Robustness of controllability under control sampling

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Works with Loïc Bourdin

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Objective

$f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \quad C^0$, and C^1 wrt (x, u)

$$\boxed{\dot{x}(t) = f(t, x(t), u(t)) \qquad u(t) \in U \subset \mathbb{R}^m}$$

$T > 0, \quad x^0, x^1 \in \mathbb{R}^n$

x^1 is L^∞_U-reachable in time T from x^0 if

$$\exists u \in L^\infty([0, T], U) \quad \text{s.t.} \quad x_u(0) = x^0 \quad \text{and} \quad x_u(T) = x^1$$

Objective

Find a general sufficient condition under which reachability in fixed time T is robust under control sampling.

Control sampling

Sampling the control u over $[0, T]$: given a partition

$$\mathbb{T} : \quad 0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$$

for some $N \in \mathbb{N}^*$, we define

$$\boxed{\mathrm{PC}^{\mathbb{T}}([0,T],U) = \{u : [0,T] \to U \text{ piecewise constant on } \mathbb{T}\}}$$

Set $\|\mathbb{T}\| = \max_{i=0,\ldots,N-1} |t_{i+1} - t_i|$ "norm" of the partition.

x^1 is $\mathrm{PC}_U^{\mathbb{T}}$-reachable in time T from x^0 if

$$\exists u \in \mathrm{PC}^{\mathbb{T}}([0,T],U) \quad \text{s.t.} \quad x_u(0) = x^0 \quad \text{and} \quad x_u(T) = x^1$$

Question

- Let $x^1 \in \mathbb{R}^n$ be L^∞_U-reachable in time T from x^0.
- Let $\mathbb{T}$ partition of $[0, T]$.

Is x^1 also $\mathrm{PC}^{\mathbb{T}}_U$-reachable in time T from x^0?

In other words: how robust is reachability in fixed time under control sampling?

Question

- Let $x^1 \in \mathbb{R}^n$ be L^∞_U-reachable in time T from x^0.
- Let $\mathbb{T}$ partition of $[0, T]$.

Is x^1 also $\mathrm{PC}^{\mathbb{T}}_U$-reachable in time T from x^0?

In other words: how robust is reachability in fixed time under control sampling?

– $\|\mathbb{T}\|$ should be sufficiently small.

– Without any specific assumption, even for small values of $\|\mathbb{T}\|$, x^1 may fail to be $\mathrm{PC}^{\mathbb{T}}_U$-reachable in time T from x^0:

- $T = n = m = 1, \qquad U = \mathbb{R}, \qquad f(x, u, t) = 1 + (u - t)^2$
- $x^1 = 1$ is L^∞_U-reachable in time T from $x^0 = 0$ with the (**unique**) control $u(t) = t$.
- $\forall \mathbb{T}$ partition of $[0, T]$, $\quad x^1$ is not $\mathrm{PC}^{\mathbb{T}}_U$-reachable in time T from x^0.

Question

- Let $x^1 \in \mathbb{R}^n$ be L^∞_U-reachable in time T from x^0.
- Let $\mathbb{T}$ partition of $[0, T]$.

Is x^1 also $\mathrm{PC}^{\mathbb{T}}_U$-reachable in time T from x^0?

In other words: how robust is reachability in fixed time under control sampling?

Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):

- U is convex;
- x^1 is L^∞_U-reachable in time T from x^0 with a control $u \in L^\infty([0, T], U)$;
- u is weakly U-regular.

Then

$$\exists \delta > 0 \quad \text{s.t.} \quad \forall \mathbb{T} \text{ partition of } [0, T],\ \|\mathbb{T}\| \leqslant \delta, \qquad x^1 \text{ is } \mathrm{PC}^{\mathbb{T}}_U\text{-reachable in time } T \text{ from } x^0$$

(stronger than: $\exists \mathbb{T}$ partition of $[0, T]$ s.t. x^1 is $\mathrm{PC}^{\mathbb{T}}_U$-reachable in time T from x^0)

Recap on reachability results

End-point mapping

$x^0 \in \mathbb{R}^n$ and $T > 0$ fixed.

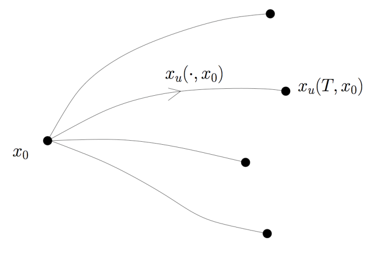
End-point mapping (in time T from x^0)

$E : L^\infty([0,T], U) \to \mathbb{R}^n$ (C^1 mapping) defined by

$$E(u) = x_u(T)$$

where

$$\dot{x}_u(t) = f(t, x_u(t), u(t)), \qquad x_u(0) = x^0$$



L^∞_U-reachable set (in time T from x^0): $\quad E(L^\infty([0,T], U))$

Without control constraints

Definition

A control u is strongly regular if the Fréchet differential $dE(u) : L^\infty([0, T], \mathbb{R}^m) \to \mathbb{R}^n$ is surjective.

A control u is weakly singular if it is not strongly regular.

Without control constraints

Definition

A control u is strongly regular if the Fréchet differential $dE(u) : L^\infty([0,T],\mathbb{R}^m) \to \mathbb{R}^n$ is surjective.

A control u is weakly singular if it is not strongly regular.

Proposition

Assume that $U = \mathbb{R}^m$.

u is strongly regular $\Rightarrow$ $x_u(T)$ belongs to the interior of the $L^\infty_{\mathbb{R}^m}$-reachable set.

(implicit function theorem $\Rightarrow$ E locally open)

Without control constraints

Proposition

u is weakly singular $\Leftrightarrow$ (x_u, u) admits a nontrivial weak extremal lift

i.e. (Pontryagin maximum principle), there exists $p(\cdot) : [0, T] \to \mathbb{R}^n \setminus \{0\}$ (adjoint vector) such that

$$\dot{x}(t) = \nabla_p H(t, x_u(t), p(t), u(t)), \qquad \dot{p}(t) = -\nabla_x H(t, x_u(t), p(t), u(t)),$$

$$\nabla_u H(t, x_u(t), p(t), u(t)) = 0$$

where $H(t, x, p, u) = \langle p, f(t, x, u) \rangle$.

With convex control constraints

Assume that U is convex.

Definition

$u \in L^\infty([0,T],U)$ is strongly U-regular if $\quad dE(u)(\mathcal{T}_{L^\infty_U}[u]) = \mathbb{R}^n \quad$ where

$$\mathcal{T}_{L^\infty_U}[u] = \mathbb{R}_+(L^\infty([0,T],U) - u) = \Big\{ \alpha(v-u) \quad \mid \quad \alpha \geqslant 0,\ v \in L^\infty([0,T],U) \Big\}$$

is the (convex) tangent cone to $L^\infty([0,T],U)$ at u.

u is weakly U-singular when it is not strongly U-regular, i.e., when $dE(u)(\mathcal{T}_{L^\infty_U}[u])$ is a proper convex subcone of $\mathbb{R}^n$.

Proposition

u strongly U-regular $\quad \Rightarrow \quad x_u(T)$ belongs to the interior of the L^∞_U-reachable set.

(conic implicit function theorem)

With convex control constraints

Assume that U is convex.

Definition

$u \in L^\infty([0,T],U)$ is strongly U-regular if $\quad dE(u)(\mathcal{T}_{L^\infty_U}[u]) = \mathbb{R}^n \quad$ where

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is the (convex) tangent cone to $L^\infty([0,T],U)$ at u.

u is weakly U-singular when it is not strongly U-regular, i.e., when $dE(u)(\mathcal{T}_{L^\infty_U}[u])$ is a proper convex subcone of $\mathbb{R}^n$.

Remark

strongly U-regular $\Rightarrow$ strongly regular.

The converse is wrong: $T = n = m = 1,\ U = [-1,1],\ f(x,u,t) = u$

$u \equiv 1$ is strongly regular and weakly U-singular.

With convex control constraints

Proposition

u is weakly U-singular $\Leftrightarrow$ (x_u, u) admits a nontrivial weak U-extremal lift

i.e. (Pontryagin maximum principle), there exists $p(\cdot) : [0, T] \to \mathbb{R}^n \setminus \{0\}$ (adjoint vector) such that

$$\dot{x}(t) = \nabla_p H(t, x_u(t), p(t), u(t)), \qquad \dot{p}(t) = -\nabla_x H(t, x_u(t), p(t), u(t)),$$

$$\nabla_u H(t, x_u(t), p(t), u(t)) \in \mathcal{N}_U[u(t)] \quad \text{(normal cone)}$$

where $H(t, x, p, u) = \langle p, f(t, x, u) \rangle$.

With general control constraints

Definition

$u \in L^\infty([0,T],U)$ is weakly U-regular if $\quad Pont_U[u] = \mathbb{R}^n \quad$ (Pontryagin cone).

u is strongly U-singular when it is not weakly U-regular, i.e., when $Pont_U[u]$ is a proper convex subcone of $\mathbb{R}^n$.

With general control constraints

Definition

$u \in L^\infty([0,T],U)$ is weakly U-regular if $\quad Pont_U[u] = \mathbb{R}^n \quad$ (Pontryagin cone).

u is strongly U-singular when it is not weakly U-regular, i.e., when $Pont_U[u]$ is a proper convex subcone of $\mathbb{R}^n$.

$Pont_U[u]$ is the smallest convex cone containing all U-variation vectors $w^u_{(\tau,\omega)}(T)$.

<u>Needle-like control variation</u> $\; u^\alpha_{(\tau,\omega)} \in L^\infty([0,T],U)$ defined by

$$u^\alpha_{(\tau,\omega)}(t) = \begin{cases} \omega & \text{along } [t, t+\alpha) \\ u(t) & \text{elsewhere} \end{cases}$$

for $\omega \in U$ and $\alpha > 0$ small. Then

$$\boxed{\lim_{\alpha \to 0^+} \frac{E(u^\alpha_{(\tau,\omega)}) - E(u)}{\alpha} = w^u_{(\tau,\omega)}(T)}$$

where $w^u_{(\tau,\omega)}$ is the unique solution on $[\tau, T]$ of the *variational system*

$$\dot{w}(t) = \nabla_x f(t, x_u(t), u(t))\, w(t), \qquad w(\tau) = f(\tau, x_u(\tau), \omega) - f(\tau, x_u(\tau), u(\tau)).$$

With general control constraints

Definition

$u \in L^\infty([0,T],U)$ is weakly U-regular if $\quad Pont_U[u] = \mathbb{R}^n \quad$ (Pontryagin cone).

u is strongly U-singular when it is not weakly U-regular, i.e., when $Pont_U[u]$ is a proper convex subcone of $\mathbb{R}^n$.

Proposition

u weakly U-regular $\Rightarrow$ $x_u(T)$ belongs to the interior of the L^∞_U-reachable set.

(application of the conic implicit function theorem to the "end-point mapping restricted to some needle-like variations")

Remark

When U is convex: strongly U-regular $\Rightarrow$ weakly U-regular (converse wrong).

With general control constraints

Proposition

u is strongly U-singular $\Leftrightarrow$ (x_u, u) admits a nontrivial strong U-extremal lift

i.e. (Pontryagin maximum principle), there exists $p(\cdot) : [0, T] \to \mathbb{R}^n \setminus \{0\}$ (adjoint vector) such that

$$\dot{x}(t) = \nabla_p H(t, x_u(t), p(t), u(t)), \qquad \dot{p}(t) = -\nabla_x H(t, x_u(t), p(t), u(t)),$$

$$u(t) \in \underset{\omega \in U}{\mathrm{argmax}}\, H(t, x_u(t), p(t), \omega)$$

where $H(t, x, p, u) = \langle p, f(t, x, u) \rangle$.

Robustness of reachability under control sampling

Theorem

- Let $x^1 \in \mathbb{R}^n$ be L^∞_U-reachable in time T from x^0.
- Let $\mathbb{T}$ partition of $[0, T]$.

Is x^1 also $\mathrm{PC}^{\mathbb{T}}_U$-reachable in time T from x^0?

In other words: how robust is reachability in fixed time under control sampling?

Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):

- U is convex.
- $x^1 \in \mathbb{R}^n$ is L^∞_U-reachable in time T from x^0 with a control $u \in L^\infty([0, T], U)$.
- u is weakly U-regular (i.e., $Pont_U[u] = \mathbb{R}^n$).

Then $\exists \delta > 0 \quad \forall \mathbb{T}$ partition of $[0, T]$ s.t. $\|\mathbb{T}\| \leqslant \delta, \quad x_u(T) \in E(\mathrm{PC}^{\mathbb{T}}([0, T], U)) \quad (\mathrm{PC}_{unif})$

Theorem

Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):

- U is convex.
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Then $\exists \delta > 0 \quad \forall \mathbb{T}$ partition of $[0,T]$ s.t. $\|\mathbb{T}\| \leqslant \delta, \quad x_u(T) \in E(\mathrm{PC}^{\mathbb{T}}([0,T],U))$ (PC_{unif})

Remark

This is stronger than the property:

$\forall \delta > 0 \quad \exists \mathbb{T}$ partition of $[0,T]$ s.t. $\|\mathbb{T}\| \leqslant \delta, \quad x_u(T) \in E(\mathrm{PC}^{\mathbb{T}}([0,T],U))$ (PC)

We may have (PC) while (PC_{unif}) fails.

Theorem

Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):

- U is convex.
- $x^1 \in \mathbb{R}^n$ is L^∞_U-reachable in time T from x^0 with a control $u \in L^\infty([0,T],U)$.
- u is weakly U-regular (i.e., $Pont_U[u] = \mathbb{R}^n$).

Then $\exists \delta > 0 \quad \forall \mathbb{T}$ partition of $[0,T]$ s.t. $\|\mathbb{T}\| \leqslant \delta, \quad x_u(T) \in E(\mathrm{PC}^{\mathbb{T}}([0,T],U))$ (PC_{unif})

In the example: $T = 4, \quad n = m = 1, \quad U = \{0,1\}, \quad f(x,u,t) = u$

- $u(t) = 1$ for $t \in [0,\pi]$ and $u(t) = 0$ for $t \in [\pi, 4]$, is weakly U-regular
- $x^1 = \pi$ is L^∞_U-reachable in time T from $x^0 = 0$

and:

– $x_u(T)$ belongs to the interior of the L^∞_U-reachable set.

– (PC) is satisfied but not (PC_{unif}) (take partitions with rational sampling times: commensurability rigidity).

– U is not convex.

Theorem

Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):

- U is convex.
- $x^1 \in \mathbb{R}^n$ is L^∞_U-reachable in time T from x^0 with a control $u \in L^\infty([0,T],U)$.
- u is weakly U-regular (i.e., $Pont_U[u] = \mathbb{R}^n$).

Then $\exists \delta > 0 \quad \forall \mathbb{T}$ partition of $[0,T]$ s.t. $\|\mathbb{T}\| \leqslant \delta, \quad x_u(T) \in E(\mathrm{PC}^{\mathbb{T}}([0,T],U))$ (PC_{unif})

– Even with U convex, there exist examples where $x_u(T)$ belongs to the interior of the L^∞_U-reachable set but does not belong to the $\mathrm{PC}^{\mathbb{T}}_U$-reachable set.

– Convexity can be slightly relaxed to "U is parameterizable by a convex set".

The theorem fails in general if U is "strongly nonconnected", i.e.,

$U = U_1 \cup U_2$ *where* $U_1 \neq \emptyset$ *and* $U_2 \neq \emptyset$

$\exists \Theta : \mathbb{R}^m \to \mathbb{R}$ C^1 *s.t.* $\Theta_{|U_1} = 0$ *and* $\Theta_{|U_2} = 1$*)*

Comments on other existing results

Remarkable series of works by Sussmann, Sontag, Grasse in the 80's and 90's. (in free final time)

- Sussmann CDC 1987: focuses on (PC) (with no assumption on U), under an assumption that is similar to weak U-regularity.
- Initial idea: Sussmann JDE 1976, notion of normal reachability, i.e., reachability under piecewise constant controls with a surjective differential end-point mapping ($\leftarrow$ open mapping theorem). It is proved that:

  global controllability $\Leftrightarrow$ global normal controllability
  (in **free** final time)

- Sontag Sussmann 82-84-88: sampled-data controls on regular subdivisions. Under Lie algebra rank condition:

  global controllability $\Leftrightarrow$ sampled-data global controllability
  (in **free** final time)

- Grasse, MCSS 1992: f C^1, nontangency property at x^0,

  small-time local controllability at x^0 $\Leftrightarrow$ sampled-data STLC at x^0
  $\Leftrightarrow$ x^0 is small-time normally self-reachable

  ($\Rightarrow$ reachability by "nice controls")

Convergence in sampled-data optimal control problems

This result of robustness of reachability in fixed time under control sampling is instrumental for the following objective:

Optimal control problem

$\dot{x}(t) = f(t, x(t), u(t))$

$x(0) = x^0, \quad x(T) = x^1$

$\min C(x, u)$

where

$C(x, u) = \int_0^T f^0(t, x(t), u(t))\, dt$

Permanent controls

$u \in L^\infty_U$

- Optimal solution (x, u)
- *PMP* $\Rightarrow$ adjoint p

Sampled-data controls

$u \in \mathrm{PC}^{\mathbb{T}}_U$

- Optimal solution $(x^{\mathbb{T}}, u^{\mathbb{T}})$
- $PMP^{\mathbb{T}} \Rightarrow$ adjoint $p^{\mathbb{T}}$

(sampled-data PMP: Bourdin Trélat, MCRF 2016)

Objective: prove that $x^{\mathbb{T}} \xrightarrow{C^0} x, \quad p^{\mathbb{T}} \xrightarrow{C^0} p, \quad C(x^{\mathbb{T}}, u^{\mathbb{T}}) \longrightarrow C(x, u)$ as $\|\mathbb{T}\| \to 0$.

Convergence in sampled-data optimal control problems

Theorem (Bourdin Trélat, ongoing)

This is true under the following assumptions:

1. U is compact and trajectories live in a (big) compact.
2. For all (t,x), the epigraph of the extended velocity set

$$\left\{ \begin{pmatrix} f(t,x,u) \\ f^0(t,x,u)+\gamma \end{pmatrix} \;\middle|\; \gamma \geqslant 0,\ u \in U \right\}$$

   is convex.
3. Uniqueness of the optimal solution (x,u).
4. Unique normal extremal lift (x,p,u).

} (1), (2): classical assumptions ensuring existence of minimizers

} (3), (4): "generic" assumptions

(in the absence of uniqueness in (3) and/or (4): convergence of subsequences)

Key points of the proof:

- Assumption (4) (normality) $\Rightarrow$ robustness of reachability under control sampling
  $\Rightarrow$ existence of optimal solution $(x^{\mathbb{T}}, u^{\mathbb{T}})$
- Technical fact: convergence of Pontryagin cones of $OCP^{\mathbb{T}}$ to OCP
  (kind of grad-Γ-convergence)

Linear quadratic case

As a particular case:

- Linear system: $f(t,x,u) = A(t)x + B(t)u + r(t)$
- Quadratic cost: $f^0(t,x,u) = (x-\bar{x}(t))^\top Q(t)(x-\bar{x}(t)) + (u-\bar{u}(t))^\top R(t)(u-\bar{u}(t))$

In fixed finite horizon T:

sampled-data difference Riccati theory in [Bourdin Trélat, Automatica 2017]

In infinite horizon: $f(t,x,u) = Ax + Bu, \quad f^0(t,x,u) = x^\top Qx + u^\top Ru$

We have proved in [Bourdin Trélat, SICON 2021] the commutation of the diagram:

$$
\begin{array}{cccc}
\text{sampled-data difference Riccati} & E_T^{\mathbb{T}} & \xrightarrow{T\to+\infty} & E_\infty^{\mathbb{T}} & \text{sampled-data algebraic Riccati} \\
& \Big\downarrow {\scriptstyle \|\mathbb{T}\|\to 0} & & \Big\downarrow {\scriptstyle \|\mathbb{T}\|\to 0} & \\
\text{(permanent) differential Riccati} & E_T & \xrightarrow[T\to+\infty]{} & E_\infty & \text{(permanent) algebraic Riccati}
\end{array}
$$

Long-term open issue

Optimal control problem

Full discretization
Euler, Runge-Kutta, etc.

↓

Dualization
Kuhn-Tucker,
then Newton's method

direct methods

Dualization
Pontryagin Maximum Principle

↓

Full discretization
Euler, Runge-Kutta, etc,
then Newton (shooting method)

indirect methods

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↓

Discretization

Euler, Runge-Kutta, etc,
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?

direct methods

indirect methods

No commutation in general.
Commutation for Runge-Kutta methods with positive coefficients (Hager, 2000).

Counterexample (Hager, 2000)

LQ optimal control problem

$$\dot{x}(t) = \frac{1}{2}x(t) + u(t), \qquad x(0) = 1, \qquad \min \frac{1}{2}\int_0^1 (2x(t)^2 + u(t)^2)\, dt$$

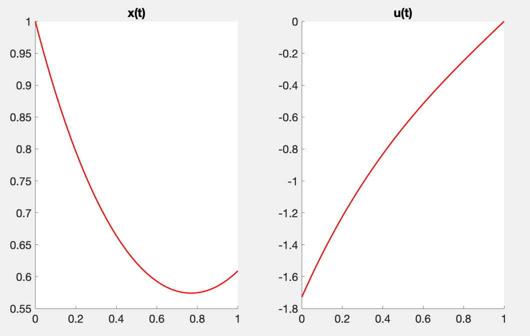
Counterexample (Hager, 2000)

LQ optimal control problem

$$\dot{x}(t) = \frac{1}{2}x(t) + u(t), \qquad x(0) = 1, \qquad \min \frac{1}{2}\int_0^1 (2x(t)^2 + u(t)^2)\, dt$$

The optimal solution is (differential Riccati)

$$\boxed{x(t) = \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)}, \qquad u(t) = \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)}} \quad \text{with optimal cost} \quad \frac{2(-2 + e^6 + e^3)}{4 + 4e^3 + e^6} \simeq 1.728$$



Counterexample (Hager, 2000)

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Full discretization with mid-point rule:

$$\begin{aligned} x_{k+1/2} &= x_k + \frac{h}{2}\left(\frac{1}{2}x_k + u_k\right) \\ x_{k+1} &= x_k + h\left(\frac{1}{2}x_{k+1/2} + u_{k+1/2}\right), \quad x_0 = 1 \end{aligned} \qquad \min \frac{1}{2}\sum_{k=0}^{N-1}\left(2x_{k+1/2}^2 + u_{k+1/2}^2\right)$$

Counterexample (Hager, 2000)

LQ optimal control problem

$$\dot{x}(t) = \frac{1}{2}x(t) + u(t), \qquad x(0) = 1, \qquad \min \frac{1}{2}\int_0^1 (2x(t)^2 + u(t)^2)\,dt$$

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The optimal solution is

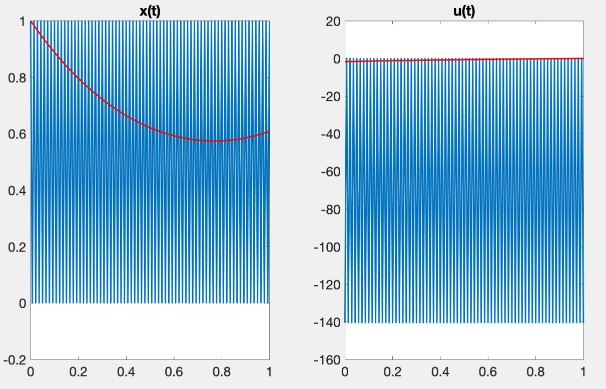
$$\boxed{u_k = -\frac{4+h}{2h}, \quad u_{k+1/2} = 0, \qquad x_k = 1, \quad x_{k+1/2} = 0} \quad \text{with optimal cost } 0$$

Counterexample (Hager, 2000)

LQ optimal control problem

$$\dot{x}(t) = \frac{1}{2}x(t) + u(t), \qquad x(0) = 1, \qquad \min \frac{1}{2}\int_0^1 (2x(t)^2 + u(t)^2)\, dt$$

Numerical simulation for $N = 70$:



Counterexample (Hager, 2000)

LQ optimal control problem

$$\dot{x}(t) = \frac{1}{2}x(t) + u(t), \qquad x(0) = 1, \qquad \min \frac{1}{2}\int_0^1 (2x(t)^2 + u(t)^2)\, dt$$

<u>But if we discretize with Euler:</u>

$$x_{k+1} = x_k + h\left(\frac{1}{2}x_k + u_k\right), \quad x_0 = 1 \qquad \min \frac{1}{2}\sum_{k=0}^{N-1}\left(2x_k^2 + u_k^2\right)$$

then everything is going fine: