Digital Optimal Control for Muscular Force Response to Functional Electrical Stimulations

Bernard Bonnard, Loïc Bourdin & Jérémy Rouot

INRIA, McTAO, XLIM & LMBA

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Guidelines

1. Models: Isometric and Non-isometric case
2. Optimal Control Problems
3. Geometric analysis of the system
4. Optimal Control Methods
5. Numerical results
6. References
Models of Muscle Contraction

**Excitation signal** $u$. Dirac impulses $\delta$ at times $t = 0, t_1, t_2, \ldots, t_N$.

$$u(t) = \sum_{i=0}^{N} \eta_i \delta(t - t_i), \quad \eta_i \in [0, 1]$$

**Pulses train** : $0 = t_0 < t_1 < \cdots < t_N < T$
- Number of impulses : $N + 1$ (fixed),
- Amplitudes : $\eta_i \in [0, \ldots, 1]$,
- Constraints : $t_i - t_{i-1} \geq I_{\text{min}} > 0, \; i = 1, \ldots, n$.

**FES signal** : $E_s$ defined from the linear dynamics

$$\dot{E}_s(t) + \frac{E_s(t)}{\tau_c} = \frac{1}{\tau_c} \sum_{i=0}^{N} R_i \eta_i \delta(t - t_i)$$

$$E_s(0) = 0$$

where

$$R_i := \begin{cases} 
1, & \text{for } i = 0, \\
1 + (\bar{R} - 1) \exp \left( - \frac{t_i - t_{i-1}}{\tau_c} \right), & \text{for } i = 1, \ldots, N,
\end{cases}$$

takes into account the memory effect due to successive contractions.
Ding et al model without fatigue  Evolution of the $Ca^{2+}$ concentration

$$\dot{C}_N(t) = -\frac{C_N(t)}{\tau_c} + E_s(t),$$

For the isometric model, the muscular force response (without considering fatigue) is given by

$$\dot{F}(t) = -F(t) \ m_2(t) + A \ m_1(t),$$

where

$$m_1(t) := \frac{C_N(t)}{K_m + C_N(t)}, \text{ and } m_2(t) := \frac{1}{\tau_1 + \tau_2 m_1(t)}.$$ 

are the Michaelis–Menten–Hill functions.

Model with 6 parameters :

- $\bar{R}, \tau_c$
- $A, K_m, \tau_1, \tau_2$ : fatigue parameters

Variants of this model are discussed in E. Wilson, *Force Response of Locust Skeletal Muscle*, (2011)
Ding et al model with fatigue\textsuperscript{1}

The fatigue parameters evolve according to the dynamics:

\[
\dot{A}(t) = -\frac{A(t) - A_{\text{rest}}}{\tau_{\text{fat}}} + \alpha_A F(t),
\]

\[
\dot{K}_m(t) = -\frac{K_m(t) - K_{m,\text{rest}}}{\tau_{\text{fat}}} + \alpha_{K_m} F(t),
\]

\[
\dot{\tau}_1(t) = -\frac{\tau_1(t) - \tau_{1,\text{rest}}}{\tau_{\text{fat}}} + \alpha_{\tau_1} F(t),
\]

which depends upon the characteristic time constant $\tau_{\text{fat}}$.

---

Non-isometric model\(^2\) We extend the previous model to model articular displacement modifying force parameters.

The force produces an joint range given by a pendulum where \(\theta\) is the joint angle.

The new dynamics is written

\[
\dot{F}(t) = m_1(t) (G(t) + A(t)) - m_2(t) F(t)
\]

\[
A(t) = A_{90}(t) \left[ a(90 - \theta(t))^2 + b(90 - \theta(t)) + 1 \right]
\]

\[
G(t) = v_1 \theta(t) e^{-v_2 \theta(t)} \dot{\theta}(t),
\]

\[
\ddot{\theta}(t) = \frac{L}{I} (F_{ext} \cos \theta(t) - F(t)),
\]

---

The fatigue variables $A_{90}, K_m, \tau_1$ satisfy the modified dynamics

\[
\begin{align*}
\dot{A}_{90}(t) &= -\frac{A_{90}(t) - A_{90,rest}}{\tau_{fat}} + (\alpha_{A_{90}} + \beta_{A_{90}} \dot{\theta}(t)) F(t) \\
\dot{K}_m(t) &= -\frac{K_m(t) - K_{m,rest}}{\tau_{fat}} + (\alpha_{K_m} + \beta_{K_m} \dot{\theta}(t)) F(t) \\
\dot{\tau}_1(t) &= -\frac{\tau_1(t) - \tau_{1,rest}}{\tau_{fat}} + (\alpha_{\tau_1} + \beta_{\tau_1} \dot{\theta}(t)) F(t).
\end{align*}
\]

(1)

with the same time constant $\tau_{fat}$ but a right hand side in $\dot{\theta}$.

**Formulation**

The dynamics can be written

\[
\dot{x}(t) = f_0(x(t)) + f_1(x(t)) u(t)
\]

with $x = (C_N, F, \theta, \dot{\theta}, A_{90}, K_m, \tau_1)$ is the state and the control is **permanent** (bounded measurable maps).
However the physical case imposes digital control where the applied control is the FES signal:

\[ E_S(t) = \frac{e^{-1/\tau_c}}{\tau_c} \sum_{i=0}^{n} R_i e^{t_i/\tau_c} \eta_i H(t - t_i), \]

**Discrete variables:**

- \((t_1, \ldots, t_N)\) must satisfy: \(t_i - t_{i-1} \geq I_{\text{min}} \quad i = 1, \ldots, n\)
- \((\eta_0, \eta_1, \ldots, \eta_N)\) must satisfy: \(\eta_i \in [0, 1], \quad i = 0, \ldots, n.\)

We obtain a force response

\[ F: (t_1, t_N, \eta_0, \ldots, \eta_N) \rightarrow F(t) \]

which depends upon finite number of parameters and is smooth for \(t \neq t_i\).

**Remark** Non standard framework:

- time dependence
- \(\delta(t - t_i)\) produces a head effect on \([t_i, t_{i+1}]\) and a tail effect for \(t \in [t_i, T]\).
Optimal control problems

- **Endurance program**: Reach a reference force
  
  \[ \int_0^T |F - F_{ref}|^2 \, dt \to \min \]

  We also can optimize many pulses trains alternating excitation and rest periods on an interval \([0, t_f]\).

- **Punch program**: Maximize the final force:
  
  \[ F(T) \to \max \]

- **Non-isometric case**: We would like to track a reference \(\theta\)-trajectory
  
  \[ t \mapsto \theta_r(t), \quad t \in [0, t_f]. \]
The aim is to analyze the dynamics of the control system with geometric arguments.

**Integrability**

**Proposition**

The dynamics of the concentration is linear and

\[
c_N(t) = \frac{1}{\tau_c} \sum_{i=0}^{N} R_i \eta_i (t - t_i) H(t - t_i) \exp \left( -\frac{t - t_i}{\tau_c} \right)
\]

Nonlinear dependance between \( c_N \) and the time control parameters \( (t_i) \).
Proposition

In the isometric case and with fixed fatigue parameters, the force equation can be integrated by quadratures using time reparameterization.

\[
\frac{dF}{ds}(s) = m_3(s) - F(s), \quad m_3 = Am_1/m_2, \quad ds = m_2(t)dt.
\]

We can observe that for a variation of the fatigue parameters, the model is linear but not controllable.
⇒ Reduce the number of variables.

Using sensitivity analysis, we can keep the most sensitive fatigue variable :\( A \). This is done using the notion of Jacobi fields.
Sensitivity analysis

- We extend the dynamics:

\[
\begin{align*}
\dot{x} &= f(x, \lambda, u) \\
\dot{\lambda} &= 0, \quad \lambda: \text{parameters.}
\end{align*}
\]

\[\implies \dot{x} = \tilde{f}(\tilde{x}, u)\]

- Hamiltonian lift: \(\tilde{H}(\tilde{x}, \tilde{p}, u) = \tilde{p} \cdot \tilde{f}\).

- Projection \(\pi_j : x \mapsto x_j\).
Sensitivity analysis

- We extend the dynamics:

\[
\begin{aligned}
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- Hamiltonian lift: \(\tilde{H}(\tilde{x}, \tilde{p}, u) = \tilde{p} \cdot \tilde{f}\).

- Projection \(\pi_j : x \mapsto x_j\).

**Definition (Jacobi fields)**

The Jacobi equation is

\[
\dot{\delta z}(t) = \frac{\partial}{\partial z} \tilde{H}(z(t), u(t)) \delta z(t)
\]

The **Jacobi fields associated to** \(x_i\)-variation \(i = 1, \ldots, n\) are the solutions \(J_i(t), i = 1, \ldots, n\) with \(J_i(0) = e_i, i = 1, \ldots, n\) where \((e_i)_i\) is the \(\mathbb{R}^n \times \mathbb{R}^n\) canonical basis.
Sensitivity analysis

- We extend the dynamics:

\[
\begin{aligned}
\dot{x} &= f(x, \lambda, u) \\
\dot{\lambda} &= 0, \quad \lambda: \text{parameters.} \\
\Rightarrow \quad \dot{x} &= \tilde{f}(\tilde{x}, u)
\end{aligned}
\]

- Hamiltonian lift: \( \tilde{H}(\tilde{x}, \tilde{p}, u) = \tilde{p} \cdot \tilde{f} \).

- Projection \( \pi_j : x \mapsto x_j \).

Definition (Sensitivity)

*The sensitivity of the fatigue variables \( x_i, i = 3, 4, 5 \) w.r.t. the force is defined by*

\[
\max_{t \in [0, T]} |\Pi_F(J_i(t))|, \quad i = 3, 4, 5 \quad (n = 5)
\]

*where \( \Pi_F \) is the projection \( z \mapsto x_2 \) (on the force variable).*
Sensitivity analysis. Time evolution of the Jacobi fields component $\delta F(\cdot)$.

The fatigue variable $A$ is the most relevant for the given reference trajectory.
Methods in optimal control

- **PMP**: permanent case with a control $u(t)$.
- **Digital case**: with a control $(t_1, \ldots, t_N, \eta_0, \ldots, \eta_N)$ in finite dimension, we extend the necessary conditions obtained in [Bourdin–Trélat] ³.

**Standard techniques**
- $L^\infty$ variations on $\eta_i$
- variations on $t_i$ associated to the $L^1$-topology

**Numerical methods**
- **Direct methods**: optimization algorithms adapted to the discretization and the sampling times
- optimization algorithms based on an approximation of the Input–Output mapping:

$$ (t_1, \ldots, t_N, \eta_0, \ldots, \eta_N) \rightarrow \tilde{F} $$

the dynamics is not integrated and this allow much better robustness and performance.

This can be used for the endurance & punch program.


To discuss

\[ \dot{q} = F_0 + uF_1 \]

\[ D_{L.A.} = \{F_0, F_1\}_{L.A.} : \text{Algèbre de Lie engendrée} \]

\[ D_{L.A.} = \cup_{k \geq 1} D_k, \quad D_1 = \text{span}\{F_0, F_1\} = \text{span}(D), \quad D_k = \text{span}\{D_{k-1} + [D, D_{k-1}]\} \]

Calcul d’une base de Hall de $D$ pour engendrer $D_{L.A.}$.

Applications

- Singular trajectories
- Refine accessibility properties of the system using Baker–Campbell–Hausdorff
- Implement an smart electro-stimulator
Choose the program’s algorithm

Example case: the Endurance Program

Scan the muscle to estimate the parameters

$\bar{R}, \tau_1, \tau_2, \tau_c, A_{rest}, \alpha, A_{fat}$ (see Table 1)

Define the maximal muscle strength (Tetanus)

$F_{\text{ref}}(t)$ with $\rho > 1$

Define $F_{\text{ref}}$

Calculate the accessory control $m_1$

$A_{rest}m_1^2 + A_{rest}m_1 - F_{\text{ref}} = 0$

Calculate the reference concentration $c_{N,\text{ref}}$

$C_{N,\text{ref}}$

Optimize the pulses train to minimize the cost function: $\int_0^T (c_N - c_{N,\text{ref}})^2 \, dt$

- $F$, Runtime: 0.235s ($n = 10$)
- $F_{\text{ref}} = 0.088$
- $c_{N,\text{ref}} = 0.0488$

**Figure** – Endurance program for a smart electrostimulator.
Numerical methods for the electrostimulator
Recall the Ding et al. model

**Ding et al isometric model**

\[
\dot{C}_N(t) = -\frac{C_N(t)}{\tau_c} + E_s(t),
\]

\[
\dot{F}(t) = -F(t) \, m_2(t) + A \, m_1(t),
\]

where

\[
m_1(t) := \frac{C_N(t)}{K_m + C_N(t)}, \quad \text{and} \quad m_2(t) := \frac{1}{\tau_1 + \tau_2 m_1(t)}.
\]

are the Michaelis–Menten–Hill functions.

Model with 6 parameters:

- \(\bar{R}, \tau_c\)
- \(A, K_m, \tau_1, \tau_2\) : fatigue parameters
"Sampled-data" optimal control

Mayer formulation

\[
\min \phi(x(T), x^0(T)) := \tilde{\phi}(x(T)) + \int_0^T f^0(x(s)) \, ds
\]

\[
\dot{x}(t) = f_1(x(t)) + f_2(t) \sum_{i=1}^{n} R_i \eta_i e^{\frac{t}{rc}} H(t - t_i),
\]

\[
x(0) = x_0,
\]

\[
\dot{x}^0(t) = f^0(x(t)), \quad x^0(0) = 0
\]

s.t.

\[
(\eta_0, \eta_1, \ldots, \eta_N, t_1, \ldots, t_N) \in \mathbb{R}^{2n+1},
\]

\[
\eta_i \in [0, 1], \quad \forall i = 0, \ldots, N,
\]

\[
t_0 = 0 < t_1 < t_2 < \ldots < t_N < T = t_{n+1},
\]

\[
t_{i+1} - t_i \geq \Delta, \quad \forall i = 0, \ldots, n,
\]

where \( x = (c_N, F, A, \tau_1, \tau_2, K_m) \), typically \( \tilde{\phi}(x) = F \) & \( f^0 = 0 \) or \( \tilde{\phi}(x) = 0 \) & \( f^0(x) = (F - F_{ref})^2 \).
Indirect Methods
Necessary optimality conditions

Recap: Permanent control case (Pontryagin, 1962)

\[ \min_{u \in U} \varphi(x(T)), \]
\[ \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \]

\( U \): Admissible controls = bounded measurable mappings.

Necessary optimality conditions

Recap: Permanent control case (Pontryagin, 1962)\(^4\)

\[
\min_{u \in \mathcal{U}} \varphi(x(T)),
\]

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0
\]

\(\mathcal{U}\): Admissible controls = \textbf{bounded measurable mappings}.

Let \(x^*\) a reference optimal trajectory associated to \(u^*\).

---

Necessary optimality conditions

\[ x_{\varepsilon}, u_{\varepsilon} \]

\[ w(s) \]

\[ x^*(s) \]

\[ x^*(T) \]

\[ w(T) \]

\[ x^*, u^* \]

\[ w(s)^T \]

\[ w(s)^T \]

\[ x^*(T) \]

\[ u_{\varepsilon} \]

\[ u^* \]

\[ s \]

\[ s + \varepsilon \]
Necessary optimality conditions

- $L^1$-perturbation:

$$u_\varepsilon(t) := \begin{cases} v \in U \subset \mathbb{R}^m & \text{on } [s, s + \varepsilon[, \ (s \in [0, T[) \\ u^*(t) & \text{on } [s + \varepsilon, T[ \end{cases}$$

- Corresponding variation vector $w$ s.t.:

$$x(t, u_\varepsilon) = x(t, u^*) + \varepsilon \mathbf{w}(t) + o(\varepsilon)$$

$$\dot{w}(t) = \nabla_x f(x^*(t), u^*(t)) \mathbf{w}(t),$$

$$w(s) = f(x^*(s), v) - f(x^*(s), u^*(s))$$

Denote by $\Phi(\cdot, \cdot)$ the state-transition matrix of $\nabla_x f(x^*, u^*)$:

$$w(T) = \Phi(T, s) w(s).$$
From optimality of \((x^*, u^*)\),

\[
0 \leq \frac{\varphi(x_\varepsilon(T)) - \varphi(x^*(T))}{\varepsilon} \xrightarrow{\varepsilon \to 0} \langle \nabla \varphi(x^*(T)), w(T) \rangle
\]
From optimality of \((x^*, u^*)\),

\[
0 \leq \frac{\varphi(x_\varepsilon(T)) - \varphi(x^*(T))}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \langle \nabla \varphi(x^*(T)), w(T) \rangle
\]

Introducing the co-state vector \(p(t)\) s.t.:

\[
\dot{p}(t) = -\nabla_x f(x^*(t), u^*(t))^\top p(t),
\]

\[
p(T) = -\nabla \varphi(x^*(T)).
\]
From optimality of \((x^*, u^*)\),

\[
0 \leq \frac{\varphi(x_{\varepsilon}(T)) - \varphi(x^*(T))}{\varepsilon} \quad \varepsilon \to 0 \quad \langle \nabla \varphi(x^*(T)), w(T) \rangle
\]

Introducing the co-state vector \(p(t)\) s.t. :

\[
\dot{p}(t) = -\nabla_x f(x^*(t), u^*(t))^\top p(t), \\
p(T) = -\nabla \varphi(x^*(T)).
\]

Using \(w(T) = \Phi(T, s) \quad w(s)\) and \(p(s) = \Phi(T, s)^\top \quad p(T)\) we finally get :

\[
\forall v \in U, \quad \langle p(s), f(x^*(s), v) - f(x^*(s), u^*(s)) \rangle \leq 0
\]

which is the so-called maximization condition of the Pontryagin maximum principle.
Necessary optimality conditions

Non Permanent control case (Bourdin, Trélat, 2016)\(^4\).

\[
\min_{u \in \mathcal{U}} \varphi(x(T)),
\]

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0
\]

\(\mathcal{U}\) : Admissible controls = **piecewise constant mappings**.
Let \(x^*\) a reference optimal trajectory associated to \(u^*\).

\[\]

---

Necessary optimality conditions

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---

Necessary optimality conditions

Non Permanent control case (Bourdin, Trélat, 2016)\(^4\).

\[
\begin{align*}
\min_{u \in \mathcal{U}} & \quad \varphi(x(T)), \\
\dot{x}(t) & = f(x(t), u(t)), \quad x(0) = x_0
\end{align*}
\]

\(\mathcal{U}\) : Admissible controls = piecewise constant mappings.

Let \(x^*\) a reference optimal trajectory associated to \(u^*\).

- **\(L^\infty\)-perturbation** : \(u_\varepsilon := u^* + \varepsilon(\xi - u^*)\) (\(\xi\) is valued in \(U\) has the same sampling times as \(u^*\)).
- This time, the corresponding variation vector \(\mathcal{W}\) satisfies :
  \[
  \dot{w} = \nabla_x f(x^*, u^*) \cdot w + \nabla_u f(x^*, u^*) (\xi - u^*),
  \]
  \[
  w(0) = 0
  \]

hence,

\[
\begin{align*}
w(T) & = \int_{0}^{T} \Phi(T, s) \nabla_u f(x^*(s), u^*(s)) (\xi(s) - u^*(s)) \, ds.
\end{align*}
\]

---

\[ w(T) = \int_0^T \Phi(T, s) \nabla_u f(x^*(s), u^*(s)) (\xi(s) - u^*(s)) \, ds. \]

Using it, together with \( 0 \leq \langle \nabla \varphi(x^*(T)), w(T) \rangle \), yield

\[ \int_0^T \langle p(s), \nabla_u f(x^*(s), u^*(s)) (\xi(s) - u^*(s)) \rangle \, ds \leq 0. \]
\[ w(T) = \int_0^T \Phi(T, s) \nabla u f(x^*(s), u^*(s)) (\xi(s) - u^*(s)) \, ds. \]

Using it, together with \( 0 \leq \langle \nabla \varphi(x^*(T)), w(T) \rangle \), yield

\[
\int_0^T \langle p(s), \nabla u f(x^*(s), u^*(s)) (\xi(s) - u^*(s)) \rangle \, ds \leq 0.
\]

Finally, taking \( \xi = \mathbf{v} \in U \) over \([t^*_i, t^*_{i+1}][\) and \( \xi(t) := u^*(t) \) elsewhere, we get

\[
\left\langle \int_{t^*_i}^{t^*_{i+1}} \nabla u H(x^*(s), p(s), u^*_i) \, ds, \mathbf{v} - u^*_i \right\rangle \leq 0,
\]

for all \( \mathbf{v} \in U \) and all \( i = 0, \ldots, N \), where \( u^*_i \) corresponds to the value of \( u^* \) over the interval \([t_i, t_{i+1}][\).
Remarks

- Same weaker maximization condition than the **discrete Pontryagin maximum principle** (Boltyanskii, 1978)\(^5\)

- Another proof with different approach by Dmitruk and Kaganovich (2011)\(^6\)

- Not directly applicable to our model since the sampling times are involved in the dynamics.

---


Application to the force-fatigue model

**Theorem**

If \((\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)\) is optimal, then there exists \(p\) satisfying the co-state equation and the transversality condition.
Application to the force-fatigue model

**Theorem**

If \((\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)\) is optimal, then there exists \(p\) satisfying the co-state equation and the transversality condition. Moreover, the necessary conditions are:

(i) the inequality

\[
\left( \int_{t_i^*}^{T} p_1(s) b(s) \, ds \right) \tilde{\eta}_i \leq 0,
\]

for all \(i = 0, \ldots, n\) and all admissible perturbation \(\tilde{\eta}_i\) of \(\eta_i^*\);

(ii) and the inequality

\[
NC_i := \left( -p_1(t_i^*) b(t_i^*) G(t_{i-1}^*, t_i^*) \eta_i^* + b(-t_i^*) \eta_i^* \int_{t_i^*}^{T} p_1(s) b(s) \, ds 
+ b(-t_i^*) (\bar{R} - 1)\eta_{i+1}^* \int_{t_i^*}^{T} p_1(s) b(s) \, ds \right) \tilde{t}_i \leq 0,
\]

for all \(i = 1, \ldots, n\) and all admissible perturbation \(\tilde{t}_i\) of \(t_i^*\).
1 **Open-loop control.**

**Direct methods:** not based on necessary optimality conditions. Inefficiency wrt to non-smooth dynamics.

**Indirect methods:**
- Shooting algorithm to solve the *boundary value problem* coming from the necessary conditions
- Newton-like algorithm to find a zero of the shooting function
- Direct method to give an initialization
- Adapted integration scheme

2 **Closed-loop control.** **Adaptive control algorithms** where the fatigue is estimated by a non-linear observer.
Numerical methods

1. Open-loop control.

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- Shooting algorithm to solve the *boundary value problem* coming from the necessary conditions
- Newton-like algorithm to find a zero of the shooting function
- Direct method to give an initialization
- Adapted integration scheme.

2. Closed-loop control. **Adaptive control algorithms** where the fatigue is estimated by a non-linear observer.

⇒ **Complementaries of the methods**:

  - *open-loop*: compute a pulses train to reach the maximal force ($T \sim 1\text{s}$),
  - *closed-loop*: stabilization near a reference force with rest and stimulation periods ($T \gg 10\text{s}$)
Direct method

Idea.

Sampled-data optimal control problem $\iff$ Finite-dimensional optimization problem

Algorithms

primal-dual interior point algorithm
derivatives are computed by automatic differentiation.

$\Rightarrow$ handle constraints on the state/control, in general less precise than indirect methods, not that robust to initialization and efficient with our model.
Direct method

**Idea.**

Sampled-data optimal control problem $\iff$ Finite-dimensional optimization problem

**Method.** Transform the optimal control problem in a nonlinear finite-dimensional optimization problem (NLP) via discretization in time of the state. $t_i, i = 1, \ldots, N$ are the optimization variables of the NLP.

**Algorithms**

- primal-dual interior point algorithm
- derivatives are computed by automatic differentiation.

$\Rightarrow$ **handle constraints on the state/control**, in general less precise than indirect methods, not that robust to initialization and efficient with our model.
**Direct method**:

\[
\max_{t_i} F(T), \quad N = 10, \quad 10ms \leq t_{i+1} - t_i, \quad i = 0, \ldots, N.
\]
Indirect method.

Exploit the **geometric structure** of the solutions via the necessary conditions. *Preliminary results*: relax the inequalities in the optimality conditions to obtain a boundary value problem.

$\implies$ Fast convergence and high accuracy/precision.

**Multiple shooting method**: $(n + 2nN + N)$ unknowns:

\[
p(0), \quad Z_i = (x(t_i), p(t_i)), \quad i = 1, \ldots, N, \quad \sigma = (t_1, \ldots, t_N).
\]
Multiple shooting method: \((n + 2nN + N)\) unknowns:

\[ p(0), \quad Z_i = (x(t_i), p(t_i)), \ i = 1, \ldots, N, \quad \sigma = (t_1, \ldots, t_N). \]
**Multiple shooting method** : \((n + 2nN + N)\) unknowns :

\[ p(0), \quad Z_i = (x(t_i), p(t_i)), \quad i = 1, \ldots, N, \quad \sigma = (t_1, \ldots, t_N). \]

**Shooting function.** Find a zero of the function \(S(p_0, Z_1, \ldots, Z_N, \sigma)\) so that

- the initial condition \(x(0) = x_0\),
- the continuity conditions \(Z_i^- = Z_i^+, \quad i = 1, \ldots, N\),
- the necessary conditions \(NC_i \leq 0 \quad i = 1, \ldots, N\),

are satisfied.
Multiple shooting method: \((n + 2nN + N)\) unknowns:

\[
p(0), \quad Z_i = (x(t_i), p(t_i)), i = 1, \ldots, N, \quad \sigma = (t_1, \ldots, t_N).
\]

**Shooting function.** Find a zero of the function \(S(p_0, Z_1, \ldots, Z_N, \sigma)\) so that

- the initial condition \(x(0) = x_0\),
- the continuity conditions \(Z_i^- = Z_i^+, i = 1, \ldots, N\),
- the necessary conditions \(NC_i \leq 0\) \(i = 1, \ldots, N\),

are satisfied.

**Shooting algorithm.** Sensitive to initialization.

**Initialization:** compute a solution \((\tilde{x}, \tilde{u})\) with a direct method, by continuation or by approximation.

Starting from \((\tilde{x}(T), p(T))\) (where \(p(T) = -\nabla \varphi(\tilde{x}(T))\) is known), integrate backward the co-state dynamics to obtain \(p(0)\).

**Tools:** Julia’s libraries:

- Extended precision for float (ArbNumerics.jl)
- **Stiff** numerical integrator (DifferentialEquations.jl)
**Figure** – Quality of the optimal solution computed with multiple shooting with respect to its perturbations. The quality is measured from the necessary conditions and the value of the cost.
Closed-loop algorithm

- **Sensitivity analysis**: select the relevant fatigue variable for estimation
- **Detectability**: construct an observer to estimate the chosen fatigue variable
- Adaptive control algorithm (MPC) based on the observer
Conclusion on these methods

- We recover the solution of the direct method with the indirect methods (using the initialization provided by the direct method)
- No implementation of the control constraints for the indirect methods
- Both methods are inefficient (in terms of robustness and time complexity) for the design of an electrostimulator \(^7 (n \approx 8)\).

Alternatives for real time computation

- Approximation of the Input–Output mapping :

\[
(t_1, \ldots, t_N, \eta_0, \ldots, \eta_N) \rightarrow \tilde{F}
\]

- Try adapted dicretization for non smooth dynamics → "event driven methods".

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7. On-going collaboration with Segula Technology (CIFRE & Peps AMIES)
Definition

A lobe at $t_k$ is the representative curve of the function

$$\ell_k : \mathbb{R} \ni t \mapsto R_k \eta_k \frac{t-t_k}{\tau_c} e^{-(t-t_k)/\tau_c} H(t-t_k).$$
Lemma

For $t \neq t_i$, $c_N(t)$ is in the polynomial-exponential category. This category is stable with respect to derivation and integration.

Property

Let $k \in \{0, \ldots, n\}$.

- If $t_{k+1} < t_k + 2\tau_c$ then $m_1|[t_k, t_{k+1}]$ is concave and $m_2|[t_k, t_{k+1}]$ is convex.

\[
\arg\max c_N|[t_k, t_{k+1}] = \arg\max m_1|[t_k, t_{k+1}] = \arg\max m_2|[t_k, t_{k+1}]
\]

\[
= \begin{cases} 
\tau_c & \text{si } k = 0 \\
\tau_c + \frac{\sum_{i=1}^k R_i \eta_i t_i e^{t_i/\tau_c}}{\sum_{i=0}^k R_i \eta_i e^{t_i/\tau_c}} & \text{sinon}.
\end{cases}
\]
Consider a finer partition of \((t_k)_{1 \leq k \leq n}\)

\[ t_{k+j/p}, \ j = 0, \ldots, p - 1, \ k = 0, \ldots, n \]

Polynomial approximation of \(m_1\) and \(m_2\) on each interval \([t_{k+j/p}, t_{k+(j+1)/p}]\).

**Example**

Take \(p = 2, \ t_{k+1/2} = \text{argmax} \ c_N(t), \ k = 0, \ldots, n \) and for \(i = 1, 2, \ j = 0, 1 : \)

\[ t \in [t_k, t_{k+1}] \]

\[ \tilde{m}_i(t) = a_{i,j,k} (t - t_{k+j/2}) + b_{i,j,k}, \ k = 0, \ldots, n. \]

Imposing \(\tilde{m}_i(t_{k+j/2}) = m_i(t_{k+j/2})\) and \(\tilde{m}_i(t_{k+(j+1)/2}) = m_i(t_{k+(j+1)/2})\), we get:

\[ a_{i,j,k} = \frac{m_i(t_{k+(j+1)/2}) - m_i(t_{k+j/2})}{t_{k+(j+1)/2} - t_{k+j/2}}, \ b_{i,j,k} = m_i(t_{k+j/2}). \]
Definition

As an approximation of $F$ for the Ding et al model without fatigue, take

$$
\tilde{F}(t) = A \int_{0}^{t} \tilde{M}(t) \tilde{M}^{-1}(s) \tilde{m}_1(s) \, ds,
$$

where $\tilde{M}(t) = \exp \left( - \int_{0}^{t} \tilde{m}_2(s) \, ds \right)$.

Decomposing the sum over each interval $[t_{k_t + j_t/p}, t_{k_t + (j_t + 1)/p}]$, $k_t = 0, \ldots, n$, $j_t = 0, \ldots, p - 1$, we have:

$$
\frac{\tilde{F}(t)}{A} = \sum_{i=0}^{k_t-1} \sum_{j=0}^{p-1} \int_{t_{i+(j+1)/p}}^{t_{i+j/p}} \tilde{M}(t) \tilde{M}^{-1}(s) \tilde{m}_1(s) \, ds
$$

$$
+ \sum_{j=0}^{j_t-1} \int_{t_{k_t+j/p}}^{t_{k_t+(j+1)/p}} \tilde{M}(t) \tilde{M}^{-1}(s) \tilde{m}_1(s) \, ds
$$

$$
+ \int_{t_{k_t+j_t/p}}^{t} \tilde{M}(t) \tilde{M}^{-1}(s) \tilde{m}_1(s) \, ds,
$$

(2)
Application

\[\tilde{m}_1(t) = \begin{cases} m_1(t_{k+1/2}) & \text{if } t \in [t_k, t_{k+1/2}] \\ a_{1,j,k}(t - t_{k+1}) + b_{1,j,k} & \text{if } t \in [t_{k+1/2}, t_{k+1}] \end{cases}, \]

\[\tilde{m}_2(t) = \begin{cases} \frac{m_2(t_k) + m_2(t_{k+1/2})}{2} & \text{if } t \in [t_k, t_{k+1/2}] \\ \frac{2}{m_2(t_{k+1/2}) + m_2(t_{k+1})} & \text{if } t \in [t_{k+1/2}, t_{k+1}] \end{cases}, \]

where \( t_{k+1/2} = \arg\max_{t \in [t_k, t_{k+1}]} c_N(t) \), \( a_{1,j,k} = (m_1(t_{k+1}) - m_1(t_{k+1/2}))/((t_{k+1} - t_{k+1/2}) \)

and \( b_{1,j,k} = m_1(t_{k+1}). \)

Then \( \tilde{F} \) has a closed form and its expression can be generated \textit{offline} with \textit{formal computation}.
\[ \sigma = (t_1, \ldots, t_n, \eta_1, \ldots, \eta_n) \]

**Finite dimensional optimization problem**

\[
\min_{\sigma} \quad \Theta(\sigma) \\
\mathcal{G}(\sigma) \leq 0,
\]

where \( \mathcal{G}(\sigma) = (\Xi_1(\sigma), \ldots, \Xi_{3n+5}(\sigma)) \) with:

\[
\Xi_i(\sigma^*) = t_{i-1}^* - t_i^* + I_{\min}, \quad i = 1, \ldots, n, \\
\Xi_{n+1}(\sigma^*) = t_n^* - T, \\
\Xi_{n+2+i}(\sigma^*) = -\eta_i^*, \quad i = 0, \ldots, n + 1, \quad \Xi_{2n+4+i}(\sigma^*) = \eta_i^* - 1, \quad i = 0, \ldots, n + 1.
\]

**Algorithm**: Primal-dual interior point method allows real time computation (standard computer).
\( \Theta(\sigma) := -\tilde{F}(T) \)

\[ F(T) = 0.34093 \]

\( \tau_1^* = 0.07023, \Xi_1 = -0.04023 \)
\( \tau_2^* = 0.1255, \Xi_2 = -0.02523 \)
\( \tau_3^* = 0.1719, \Xi_3 = -0.01645 \)
\( \tau_4^* = 0.2123, \Xi_4 = -0.01043 \)
\( \tau_5^* = 0.2476, \Xi_5 = -0.008225 \)
\( \tau_6^* = 0.2785, \Xi_6 = -0.0009607 \)
\( \tau_7^* = 0.3085, \Xi_7 = 0 \)
\( T = 0.3502, \Xi_8 = -0.04168 \)

**Figure** – Optimal solution computed with the approximation. Integration of the true \( F \) with respect to this solution.
$$\Theta(\sigma) = \sum_{k=0}^{n} \left( \tilde{F}(t_{k+1}) - F_{ref} \right)^2 (t_{k+1} - t_k).$$

\begin{align*}
\text{--- : } F, & \quad \text{--- : } \tilde{F}, \quad \text{Runtime: } 0.305\text{s} \quad (n = 7, p = 2).
\end{align*}

\textbf{FIGURE} – Optimal solution computed with the approximation. Integration of the true \( F \) with respect to this solution
Remarks

- Mathematical structure of $m_1, m_2$.

- We can tune $\tilde{m}_1, \tilde{m}_2$ to obtain upper (or lower) approximation of $F$.

- Using naive discretization and offline computation of $\tilde{F}$ leads to inefficient method.

- Error estimate $k = 0, \ldots, n$:
  \[
  |F(t_k) - \tilde{F}(t_k)| / A \leq \int_0^{t_k} |m_1(s) - \tilde{m}_1(s)| \, ds + t_k \int_0^{t_k} |m_2(s) - \tilde{m}_2(s)| \, ds.
  \]
Conclusion

- Compare with fast MPC
- Non smooth integrator ("event driven methods")
- Fatigue model
- Nonisometric case

With explicit Euler scheme:

\[
\tilde{F}(t_{i+(j+1)/p}) = \tilde{F}(t_{i+j/p}) c_{i,j} + A d_{i,j},
\]

where \( c_{i,j} = (1 - h_{i,j} m_2(t_{i+j/p})) \), \( h_{i,j} = t_{i+(j+1)/p} - t_{i+j/p} \) and \( d_{i,j} = m_1(t_{i+j/p}) \) for \( i = 0, \ldots, n \) and \( j = 0, \ldots, p - 1 \).

Then

\[
\tilde{F}(t_{k_t+j_t/p})/A = \sum_{j=0}^{j_t-1} h_{k_t,j} d_{k_t,j} \prod_{j'=j+1}^{j_t-1} c_{k_t,j'} + \sum_{i=0}^{k_t-1} \sum_{j=0}^{p-1} h_{i,j} d_{i,j} \left( \prod_{j'=0}^{p-1} c_{i',j'} \prod_{j'=j+1}^{p-1} c_{i,j'} \prod_{j'=0}^{j_t-1} c_{k_t,j'} \right).
\]