Robustness of controllability under control sampling

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Works with Loïc Bourdin

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Objective

\[ f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \quad C^0, \text{ and } C^1 \text{ wrt } (x, u) \]

\[ \dot{x}(t) = f(t, x(t), u(t)) \quad u(t) \in U \subset \mathbb{R}^m \]

\[ T > 0, \quad x^0, x^1 \in \mathbb{R}^n \]

\[ x^1 \text{ is } L^\infty_U \text{-reachable in time } T \text{ from } x^0 \text{ if } \]

\[ \exists u \in L^\infty([0, T], U) \quad \text{s.t.} \quad x_u(0) = x^0 \quad \text{and} \quad x_u(T) = x^1 \]

Objective

Find a general sufficient condition under which reachability in fixed time \( T \) is robust under control sampling.
Sampling the control \( u \) over \([0, T]\): given a partition

\[
T : \quad 0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T
\]

for some \( N \in \mathbb{N}^* \), we define

\[
PC_T([0, T], U) = \{ u : [0, T] \to U \text{ piecewise constant on } T \}
\]

Set \( ||T|| = \max_{i=0, \ldots, N-1} |t_{i+1} - t_i| \) “norm” of the partition.

\( x^1 \) is \( PC_U \)-reachable in time \( T \) from \( x^0 \) if

\[
\exists u \in PC_T([0, T], U) \text{ s.t. } x_u(0) = x^0 \text{ and } x_u(T) = x^1
\]
Let $x^1 \in \mathbb{R}^n$ be $L^\infty_U$-reachable in time $T$ from $x^0$.

Let $\mathbb{T}$ partition of $[0, T]$.

Is $x^1$ also $PC^{\mathbb{T}}_U$-reachable in time $T$ from $x^0$?

In other words: how robust is reachability in fixed time under control sampling?
Let $x^1 \in \mathbb{R}^n$ be $L_U^\infty$-reachable in time $T$ from $x^0$.

Let $\mathbb{T}$ partition of $[0, T]$.

Is $x^1$ also $PC_U^\mathbb{T}$-reachable in time $T$ from $x^0$?

In other words: how robust is reachability in fixed time under control sampling?

- $\|\mathbb{T}\|$ should be sufficiently small.
- Without any specific assumption, even for small values of $\|\mathbb{T}\|$, $x^1$ may fail to be $PC_U^\mathbb{T}$-reachable in time $T$ from $x^0$:
  - $T = n = m = 1$, $U = \mathbb{R}$, $f(x, u, t) = 1 + (u - t)^2$
  - $x^1 = 1$ is $L_U^\infty$-reachable in time $T$ from $x^0 = 0$ with the (unique) control $u(t) = t$.
  - $\forall \mathbb{T}$ partition of $[0, T]$, $x^1$ is not $PC_U^\mathbb{T}$-reachable in time $T$ from $x^0$. 
Question

Let $x^1 \in \mathbb{R}^n$ be $L_\infty^U$-reachable in time $T$ from $x^0$.

Let $T$ partition of $[0, T]$.

Is $x^1$ also $PC_T^U$-reachable in time $T$ from $x^0$?

In other words: how robust is reachability in fixed time under control sampling?

Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):

- $U$ is convex;
- $x^1$ is $L_\infty^U$-reachable in time $T$ from $x^0$ with a control $u \in L^\infty([0, T], U)$;
- $u$ is weakly $U$-regular.

Then

$$\exists \delta > 0 \text{ s.t. } \forall T \text{ partition of } [0, T], \|T\| \leq \delta, \quad x^1 \text{ is } PC_T^U\text{-reachable in time } T \text{ from } x^0$$

(stronger than: $\exists T \text{ partition of } [0, T] \text{ s.t. } x^1 \text{ is } PC_T^U\text{-reachable in time } T \text{ from } x^0$)
Recap on reachability results
$x^0 \in \mathbb{R}^n$ and $T > 0$ fixed.

**End-point mapping (in time $T$ from $x^0$)**

$E : L^\infty([0, T], U) \rightarrow \mathbb{R}^n$ ($C^1$ mapping) defined by

$$E(u) = x_u(T)$$

where

$$\dot{x}_u(t) = f(t, x_u(t), u(t)), \quad x_u(0) = x^0$$

$L^\infty_U$-reachable set (in time $T$ from $x^0$): $E(L^\infty([0, T], U))$
Without control constraints

Definition

A control $u$ is **strongly regular** if the Fréchet differential $dE(u) : L^\infty([0, T], \mathbb{R}^m) \rightarrow \mathbb{R}^n$ is surjective.

A control $u$ is **weakly singular** if it is not strongly regular.
Without control constraints

**Definition**

A control \( u \) is **strongly regular** if the Fréchet differential 
\[ dE(u) : L^\infty([0, T], \mathbb{R}^m) \rightarrow \mathbb{R}^n \]

is surjective.

A control \( u \) is **weakly singular** if it is not strongly regular.

**Proposition**

Assume that \( U = \mathbb{R}^m \).

\( u \) is strongly regular \( \Rightarrow \) \( x_u(T) \) belongs to the interior of the \( L^\infty_{\mathbb{R}^m} \)-reachable set.

(implicit function theorem \( \Rightarrow \) \( E \) locally open)
Without control constraints

**Proposition**

*u* is weakly singular $\Leftrightarrow (x_u, u)$ admits a nontrivial weak extremal lift

i.e. (Pontryagin maximum principle), there exists $p(\cdot) : [0, T] \rightarrow \mathbb{R}^n \setminus \{0\}$ (adjoint vector) such that

$$
\dot{x}(t) = \nabla_p H(t, x_u(t), p(t), u(t)), \quad \dot{p}(t) = -\nabla_x H(t, x_u(t), p(t), u(t)),
$$

$$
\nabla_u H(t, x_u(t), p(t), u(t)) = 0
$$

where $H(t, x, p, u) = \langle p, f(t, x, u) \rangle$. 

With convex control constraints

Assume that $U$ is convex.

**Definition**

$u \in L^\infty([0, T], U)$ is strongly $U$-regular if

$$dE(u)(T_{L^\infty_U}[u]) = \mathbb{R}^n$$

where

$$T_{L^\infty_U}[u] = \mathbb{R}_+(L^\infty([0, T], U) - u) = \left\{ \alpha(v - u) \mid \alpha \geq 0, \ v \in L^\infty([0, T], U) \right\}$$

is the (convex) tangent cone to $L^\infty([0, T], U)$ at $u$.

$u$ is weakly $U$-singular when it is not strongly $U$-regular, i.e., when $dE(u)(T_{L^\infty_U}[u])$ is a proper convex subcone of $\mathbb{R}^n$.

**Proposition**

$u$ strongly $U$-regular $\Rightarrow$ $x_u(T)$ belongs to the interior of the $L^\infty_U$-reachable set.

*(conic implicit function theorem)*
With convex control constraints

Assume that \( U \) is convex.

**Definition**

\( u \in L^\infty([0, T], U) \) is **strongly \( U \)-regular** if  
\[
\text{d}E(u)(\mathcal{T}_{L^\infty_U}[u]) = \mathbb{R}^n \quad \text{where}
\]

\[
\mathcal{T}_{L^\infty_U}[u] = \mathbb{R}_+(L^\infty([0, T], U) - u) = \left\{ \alpha(v - u) \mid \alpha \geq 0, \ v \in L^\infty([0, T], U) \right\}
\]

is the (convex) tangent cone to \( L^\infty([0, T], U) \) at \( u \).

\( u \) is **weakly \( U \)-singular** when it is not strongly \( U \)-regular, i.e., when \( \text{d}E(u)(\mathcal{T}_{L^\infty_U}[u]) \) is a proper convex subcone of \( \mathbb{R}^n \).

**Remark**

strongly \( U \)-regular \( \Rightarrow \) strongly regular.

The converse is wrong:  
\[
T = n = m = 1, \ U = [-1, 1], \ f(x, u, t) = u \\
u \equiv 1 \text{ is strongly regular and weakly } U \text{-singular.}
With convex control constraints

**Proposition**

$u$ is weakly $U$-singular $\iff (x_u, u)$ admits a nontrivial weak $U$-extremal lift

i.e. (Pontryagin maximum principle), there exists $p(\cdot) : [0, T] \to \mathbb{R}^n \setminus \{0\}$ (adjoint vector) such that

$$
\dot{x}(t) = \nabla_p H(t, x_u(t), p(t), u(t)), \quad \dot{p}(t) = -\nabla_x H(t, x_u(t), p(t), u(t)),
$$

$$
\nabla_u H(t, x_u(t), p(t), u(t)) \in N_U[u(t)] \quad \text{(normal cone)}
$$

where $H(t, x, p, u) = \langle p, f(t, x, u) \rangle$. 
With general control constraints

**Definition**

\( u \in L^\infty([0, T], U) \) is **weakly U-regular** if \( \text{Pont}_U[u] = \mathbb{R}^n \) (Pontryagin cone).

\( u \) is **strongly U-singular** when it is not weakly U-regular, i.e., when \( \text{Pont}_U[u] \) is a proper convex subcone of \( \mathbb{R}^n \).
With general control constraints

**Definition**

\( u \in L^\infty([0, T], U) \) is weakly \( U \)-regular if \( \text{Pont}_U[u] = \mathbb{R}^n \) (Pontryagin cone).

\( u \) is strongly \( U \)-singular when it is not weakly \( U \)-regular, i.e., when \( \text{Pont}_U[u] \) is a proper convex subcone of \( \mathbb{R}^n \).

\( \text{Pont}_U[u] \) is the smallest convex cone containing all \( U \)-variation vectors \( w^u_{(\tau, \omega)}(T) \).

**Needle-like control variation** \( u^\alpha_{(\tau, \omega)} \in L^\infty([0, T], U) \) defined by

\[
    u^\alpha_{(\tau, \omega)}(t) = \begin{cases} 
        \omega & \text{along } [t, t + \alpha) \\
        u(t) & \text{elsewhere}
    \end{cases}
\]

for \( \omega \in U \) and \( \alpha > 0 \) small. Then

\[
    \lim_{\alpha \to 0^+} \frac{E(u^\alpha_{(\tau, \omega)}) - E(u)}{\alpha} = w^u_{(\tau, \omega)}(T)
\]

where \( w^u_{(\tau, \omega)} \) is the unique solution on \([\tau, T]\) of the variational system

\[
    \dot{w}(t) = \nabla_x f(t, x_u(t), u(t)) \ w(t), \quad w(\tau) = f(\tau, x_u(\tau), \omega) - f(\tau, x_u(\tau), u(\tau)).
\]
**Definition**

\( u \in L^\infty([0, T], U) \) is **weakly U-regular** if \( \text{Pont}_U[u] = \mathbb{R}^n \) (Pontryagin cone).

\( u \) is **strongly U-singular** when it is not weakly \( U \)-regular, i.e., when \( \text{Pont}_U[u] \) is a proper convex subcone of \( \mathbb{R}^n \).

**Proposition**

\( u \) weakly \( U \)-regular \( \Rightarrow \) \( x_u(T) \) belongs to the interior of the \( L^\infty_U \)-reachable set.

(application of the conic implicit function theorem to the “end-point mapping restricted to some needle-like variations”)

**Remark**

When \( U \) is convex: strongly \( U \)-regular \( \Rightarrow \) weakly \( U \)-regular (converse wrong).
With general control constraints

**Proposition**

\( u \) is strongly \( U \)-singular \( \iff \) \((x_u, u)\) admits a nontrivial strong \( U \)-extremal lift

i.e. *(Pontryagin maximum principle)*, there exists \( p(\cdot) : [0, T] \rightarrow \mathbb{R}^n \setminus \{0\} \) (adjoint vector) such that

\[
\dot{x}(t) = \nabla_p H(t, x_u(t), p(t), u(t)), \quad \dot{p}(t) = -\nabla_x H(t, x_u(t), p(t), u(t)),
\]

\[
u(t) \in \arg\max_{\omega \in U} H(t, x_u(t), p(t), \omega)
\]

where \( H(t, x, p, u) = \langle p, f(t, x, u) \rangle \).
Robustness of reachability under control sampling
Theorem

Let $x^1 \in \mathbb{R}^n$ be $L^\infty_U$-reachable in time $T$ from $x^0$.

Let $\mathbb{T}$ partition of $[0, T]$.

Is $x^1$ also $PC^\mathbb{T}_U$-reachable in time $T$ from $x^0$?

In other words: how robust is reachability in fixed time under control sampling?

Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):

- $U$ is convex.
- $x^1 \in \mathbb{R}^n$ is $L^\infty_U$-reachable in time $T$ from $x^0$ with a control $u \in L^\infty([0, T], U)$.
- $u$ is weakly $U$-regular (i.e., $P_{ou}u = \mathbb{R}^n$).

Then $\exists \delta > 0 \ \forall \mathbb{T}$ partition of $[0, T]$ s.t. $\|\mathbb{T}\| \leq \delta$, $x_u(T) \in E(PC^\mathbb{T}([0, T], U)) \ (PC_{\text{unif}})$
**Theorem (Bourdin Trélat, MCSS 2021)**

**Assumptions (≃ sharp):**
- $U$ is convex.
- $x^1 \in \mathbb{R}^n$ is $L^\infty_U$-reachable in time $T$ from $x^0$ with a control $u \in L^\infty([0, T], U)$.
- $u$ is weakly $U$-regular (i.e., $\text{Pont}_U[u] = \mathbb{R}^n$).

Then $\exists \delta > 0 \ \forall \mathcal{T}$ partition of $[0, T]$ s.t. $\|\mathcal{T}\| \leq \delta$, $x_u(T) \in E(\text{PC}^\mathcal{T}([0, T], U))$ (PC\text{unif})

**Remark**

This is stronger than the property:

$\forall \delta > 0 \ \exists \mathcal{T}$ partition of $[0, T]$ s.t. $\|\mathcal{T}\| \leq \delta$, $x_u(T) \in E(\text{PC}^\mathcal{T}([0, T], U))$ (PC)

We may have (PC) while (PC\text{unif}) fails.
Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):
- $U$ is convex.
- $x^1 \in \mathbb{R}^n$ is $L^\infty_U$-reachable in time $T$ from $x^0$ with a control $u \in L^\infty([0, T], U)$.
- $u$ is weakly $U$-regular (i.e., $\text{Pont}_U[u] = \mathbb{R}^n$).

Then $\exists \delta > 0 \ \forall \mathbb{T}$ partition of $[0, T]$ s.t. $||\mathbb{T}|| \leq \delta$, $x_u(T) \in E(\text{PC}^{\mathbb{T}}([0, T], U))$ \textbf{(PC} \text{unif)}

In the example: $T = 4$, $n = m = 1$, $U = \{0, 1\}$, $f(x, u, t) = u$
- $u(t) = 1$ for $t \in [0, \pi]$ and $u(t) = 0$ for $t \in [\pi, 4]$, is weakly $U$-regular
- $x^1 = \pi$ is $L^\infty_U$-reachable in time $T$ from $x^0 = 0$

and:
- $x_u(T)$ belongs to the interior of the $L^\infty_U$-reachable set.
- (PC) is satisfied but not (PC\text{unif}) (take partitions with rational sampling times: commensurability rigidity).
- $U$ is not convex.
Theorem (Bourdin Trélat, MCSS 2021)

Assumptions ($\simeq$ sharp):
- $U$ is convex.
- $x^1 \in \mathbb{R}^n$ is $L^\infty_U$-reachable in time $T$ from $x^0$ with a control $u \in L^\infty([0, T], U)$.
- $u$ is weakly $U$-regular (i.e., $\text{Pont}_U[u] = \mathbb{R}^n$).

Then $\exists \delta > 0$ $\forall \mathbb{T}$ partition of $[0, T]$ s.t. $||\mathbb{T}|| \leq \delta$, $x_u(T) \in E(\text{PC}^\mathbb{T}([0, T], U))$ ($\text{PC}_{\text{unif}}$)

- Even with $U$ convex, there exist examples where $x_u(T)$ belongs to the interior of the $L^\infty_U$-reachable set but does not belong to the $\text{PC}^\mathbb{T}_U$-reachable set.
- Convexity can be slightly relaxed to “$U$ is parameterizable by a convex set”.

The theorem fails in general if $U$ is “strongly nonconnected”, i.e.,
- $U = U_1 \cup U_2$ where $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$
- $\exists \Theta : \mathbb{R}^m \rightarrow \mathbb{R} \text{ } C^1$ s.t. $\Theta|_{U_1} = 0$ and $\Theta|_{U_2} = 1$
Remarkable series of works by Sussmann, Sontag, Grasse in the 80’s and 90’s.
(in free final time)

- **Sussmann** CDC 1987: focuses on (PC) (with no assumption on $U$), under an assumption that is similar to weak $U$-regularity.

- Initial idea: **Sussmann** JDE 1976, notion of normal reachability, i.e., reachability under piecewise constant controls with a surjective differential end-point mapping ($\leftarrow$ open mapping theorem). It is proved that:
  
  \[
  \text{global controllability } \iff \text{global normal controllability (in free final time)}
  \]

- **Sontag Sussmann** 82-84-88: sampled-data controls on regular subdivisions. Under Lie algebra rank condition:
  
  \[
  \text{global controllability } \iff \text{sampled-data global controllability (in free final time)}
  \]

- **Grasse, MCSS 1992**: $f \in C^1$, nontangency property at $x^0$, small-time local controllability at $x^0 \iff$ sampled-data STLC at $x^0 \iff x^0$ is small-time normally self-reachable
  
  ($\Rightarrow$ reachability by “nice controls”)}
This result of robustness of reachability in fixed time under control sampling is instrumental for the following objective:

**Optimal control problem**

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), u(t)) \\
x(0) &= x^0, \quad x(T) = x^1 \\
\min \ C(x, u)
\end{align*}
\]

where

\[
C(x, u) = \int_0^T f^0(t, x(t), u(t)) \, dt
\]

- **Permanent controls**
  - \( u \in L_\infty^U \)
  - Optimal solution \((x, u)\)
  - PMP \( \Rightarrow \) adjoint \( p \)

- **Sampled-data controls**
  - \( u \in PC_T^U \)
  - Optimal solution \((x^T, u^T)\)
  - PMP\(_T \Rightarrow \) adjoint \( p^T \)

(sampled-data PMP: Bourdin Trélat, MCRF 2016)

**Objective:** prove that

\[
\begin{align*}
\mathbf{x}^T &\xrightarrow{C^0} \mathbf{x}, \\
\mathbf{p}^T &\xrightarrow{C^0} \mathbf{p}, \\
C(\mathbf{x}^T, \mathbf{u}^T) &\rightarrow C(\mathbf{x}, \mathbf{u})
\end{align*}
\]

as \(||T|| \rightarrow 0.||\)
Theorem (Bourdin Trélat, ongoing)

This is true under the following assumptions:

1. \( U \) is compact and trajectories live in a (big) compact.
2. For all \( (t, x) \), the epigraph of the extended velocity set
   \[
   \left\{ \left( f(t, x, u), f^0(t, x, u) + \gamma \right) \quad | \quad \gamma \geq 0, \ u \in U \right\}
   \]
   is convex.
3. Uniqueness of the optimal solution \( (x, u) \).
4. Unique normal extremal lift \( (x, p, u) \).

(1), (2): classical assumptions ensuring existence of minimizers
(3), (4): “generic” assumptions

(in the absence of uniqueness in (3) and/or (4): convergence of subsequences)

Key points of the proof:

- Assumption (4) (normality) \( \Rightarrow \) robustness of reachability under control sampling
  \( \Rightarrow \) existence of optimal solution \( (x^T, u^T) \)
- Technical fact: convergence of Pontryagin cones of \( OCP^T \) to \( OCP \)
  (kind of grad-\( \Gamma \)-convergence)
As a particular case:
- Linear system: \( f(t, x, u) = A(t)x + B(t)u + r(t) \)
- Quadratic cost: \( f^0(t, x, u) = (x - \bar{x}(t))^\top Q(t)(x - \bar{x}(t)) + (u - \bar{u}(t))^\top R(t)(u - \bar{u}(t)) \)

In fixed finite horizon \( T \):
  sampled-data difference Riccati theory in [Bourdin Trélat, Automatica 2017]

In infinite horizon: \( f(t, x, u) = Ax + Bu, \quad f^0(t, x, u) = x^\top Qx + u^\top Ru \)

We have proved in [Bourdin Trélat, SICON 2021] the commutation of the diagram:
Long-term open issue

Optimal control problem

**Full discretization**
- Euler, Runge-Kutta, etc.

**Dualization**
- Kuhn-Tucker, then Newton's method

**Direct methods**

**Dualization**
- Pontryagin Maximum Principle

**Full discretization**
- Euler, Runge-Kutta, etc, then Newton (shooting method)

**Indirect methods**
Long-term open issue

Optimal control problem

**Full discretization**
Euler, Runge-Kutta, etc.

**Dualization**
Kuhn-Tucker, then Newton's method

**Dualization**
Pontryagin Maximum Principle

**Discretization**
Euler, Runge-Kutta, etc, then Newton (shooting method)

direct methods

indirect methods

No commutation in general. Commutation for Runge-Kutta methods with positive coefficients (Hager, 2000).
Counterexample (Hager, 2000)

LQ optimal control problem

\[
\dot{x}(t) = \frac{1}{2}x(t) + u(t), \quad x(0) = 1, \quad \min \frac{1}{2} \int_0^1 (2x(t)^2 + u(t)^2) \, dt
\]
Counterexample (Hager, 2000)

LQ optimal control problem

\[ \dot{x}(t) = \frac{1}{2} x(t) + u(t), \quad x(0) = 1, \quad \min \frac{1}{2} \int_0^1 (2x(t)^2 + u(t)^2) \, dt \]

The optimal solution is (differential Riccati)

\[
\begin{align*}
    x(t) &= \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)}, \\
    u(t) &= \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)}
\end{align*}
\]

with optimal cost

\[
\frac{2(-2 + e^6 + e^3)}{4 + 4e^3 + e^6} \simeq 1.728
\]
Counterexample (Hager, 2000)

LQ optimal control problem

\[
\dot{x}(t) = \frac{1}{2}x(t) + u(t), \quad x(0) = 1, \quad \min \frac{1}{2} \int_0^1 (2x(t)^2 + u(t)^2) \, dt
\]

Full discretization with mid-point rule:

\[
\begin{align*}
x_{k+1/2} &= x_k + \frac{h}{2} \left( \frac{1}{2} x_k + u_k \right) \\
x_{k+1} &= x_k + h \left( \frac{1}{2} x_{k+1/2} + u_{k+1/2} \right), \quad x_0 = 1
\end{align*}
\]

\[
\min \frac{1}{2} \sum_{k=0}^{N-1} \left( 2x_{k+1/2}^2 + u_{k+1/2}^2 \right)
\]
Counterexample (Hager, 2000)

LQ optimal control problem

\[
\dot{x}(t) = \frac{1}{2} x(t) + u(t), \quad x(0) = 1, \quad \min \frac{1}{2} \int_0^1 (2x(t)^2 + u(t)^2) \, dt
\]

Full discretization with mid-point rule:

\[
x_{k+1/2} = x_k + \frac{h}{2} \left( \frac{1}{2} x_k + u_k \right)
\]

\[
x_{k+1} = x_k + h \left( \frac{1}{2} x_{k+1/2} + u_{k+1/2} \right), \quad x_0 = 1
\]

The optimal solution is

\[
u_k = -\frac{4 + h}{2h}, \quad u_{k+1/2} = 0, \quad x_k = 1, \quad x_{k+1/2} = 0
\]

with optimal cost 0
Counterexample (Hager, 2000)

**LQ optimal control problem**

\[
\dot{x}(t) = \frac{1}{2}x(t) + u(t), \quad x(0) = 1, \quad \min \frac{1}{2} \int_{0}^{1} (2x(t)^2 + u(t)^2) \, dt
\]

Numerical simulation for \( N = 70 \):
**Counterexample (Hager, 2000)**

**LQ optimal control problem**

\[ \dot{x}(t) = \frac{1}{2} x(t) + u(t), \quad x(0) = 1, \quad \min \frac{1}{2} \int_0^1 (2x(t)^2 + u(t)^2) \, dt \]

But if we discretize with Euler:

\[ x_{k+1} = x_k + h \left( \frac{1}{2} x_k + u_k \right), \quad x_0 = 1, \quad \min \frac{1}{2} \sum_{k=0}^{N-1} \left( 2x_k^2 + u_k^2 \right) \]

then everything is going fine: