathbb{T}, \\ g(x(0), x(T)) \in S, & \\ h(x(t), t) \leq 0, & \forall t \in [0, T] \cap \mathbb{T}, \\ u(t) \in U, & \Delta_1\text{-a.e. } t \in [0, T] \cap \mathbb{T}_1. \end{array} \right. \end{aligned}$$

where $U \subset \mathbb{R}^m$ and $S \subset \mathbb{R}^\ell$ are nonempty closed convex and g submersive.

If (x^*, u^*) optimal, there exists $(p, \lambda, d\eta) \in \text{AC}(\mathbb{T}, \mathbb{R}^n) \times \{0, 1\} \times \mathcal{B}_+(\mathbb{T})$ such that

- **Nontriviality condition :** the triplet $(p, \lambda, d\eta)$ is not trivial,
- **Hamiltonian system :**

$$x^\Delta(t) = \nabla_p H(x^*(t), u^{*\square}(t), q(t), \lambda, t), \quad p^\Delta(t) = -\nabla_x H(x^*(t), u^{*\square}(t), q(t), \lambda, t),$$

- **Transversality condition :**

$$\begin{pmatrix} p(0) \\ -q(T) \end{pmatrix} = \lambda \nabla M(x^*(0), x^*(T)) + \nabla g(x^*(0), x^*(T))^\top \xi,$$

for some $\xi \in \text{N}_S[g(x^*(0), x^*(T))]$,

- **Complementary slackness condition :** $\text{supp}(d\eta) \subset \{t \in \mathbb{T} \mid h(x^*(t), t) = 0\}$,
- **If** $s \in \text{RD}_1$: $u^*(s) \in \arg \max_{v \in U} H(x^*(s), v, q(s), \lambda, s)$,
- **If** $r \in \text{RS}_1$: $\int_{[r, \sigma_1(r)) \cap \mathbb{T}} \nabla_u H(x^*(\tau), u^*(r), q(\tau), \lambda, \tau) \Delta\tau \in \text{N}_U[u^*(r)]$,

where $q \in \text{BV}(\mathbb{T}, \mathbb{R}^n)$ is defined by

$$q(t) := \begin{cases} p^\sigma(t) + \int_{[0, t] \cap \mathbb{T}} \nabla h(x^*(\tau), \tau) d\eta(\tau) & \text{if } t \neq T, \\ p(T) + \int_{[0, T] \cap \mathbb{T}} \nabla h(x^*(\tau), \tau) d\eta(\tau) & \text{if } t = T. \end{cases}$$

Ekeland penalization

- ~~ Write the inequality state constraint as $\mathfrak{h}(x) \in \mathfrak{S} := C(\mathbb{T}, \mathbb{R}_-)$.
- ~~ In the Ekeland penalized functional, add the term $d_{\mathfrak{S}}^2(\mathfrak{h}(x_{(u, x_{\text{init}})}))$.
- ~~ Take an equivalent norm on $C(\mathbb{T}, \mathbb{R})$ so that the dual norm is strictly convex and thus $d_{\mathfrak{S}}^2$ is strictly Hadamard differentiable.

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Local perturbations unsuitable : adapt implicit spike perturbations

$$\forall v \in L^\infty(\mathbb{T}_1, \mathbb{R}^m), \quad \forall \alpha > 0, \quad u_\alpha(\cdot) := \begin{cases} v(\cdot) & \text{over } \mathcal{Q}_\alpha, \\ u(\cdot) & \text{over } \text{RD}_1 \setminus \mathcal{Q}_\alpha, \\ u(\cdot) + \alpha(v(\cdot) - u(\cdot)) & \text{over } \text{RS}_1, \end{cases}$$

where $\mathcal{Q}_\alpha \subset \text{RD}_1$ is **implicitly constructed** so that

$$x_\alpha = x + \alpha w + \text{rest} \quad \text{over } [0, T] \cap \mathbb{T},$$

where

$$\begin{cases} w^\Delta(t) = \nabla_x f(\dots)w(t) + \begin{cases} f(x(t), v(t), t) - f(x(t), u(t), t), & \Delta\text{-a.e. } t \in \text{RD}_1, \\ \nabla_u f(\dots)(v^\square(t) - u^\square(t)), & \text{elsewhere} \end{cases} \\ w(0) = 0_{\mathbb{R}^n}. \end{cases}$$

Construction of \mathcal{Q}_α based on **Sierpinski theorem** (requiring a nonatomic measure).

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Classical theory : difficulties in indirect numerical methods

When $\mathbb{T} = \mathbb{T}_1 = [0, T]$, the BV-adjoint vector q satisfies the Lebesgue decomposition

$$q = q_{\text{ac}} + q_{\text{sing}} + q_{\text{jumps}}.$$

Thus q may not be well-behaved :

- pathological behaviour of the singular part q_{sing} (Cantor function),
- unknown locations in $\{h(x^*, \cdot) = 0\}$ of the jumps of q_{jumps} (possibly infinite).

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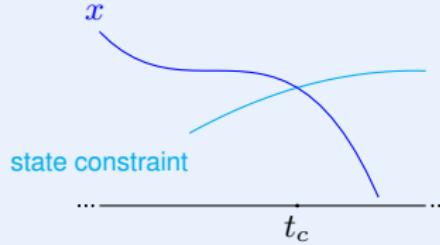
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Classical theory : two facts “with hands”

Usually a control is not constant along a boundary interval

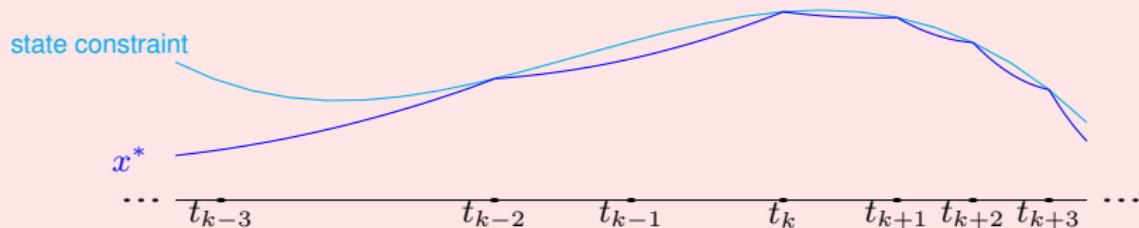


Usually the trajectory “hits” the state constraint transversely



If the control is locally constant at t_c , the trajectory “crosses” the state constraint

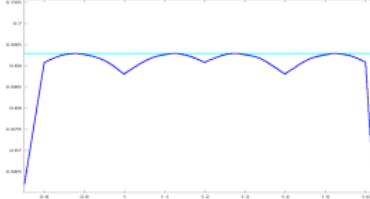
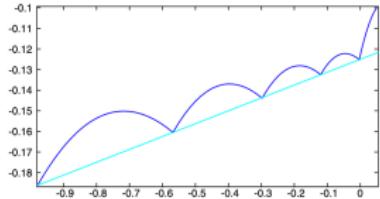
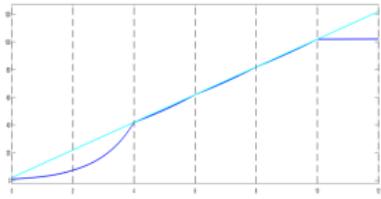
- Consider an optimal sampled-data control problem with inequality state constraints.
- Under (quite unrestrictive) conditions, **an optimal trajectory contacts the state constraints only at the sampling times exactly.**



Sampled-data controls : difficulties vanish in indirect numerical methods

In case of bouncing trajectory phenomenon, we get $\text{supp}(d\eta) \subset \{h(x^*, \cdot) = 0\} \subset \mathbb{T}_1$.

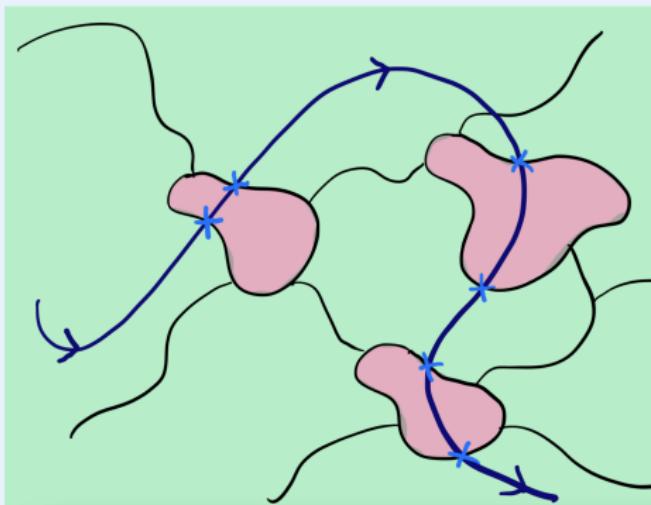
- the BV-adjoint vector q has no singular part,
- and it has a finite number of jumps localized exactly at the sampling times.



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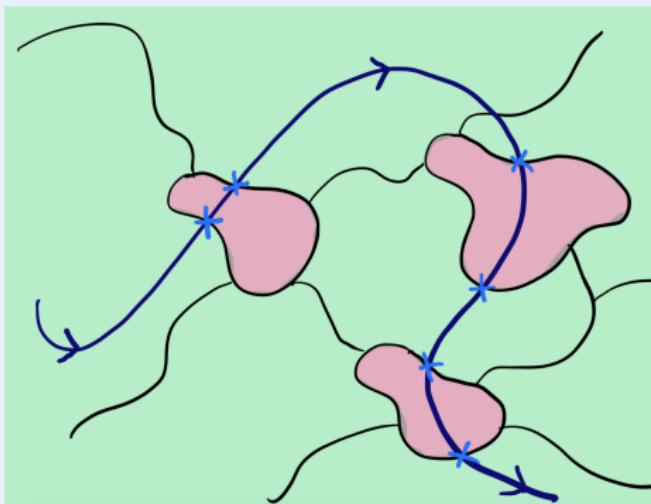
Optimal control problems with non-control zones.

In a non-control zone, the control value is frozen to the one when the trajectory crosses the boundary.



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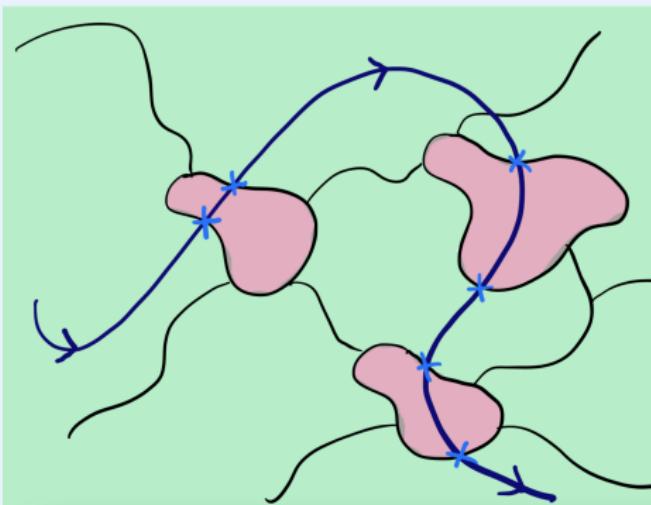
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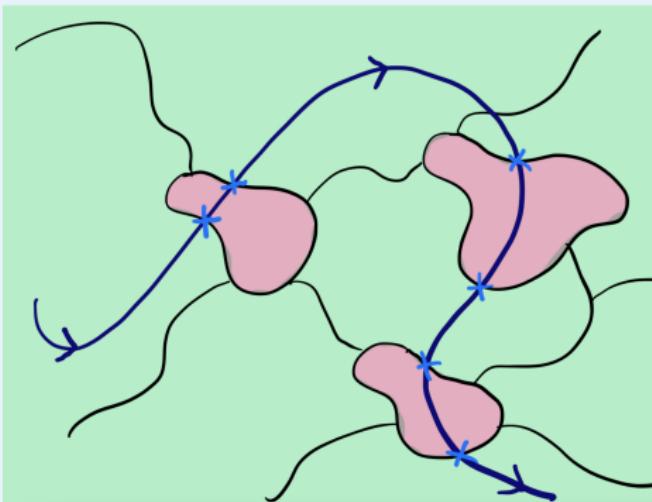
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- ~~ Nonpermanent control but the present time scale setting is not suitable !
- ~~ Techniques from hybrid optimal control theory.
- ~~ Applications to time crisis problems (two possible situations).

Challenges about optimal sampled-data controls

- With indirect numerical methods to solve optimal sampled-data control problems, how to exploit the condition

$$\int_{t_k}^{t_{k+1}} \nabla_u H(x^*(\tau), u^*(t_k), p(\tau), \lambda, \tau) d\tau \in N_U[u^*(t_k)]$$

efficiently ?

- Combinatorial optimization techniques for optimal sampled-data control problems with U finite (for example when $U = \{0, 1\}$) ?

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- Convergence results when $N \rightarrow \infty$? (weaker than $\delta \rightarrow 0$)
- Mayer cost that depends on N being free ?

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More challenges

- Sampling procedure on state (delay systems or hybrid continuous/discrete state).
- Hamilton-Jacobi theory for nonpermanent controls ?
- Nonpermanent control theory.

THANK
YOU!