Digital Optimal Control for Muscular Force Response to Functional Electrical Stimulations

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Guidelines

Models of Muscle Contraction

Excitation signal u**.** Dirac impulses $\boldsymbol{\delta}$ at times $t = 0, t_1, t_2, \ldots, t_N$.

$$u(t) = \sum_{i=0}^{N} \eta_i \boldsymbol{\delta}(t - t_i), \quad \eta_i \in [0, 1]$$

Pulses train : $0 = t_0 < t_1 < \cdots < t_N < T$

- *Number of impulses :* $N + 1$ (fixed),
- *Amplitudes :* $\eta_i \in [0, \ldots, 1]$,
- *Constraints :* $t_i - t_{i-1} \geq I_{\min} > 0$, $i = 1, \ldots, n$.

FES signal : E_s defined from the linear dynamics

$$\dot{E}_s(t) + \frac{E_s(t)}{\tau_c} = \frac{1}{\tau_c} \sum_{i=0}^{N} R_i \eta_i \boldsymbol{\delta}(t - t_i)$$

$$E_s(0) = 0$$

where

$$R_i := \begin{cases} 1, & \text{for } i = 0, \\ 1 + (\bar{R} - 1) \exp\left(-\dfrac{t_i - t_{i-1}}{\tau_c}\right), & \text{for } i = 1, \ldots, N, \end{cases}$$

takes into account the memory effect due to successive contractions.

Ding et al model without fatigue Evolution of the Ca^{2+} concentration

$$\dot{C}_N(t) = -\frac{C_N(t)}{\tau_c} + E_s(t),$$

For the isometric model, the muscular force response (without considering fatigue) is given by

$$\dot{F}(t) = -F(t)\ \boldsymbol{m_2(t)} + A\ \boldsymbol{m_1(t)},$$

where

$$\boldsymbol{m_1(t)} := \frac{C_N(t)}{K_m + C_N(t)}, \text{ and } \boldsymbol{m_2(t)} := \frac{1}{\tau_1 + \tau_2 m_1(t)}.$$

are the Michaelis–Menten–Hill functions.

Model with 6 parameters :

- $\bar{R}, \tau_c$
- A, K_m, τ_1, τ_2 : fatigue parameters

Variants of this model are discussed in E. Wilson, *Force Response of Locust Skeletal Muscle*, (2011)

Ding et al model with fatigue[1]

The fatigue parameters evolve according to the dynamics :

$$\dot{A}(t) = -\frac{A(t) - A_{\text{rest}}}{\tau_{fat}} + \alpha_A F(t),$$

$$\dot{K}_m(t) = -\frac{K_m(t) - K_{m,\text{rest}}}{\tau_{fat}} + \alpha_{K_m} F(t),$$

$$\dot{\tau}_1(t) = -\frac{\tau_1(t) - \tau_{1,\text{rest}}}{\tau_{fat}} + \alpha_{\tau_1} F(t),$$

which depends upon the characteristic time constant τ_{fat}.

1. J. Ding, A.S. Wexler and S.A. Binder-Macleod, *Development of a mathematical model that predicts optimal muscle activation patterns by using brief trains*, J. Appl. Physiol., **88** (2000) 917–925

Non-isometric model[2] We extend the previous model to model articular displacement modifying force parameters.

The force produces an joint range given by a pendulum where θ is the joint angle.

The new dynamics is written

$$
\begin{aligned}
\dot{F}(t) &= m_1(t)\,(G(t) + A(t)) - m_2(t)\,F(t) \\
A(t) &= A_{90}(t)\left[a(90-\theta(t))^2 + b(90-\theta(t)) + 1\right] \\
G(t) &= v_1\theta(t)e^{-v_2\theta(t)}\dot{\theta}(t), \\
\ddot{\theta}(t) &= \frac{L}{I}(F_{ext}\cos\theta(t) - F(t)),
\end{aligned}
$$

2. M.S. Marion, A.S. Wexler and M.L. Hull, Predicting non-isometric fatigue induced by electrical stimulation pulse trains as a function of pulse duration, Journal of neuroengineering and rehabilitation, 10 no.13 (2013)

The fatigue variables A_{90}, K_m, τ_1 satisfy the modified dynamics

$$\begin{aligned}
\dot{A}_{90}(t) &= -\frac{A_{90}(t) - A_{90,rest}}{\tau_{fat}} + (\alpha_{90} + \beta_{90}\dot{\theta}(t))\, F(t) \\
\dot{K}_m(t) &= -\frac{K_m(t) - K_{m,rest}}{\tau_{fat}} + (\alpha_{K_m} + \beta_{K_m}\dot{\theta}(t))\, F(t) \\
\dot{\tau}_1(t) &= -\frac{\tau_1(t) - \tau_{1,rest}}{\tau_{fat}} + (\alpha_{\tau_1} + \beta_{\tau_1}\dot{\theta}(t))\, F(t).
\end{aligned} \tag{1}$$

with the same time constant τ_{fat} but a right hand side in $\dot{\theta}$.

Formulation

The dynamics can be written

$$\dot{x}(t) = f_0(x(t)) + f_1(x(t))\, u(t)$$

with $x = (C_N, F, \theta, \dot{\theta}, A_{90}, K_m, \tau_1)$ is the state and the control is **permanent** (bounded measrable maps).

However the physical case imposes digital control where the applied control is the FES signal :

$$E_s(t) = \frac{e^{-1/\tau_c}}{\tau_c} \sum_{i=0}^{n} R_i e^{t_i/\tau_c} \eta_i H(t - t_i),$$

Discrete variables :

- $(t_1, \ldots, t_N)$ must satisfy : $t_i - t_{i-1} \geq I_{\min}$ $i = 1, \ldots, n$
- $(\eta_0, \eta_1, \ldots, \eta_N)$ must satisfy : $\eta_i \in [0, 1],\ i = 0, \ldots, n.$

We obtain a force response

$$F : (t_1, t_N, \eta_0, \ldots, \eta_N) \mapsto F(t)$$

which depends upon finite number of parameters and is smooth for $t \neq t_i$.

Remark Non standard framework :

- time dependence
- $\delta(t - t_i)$ produces a head effect on $[t_i, t_{i+1}]$ and a tail effect for $t \in [t_i, T]$.

Optimal control problems

- **Endurance program :** Reach a reference force

$$\int_0^T |F - F_{ref}|^2 \, \mathrm{d}t \to \min$$

  We also can optimize many pulses trains alternating excitation and rest periods on an interval $[0, t_f]$.

- **Punch program :** Maximize the final force :

$$F(T) \to \max$$

- **Non-isometric case :** We would like to track a reference θ-trajectory $t \mapsto \theta_r(t),\ t \in [0, t_f]$.

Geometric analysis

The aim is to analyze the dynamics of the control system with geometric arguments.

Integrability

Proposition

The dynamics of the concentration is linear and

$$c_N(t) = \frac{1}{\tau_c} \sum_{i=0}^{N} R_i \eta_i \, (t - t_i) \boldsymbol{H}(t - t_i) \exp\left(-\frac{t - t_i}{\tau_c}\right)$$

Nonlinear dependance between c_N and the time control parameters (t_i).

Proposition

In the isometric case and with fixed fatigue parameters, the force equation can be integrated by quadratures using time reparameterization.

$$\frac{\mathrm{d}F}{\mathrm{d}s}(s) = m_3(s) - F(s), \quad m_3 = Am_1/m2, \quad \mathrm{d}s = m_2(t)\mathrm{d}t.$$

We can observe that for a variation of the fatigue parameters, the model is linear but not controllable.
$\Rightarrow$ Reduce the number of variables.

Using **sensitivity analysis**, we can keep the most sensitive fatigue variable :A. This is done using the notion of **Jacobi fields**.

Sensitivity analysis

- We extend the dynamics :

$$\begin{cases} \dot{x} = f(x, \lambda, u) \\ \dot{\lambda} = 0, \qquad \lambda : \text{parameters.} \end{cases} \implies \dot{\tilde{x}} = \tilde{f}(\tilde{x}, u)$$

- Hamiltonian lift : $\tilde{H}(\tilde{x}, \tilde{p}, u) = \tilde{p} \cdot \tilde{f}$.
- Projection $\pi_j : x \mapsto x_j$.

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Definition (Jacobi fields)

The Jacobi equation is

$$\dot{\delta} z(t) = \frac{\partial}{\partial z} \overrightarrow{H}(z(t), u(t))\, \delta z(t)$$

The **Jacobi fields associated to x_i-variation** $i = 1, \ldots, n$ *are the solutions* $J_i(t)$, $i = 1, \ldots, n$ *with* $J_i(0) = e_i$, $i = 1, \ldots, n$ *where* $(e_i)_i$ *is the* $\mathbb{R}^n \times \mathbb{R}^n$ *canonical basis.*

Sensitivity analysis

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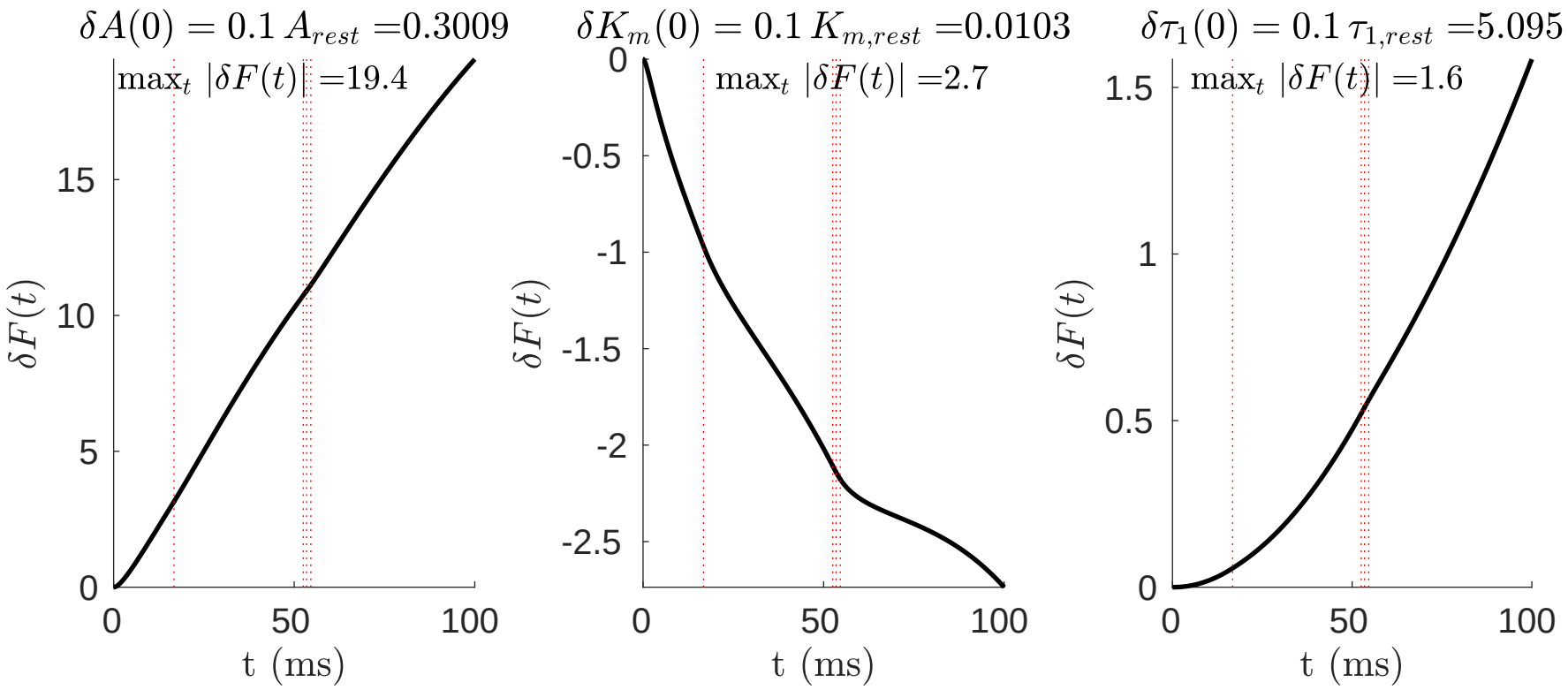
- Hamiltonian lift : $\tilde{H}(\tilde{x}, \tilde{p}, u) = \tilde{p} \cdot \tilde{f}$.
- Projection $\pi_j : x \mapsto x_j$.

Definition (Sensitivity)

The sensitivity of the **fatigue variables** $\boldsymbol{x_i}$, $i = 3, 4, 5$ **w.r.t. the force** *is defined by*

$$\max_{t \in [0,T]} |\Pi_F(J_i(t))|, \quad i = 3, 4, 5 \quad (n = 5)$$

where Π_F *is the projection* $z \to x_2$ *(on the force variable).*



Sensitivity analysis. Time evolution of the Jacobi fields component $\delta F(\cdot)$.

The fatigue variable A is the most relevant for the given reference trajectory

Methods in optimal control

- **PMP :** permanent case with a control $u(t)$.
- **Digital case** with a control $(t_1, \ldots, t_N, \eta_0, \ldots, \eta_N)$ in finite dimension, we extend the necessary conditions obtained in [Bourdin–Trélat][3].

Standard techniques

- L^∞ variations on η_i
- variations on t_i associated to the L^1-topology

Numerical methods

- **Direct methods** optimization algorithms adapted to the discretization and the sampling times
- optimization algorithms based on an approximation of the Input–Output mapping :

$$(t_1, \ldots, t_N, \eta_0, \ldots, \eta_N) \mapsto \tilde{F}$$

  the dynamics is not integrated and this allow much better robustness and performance.
  This can be used for the endurance & punch program.

3. Bourdin L., Trélat E., *Optimal sampled-data control, and generalizations on time scales*, Math. Cont. Related Fields 6, 53-94 (2016)

- Bakir T., Bonnard B., Gayrard S., Rouot J., *Finite Dimensional Approximation to Muscular Response in Force-Fatigue Dynamics using Functional Electrical Stimulation*, submitted
- Bakir T., Bonnard B., Bourdin L., Rouot J., *Pontryagin-Type Conditions for Optimal Muscular Force Response to Functional Electric Stimulations*, J Optim Theory Appl (2020) 184 :581.
- Bonnard B., Rouot J., *Geometric optimal techniques to control the muscular force response to functional electrical stimulation using a non-isometric force-fatigue model*, Journal of Geometric Mechanics, American Institute of Mathematical Sciences (AIMS), 2020
- Bourdin L., Trélat E., *Optimal sampled-data control, and generalizations on time scales*, Math. Cont. Related Fields 6, 53-94 (2016)
- Y. Wang and S. Boyd. Fast Model Predictive Control using Online Optimization. Control Systems Technology, IEEE Transactions on, 18(2) :267–278, 2010.

To discuss

$$\dot{q} = F_0 + uF_1$$

$$D_{L.A.} = \{F_0, F_1\}_{L.A.} : \text{ Algèbre de Lie engendrée}$$

$$D_{L.A.} = \cup_{k\geq 1} D_k, \quad D_1 = \text{span}\{F_0, F_1\} = \text{span}(D), \quad D_k = \text{span}\{D_{k-1} + [D, D_{k-1}]\}$$

Calcul d'une base de Hall de D pour engendrer $D_{L.A.}$.

Applications

- Singular trajectories
- Refine accessibility properties of the system using Baker–Campbell–Hausdorff
- Implement an smart electro-stimulator

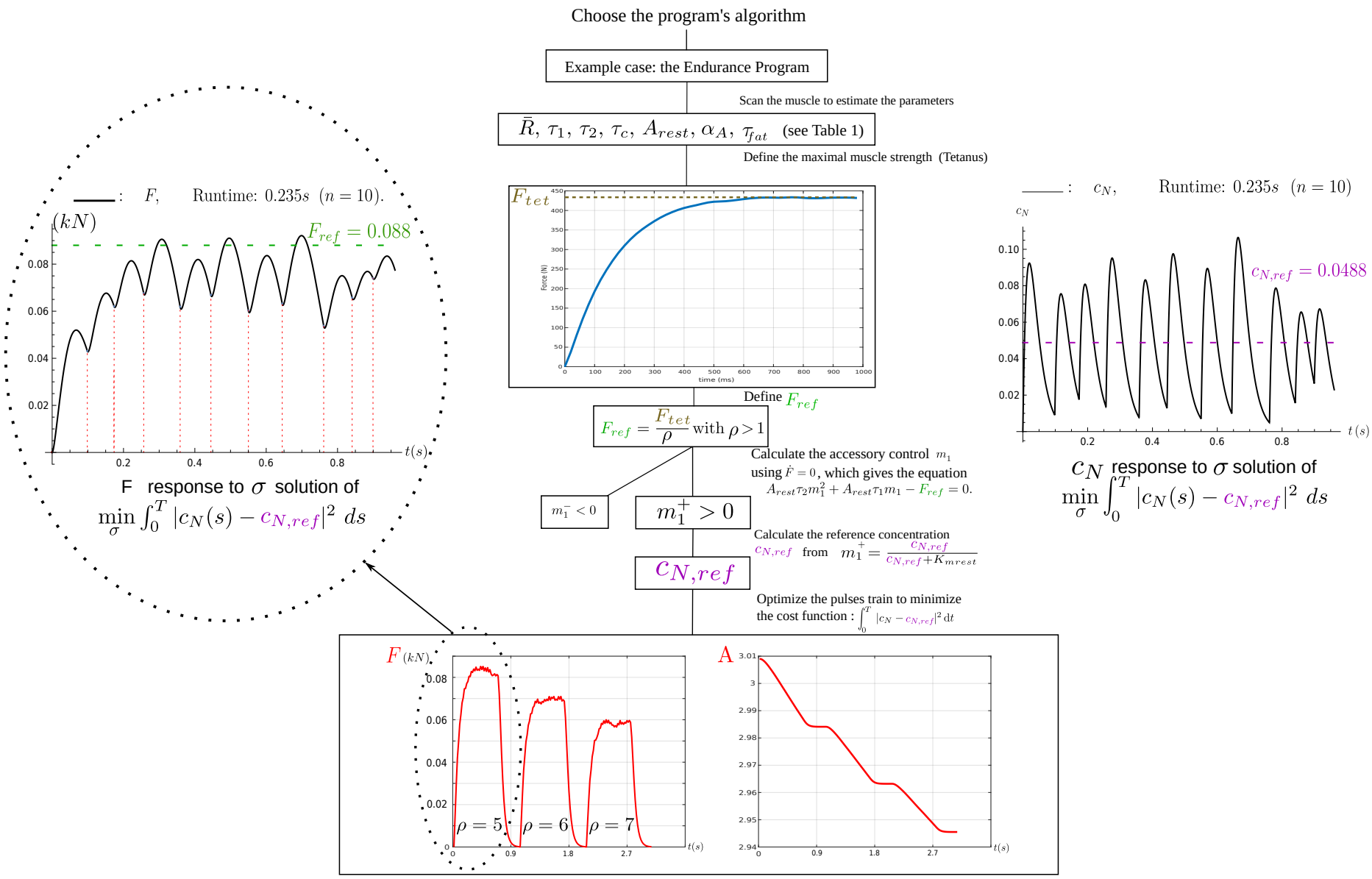


FIGURE – Endurance program for a smart electrostimulator.

Numerical methods for the electrostimulator

Recall the Ding et al. model

Ding et al isometric model

$$\dot{C}_N(t) = -\frac{C_N(t)}{\tau_c} + E_s(t),$$
$$\dot{F}(t) = -F(t)\ \boldsymbol{m_2(t)} + A\ \boldsymbol{m_1(t)},$$

where

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are the Michaelis–Menten–Hill functions.

Model with 6 parameters :

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"Sampled-data" optimal control

Mayer formulation

$$\min \quad \varphi(x(T), x^0(T)) := \tilde{\varphi}(x(T)) + \int_0^T f^0(x(s))\,\mathrm{d}s$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f_1(x(t)) + f_2(t) \sum_{i=1}^n R_i \eta_i \, e^{\frac{t_i}{\tau_c}} \, H(t - t_i), & \\ x(0) = x_0, & \\ \dot{x}^0(t) = f^0(x(t)), \quad x^0(0) = 0 & \\ (\eta_0, \eta_1, \ldots, \eta_N, t_1, \ldots, t_N) \in \mathbb{R}^{2n+1}, & \\ \eta_i \in [0,1], & \forall i = 0, \ldots, N, \\ t_0 = 0 < t_1 < t_2 < \ldots < t_N < T = t_{n+1}, & \\ t_{i+1} - t_i \geq \Delta, & \forall i = 0, \ldots, n, \end{cases}$$

where $x = (c_N, F, A, \tau_1, \tau_2, K_m)$, typically $\tilde{\varphi}(x) = F$ & $f^0 = 0$ or $\tilde{\varphi}(x) = 0$ & $f^0(x) = (F - F_{ref})^2$.

Indirect Methods

Necessary optimality conditions

Recap : Permanent control case (Pontryagin,1962) [4]

$$\min_{u \in \mathcal{U}} \varphi(x(T)),$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

$\mathcal{U}$: Admissible controls = bounded measurable mappings.

4. Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F. : *The mathematical theory of optimal processes*, John Wiley & Sons, Inc. (1962).

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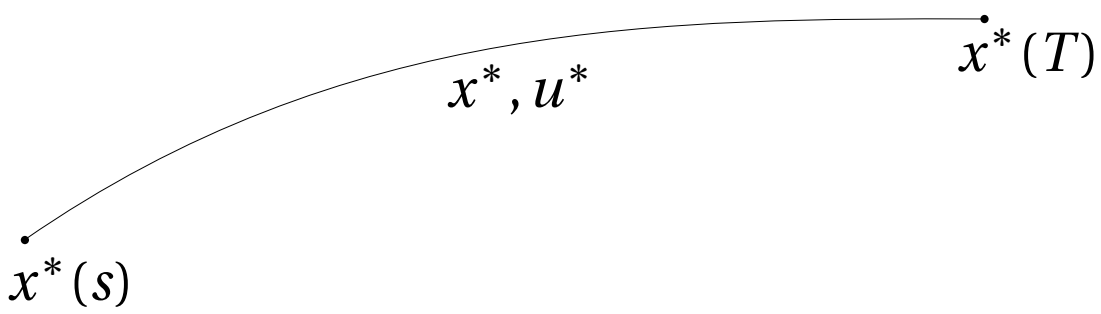
Let x^* a reference optimal trajectory associated to u^*.



4. Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F. : *The mathematical theory of optimal processes*, John Wiley & Sons, Inc. (1962).

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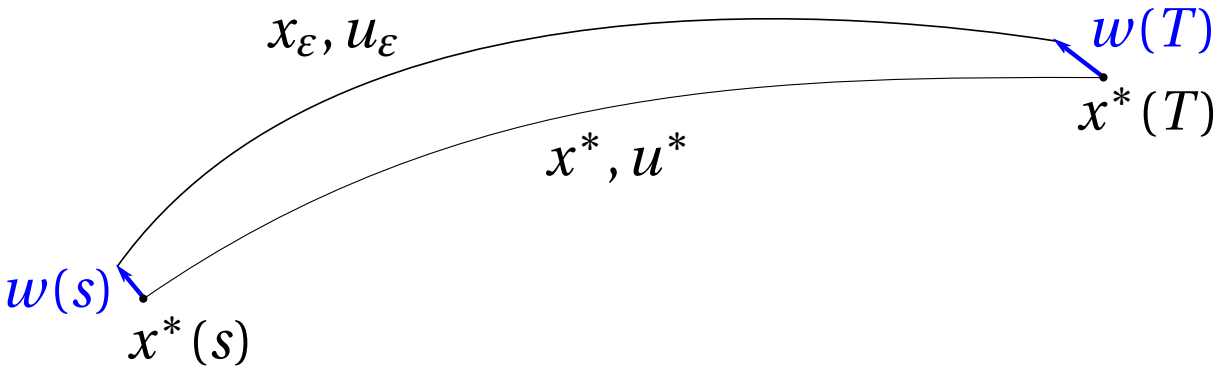

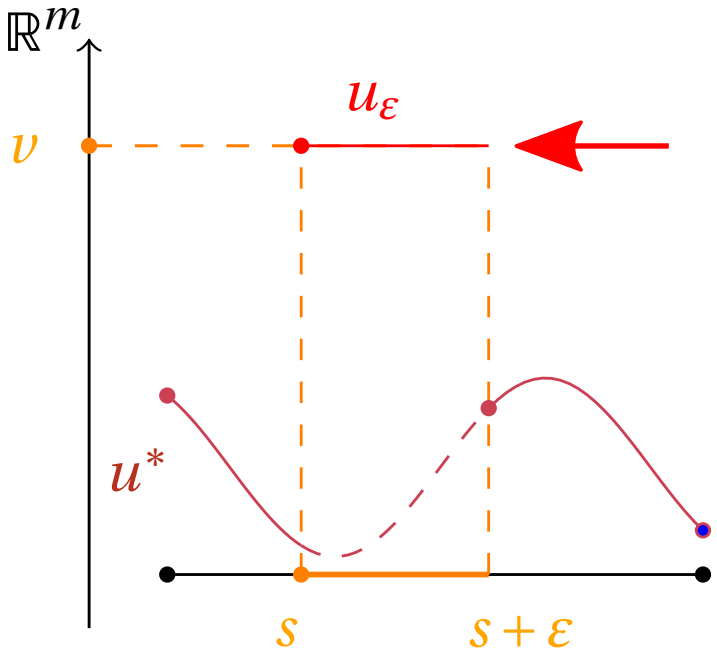

Necessary optimality conditions

- **L^1-perturbation :**

$$u_\varepsilon(t) := \begin{cases} v \in U \subset \mathbb{R}^m & \text{on } [s, s+\varepsilon[, \ (s \in [0, T[) \\ u^*(t) & \text{on } [s+\varepsilon, T[\end{cases}$$

- Corresponding variation vector w s.t. : $x(t, u_\varepsilon) = x(t, u^*) + \varepsilon\, w(t) + o(\varepsilon)$

$$\dot{w}(t) = \nabla_x f(x^*(t), u^*(t))\, w(t),$$
$$w(s) = f(x^*(s), v) - f(x^*(s), u^*(s))$$

Denote by $\Phi(\cdot, \cdot)$ the state-transition matrix of $\nabla_x f(x^*, u^*)$:

$$w(T) = \Phi(T, s)\, w(s).$$

From optimality of (x^*, u^*),

$$0 \quad \leq \quad \frac{\varphi(x_\varepsilon(T)) - \varphi(x^*(T))}{\varepsilon} \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad \left\langle \nabla\varphi(x^*(T)), w(T) \right\rangle$$

From optimality of (x^*, u^*),

$$0 \quad \leq \quad \frac{\varphi(x_\varepsilon(T)) - \varphi(x^*(T))}{\varepsilon} \quad \xrightarrow[\varepsilon \to 0]{} \quad \left\langle \nabla\varphi(x^*(T)), w(T) \right\rangle$$

Introducing the co-state vector $p(t)$ s.t. :

$$\begin{aligned} \dot{p}(t) &= -\nabla_x f(x^*(t), u^*(t))^\top p(t), \\ p(T) &= -\nabla\varphi(x^*(T)). \end{aligned}$$

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Using $w(T) = \Phi(T, s)\ w(s)$ and $p(s) = \Phi(T, s)^\top\ p(T)$ we finally get :

$$\boxed{\forall v \in U, \quad \left\langle p(s), f(x^*(s), v) - f(x^*(s), u^*(s)) \right\rangle \leq 0}$$

which is the so-called *maximization condition of the Pontryagin maximum principle.*

Necessary optimality conditions

Non Permanent control case (Bourdin, Trélat, 2016)[4]**.**

$$\min_{u \in \mathcal{U}} \varphi(x(T)),$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

$\mathcal{U}$: Admissible controls = piecewise constant mappings.
Let x^* a reference optimal trajectory associated to u^*.

4. Bourdin L., Trélat E., *Optimal sampled-data control, and generalizations on time scales*, Math. Cont. Related Fields 6, 53-94 (2016)

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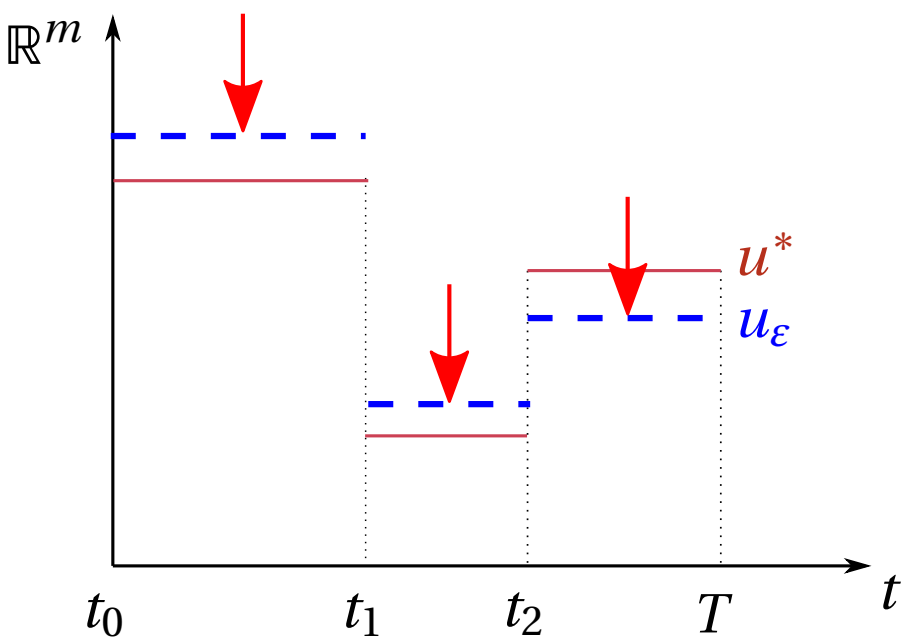

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Let x^* a reference optimal trajectory associated to u^*.

- **L^∞-perturbation :** $u_\varepsilon := u^* + \varepsilon(\xi - u^*)$ (ξ is valued in U has the same sampling times as u^*).
- This time, the corresponding variation vector w satisfies :

$$\dot{w} = \nabla_x f(x^*, u^*)\ w + \nabla_u f(x^*, u^*)\ (\xi - u^*),$$
$$w(0) = 0$$

hence,

$$w(T) = \int_0^T \Phi(T, s)\ \nabla_u f(x^*(s), u^*(s))\ (\xi(s) - u^*(s))\, \mathrm{d}s.$$

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$$w(T) = \int_0^T \Phi(T,s)\ \nabla_u f(x^*(s), u^*(s))\ (\xi(s) - u^*(s))\,\mathrm{d}s.$$

Using it, together with $0 \leq \langle \nabla\varphi(x^*(T)), w(T)\rangle$, yield

$$\int_0^T \left\langle p(s), \nabla_u f(x^*(s), u^*(s))\ (\xi(s) - u^*(s))\right\rangle \mathrm{d}s \leq 0.$$

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$$\int_0^T \left\langle p(s), \nabla_u f(x^*(s), u^*(s))\ (\xi(s) - u^*(s))\right\rangle \mathrm{d}s \leq 0.$$

Finally, taking $\xi = \boldsymbol{v} \in U$ over $[t_i^*, t_{i+1}^*[$ and $\xi(t) := u^*(t)$ elsewhere, we get

$$\boxed{\left\langle \int_{t_i^*}^{t_{i+1}^*} \nabla_u H(x^*(s), p(s), u_i^*)\,\mathrm{d}s\,,\ \boldsymbol{v} - u_i^* \right\rangle \leq 0,}$$

for all $\boldsymbol{v} \in U$ and all $i = 0, \ldots, N$, where u_i^* corresponds to the value of u^* over the interval $[t_i, t_{i+1}[$.

Remarks

- Same weaker maximization condition than the **discrete Pontryagin maximum principle** (Boltyanskii, 1978) [5]
- Another proof with different approach by Dmitruk and Kaganovich (2011) [6]
- Not directly applicable to our model since the sampling times are involved in the dynamics.

5. V.G. Boltyanskii, *Optimal control of discrete systems*, John Wiley & Sons, New York-Toronto, Ont., 1978.
6. A.V. Dmitruk, A.M. Kaganovich. *Maximum principle for optimal control problems with intermediate constraints*, Comput. Math. Model., 22(2) :180–215, 2011.

Application to the force-fatigue model

Theorem

If $(\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)$ **is optimal**, *then there exists* p *satisfying the co-state equation and the transversality condition.*

Application to the force-fatigue model

Theorem

If $(\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)$ **is optimal**, *then there exists* p *satisfying the co-state equation and the transversality condition.*
Moreover, the necessary conditions are :

(i) *the inequality*

$$\left(\int_{t_i^*}^{T} p_1(s) b(s)\, \mathrm{d}s\right) \tilde{\eta}_i \leq 0,$$

for all $i = 0, \ldots, n$ *and all admissible perturbation* $\tilde{\eta}_i$ *of* η_i^* *;*

(ii) *and the inequality*

$$\boldsymbol{NC_i} := \left(- p_1(t_i^*) b(t_i^*) G(t_{i-1}^*, t_i^*) \eta_i^* + b(-t_i^*) \eta_i^* \int_{t_i^*}^{T} p_1(s) b(s)\, \mathrm{d}s \right.$$
$$\left. + b(-t_i^*)(\bar{R} - 1) \eta_{i+1}^* \int_{t_{i+1}^*}^{T} p_1(s) b(s)\, \mathrm{d}s \right) \tilde{t}_i \leq 0,$$

for all $i = 1, \ldots, n$ *and all admissible perturbation* $\tilde{t}_i$ *of* t_i^* *.*

Numerical methods

1. *Open-loop control*.
   Direct methods : not based on necessary optimality conditions.
   inefficiency wrt to non-smooth dynamics.
   Indirect methods :
   - Shooting algorithm to solve the *boundary value problem* coming from the necessary conditions
   - Newton-like algorithm to find a zero of the shooting function
   - Direct method to give an initialization
   - Adapted integration scheme .

2. *Closed-loop control*. **Adaptive control algorithms** where the fatigue is estimated by a non-linear observer.

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2. *Closed-loop control*. **Adaptive control algorithms** where the fatigue is estimated by a non-linear observer.

$\Longrightarrow$ *Complementaries of the methods :*

open-loop : compute a pulses train to reach the maximal force (T $\sim$ 1s),

closed-loop : stabilization near a reference force with rest and stimulation periods (T $\gg$ 10s)

Direct method

Idea.

$$\begin{array}{c}\text{Sampled-data optimal}\\ \text{control problem}\end{array} \iff \begin{array}{c}\text{Finite-dimensional}\\ \text{optimization problem}\end{array}$$

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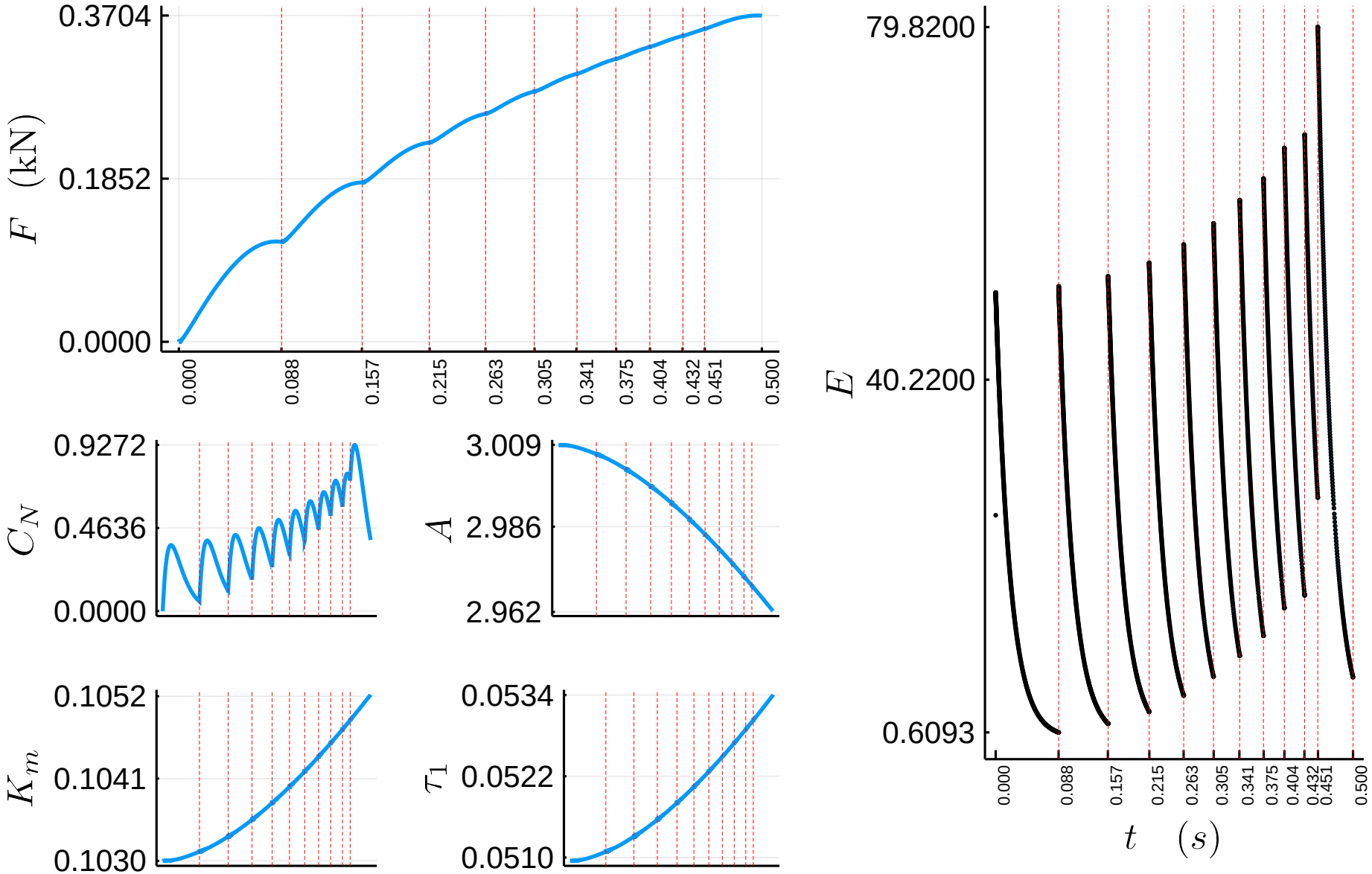
$$\text{Sampled-data optimal control problem} \iff \text{Finite-dimensional optimization problem}$$

Method. Transform the optimal control problem in a nonlinear finite-dimensional optimization problem (NLP) via discretization in time of the state.
t_i, $i = 1, \ldots, N$ are the optimization variables of the NLP.

Algorithms

- primal-dual interior point algorithm
- derivatives are computed by automatic differentiation.

$\Rightarrow$ **handle constraints on the state/control**, in general less precise than indirect methods, not that robust to initialization and efficient with our model.



Direct method : $\max_{t_i} F(T)$, $N = 10$, $10ms \leq t_{i+1} - t_i$, $i = 0, \ldots, N$.

Indirect method.

Exploit the **geometric structure** of the solutions via the necessary conditions.
Preliminary results : relax the inequalities in the optimality conditions to obtain a boundary value problem.

$\Longrightarrow$ Fast convergence and high accuracy/precision.

Multiple shooting method : $(n + 2nN + N)$ unknowns :

$$p(0), \quad Z_i = (x(t_i), p(t_i)),\ i = 1, \ldots, N, \quad \sigma = (t_1, \ldots, t_N).$$

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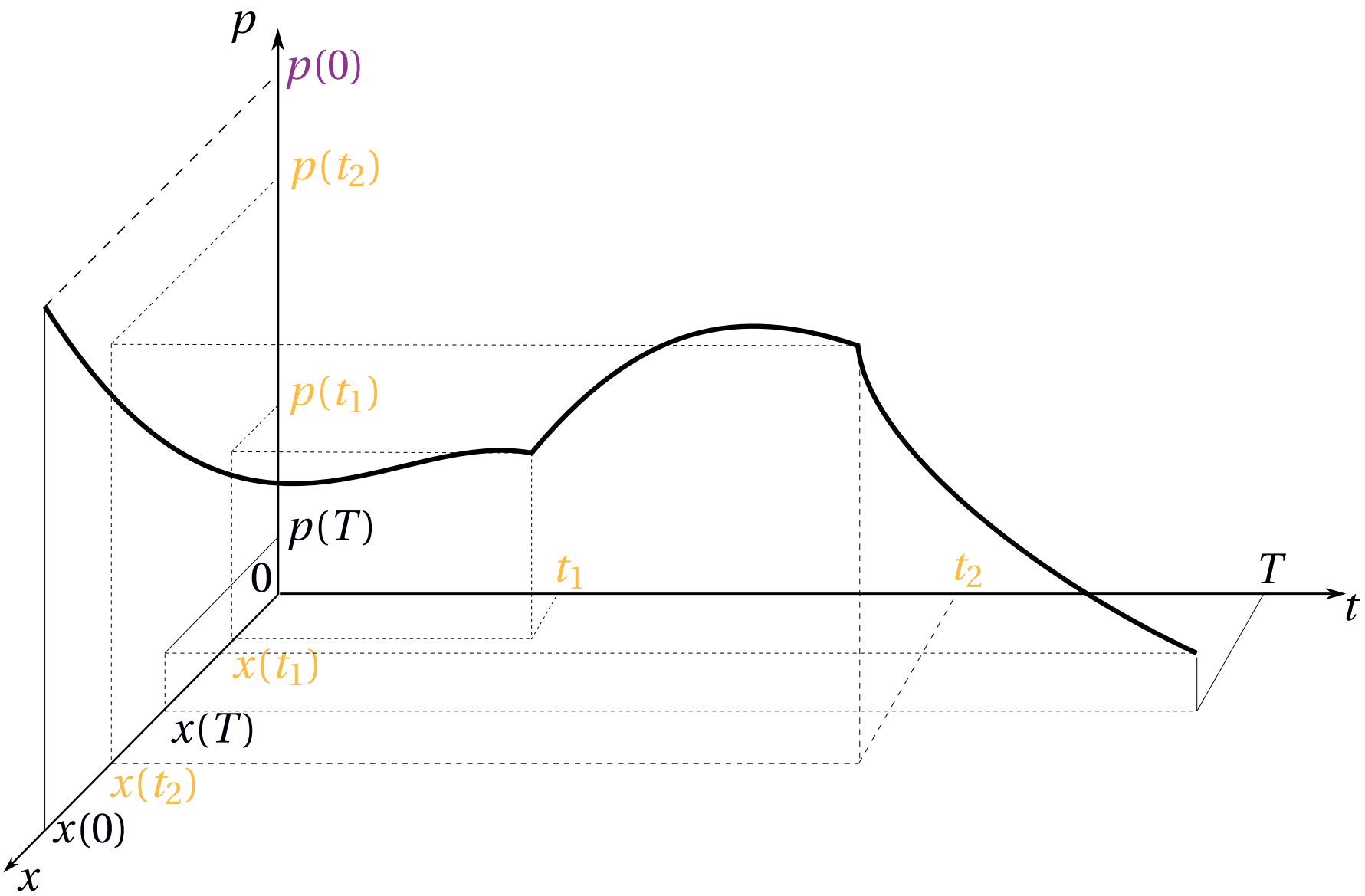

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$$p(0), \quad Z_i = (x(t_i), p(t_i)),\ i = 1, \ldots, N, \quad \sigma = (t_1, \ldots, t_N).$$

Shooting function. Find a zero of the function $\boldsymbol{S}(p_0, Z_1, \ldots, Z_N, \sigma)$ so that

- the initial condition $\boldsymbol{x(0) = x_0}$,
- the continuity conditions $\boldsymbol{Z_i^-} = \boldsymbol{Z_i^+}$, $i = 1, \ldots, N$,
- the necessary conditions $\boldsymbol{NC_i \leq 0}$ $i = 1, \ldots, N$,

are satisfied.

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are satisfied.

Shooting algorithm. Sensitive to initialization.

Initialization : compute a solution $(\tilde{x}, \tilde{u})$ with a direct method, by continuation or by approximation.

Starting from $(\tilde{x}(T), p(T))$ (where $p(T) = -\nabla\varphi(\tilde{x}(T))$ is known), **integrate backward the co-state dynamics** to obtain $p(0)$.

Tools : Julia's libraries :

- Extended precision for float (`ArbNumerics.jl`)
- **Stiff** numerical integrator (`DifferentialEquations.jl`)

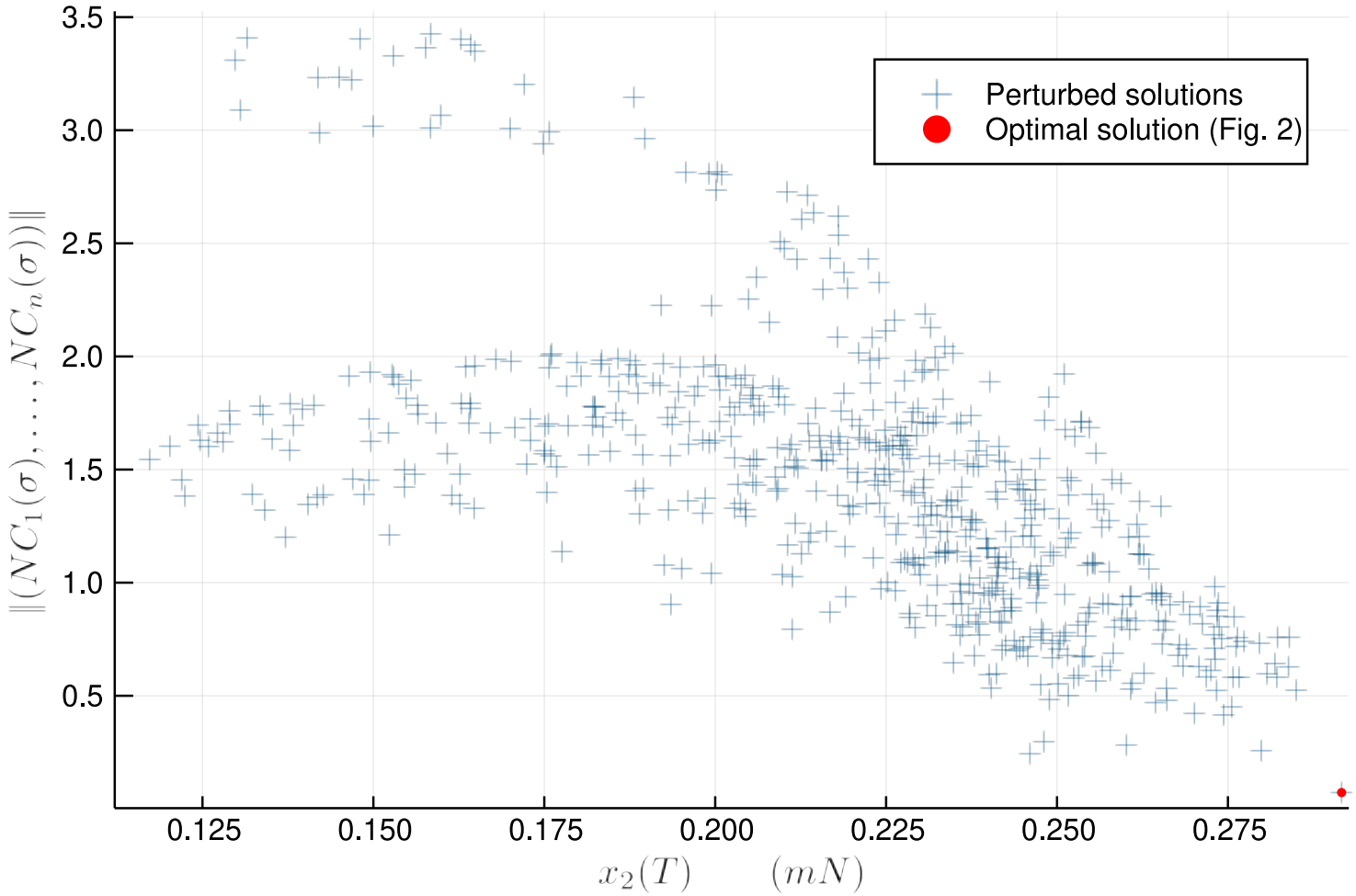


FIGURE – Quality of the optimal solution computed with multiple shooting with respect to its perturbations. The quality is measured from the necessary conditions and the value of the cost.

Closed-loop algorithm

- **Sensitivity analysis :** select the relevant fatigue variable for estimation
- **Detectability :** construct an observer to estimate the chosen fatigue variable
- Adaptive control algorithm (MPC) based on the observer

Conclusion on these methods

- We recover the solution of the direct method with the indirect methods (using the initialization provided by the direct method)
- No implementation of the control constraints for the indirect methods
- Both methods are inefficient (in terms of robustness and time complexity) for the **design of an electrostimulator**[7] ($n \simeq 8$).

Alternatives for real time computation

- Approximation of the Input–Output mapping :

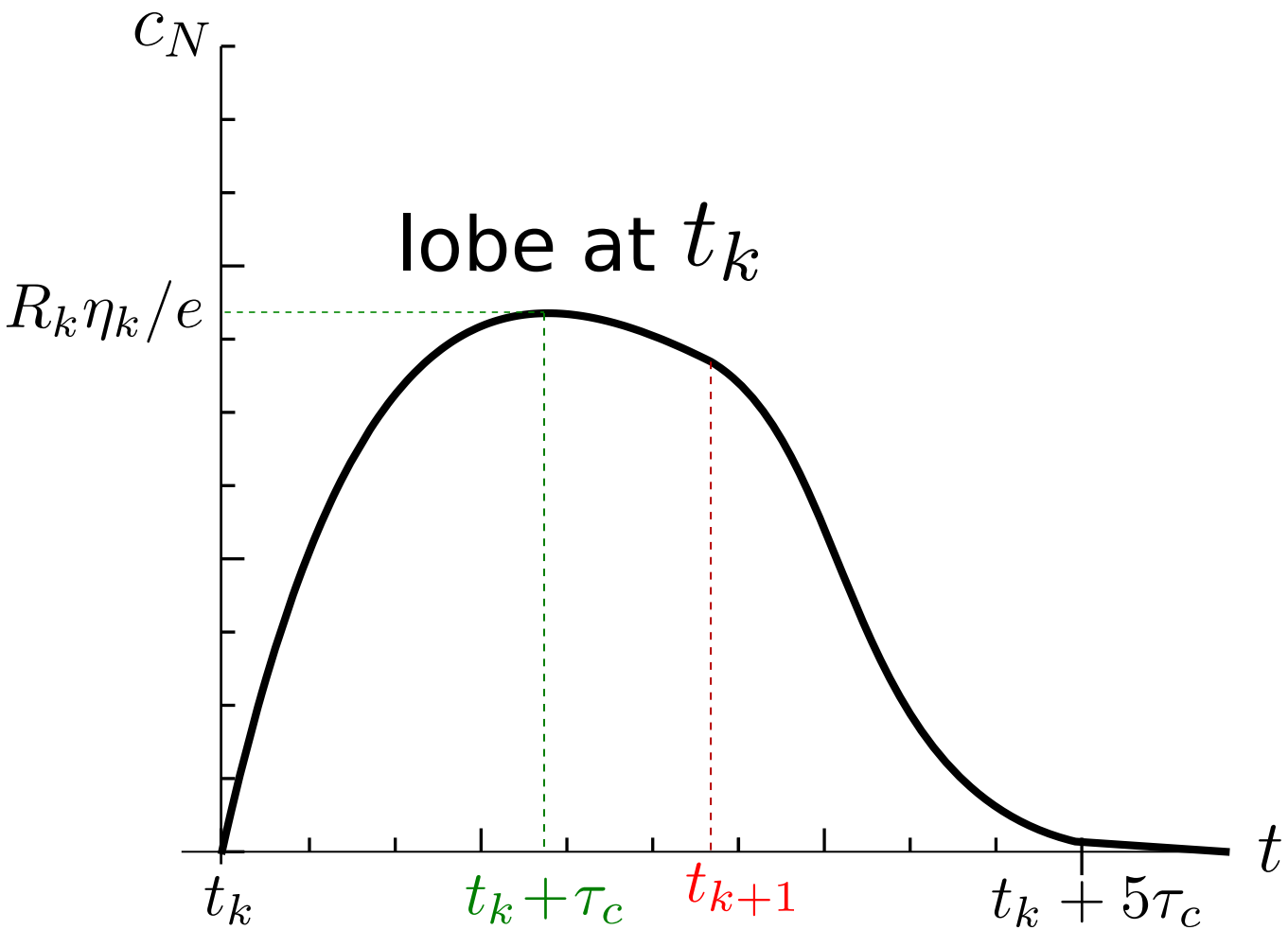
$$(t_1, \ldots, t_N, \eta_0, \ldots, \eta_N) \mapsto \tilde{F}$$

- Try adapted dicretization for non smooth dynamics → "event driven methods".

7. On-going collaboration with Segula Technology (CIFRE & Peps AMIES)

Definition

A lobe at t_k is the representative curve of the function
$\ell_k : \mathbb{R} \ni t \mapsto R_k \eta_k \frac{t-t_k}{\tau_c} e^{-(t-t_k)/\tau_c} H(t-t_k)$.



Lemma

For $t \neq t_i$, $c_N(t)$ is in the polynomial-exponential category. This category is stable with respect to derivation and integration.

Property

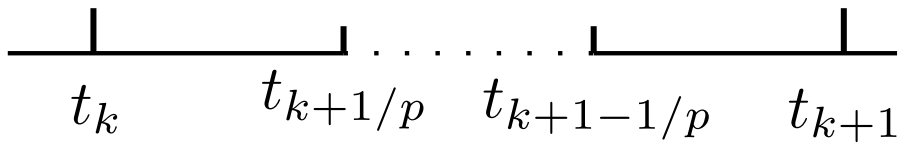
Let $k \in \{0, \ldots, n\}$.

- *If $t_{k+1} < t_k + 2\tau_c$ then $m_{1|[t_k,t_{k+1}]}$ is concave and $m_{2|[t_k,t_{k+1}]}$ is convex.*
-

$$\operatorname{argmax} c_{N|[t_k,t_{k+1}]} = \operatorname{argmax} m_{1|[t_k,t_{k+1}]} = \operatorname{argmax} m_{2|[t_k,t_{k+1}]}$$
$$= \begin{cases} \tau_c & \text{si } k = 0 \\ \tau_c + \dfrac{\sum_{i=1}^{k} R_i \eta_i \, t_i \, e^{t_i/\tau_c}}{\sum_{i=0}^{k} R_i \eta_i \, e^{t_i/\tau_c}} & \text{sinon .} \end{cases}$$

Consider a finer partition of $(t_k)_{1\leq k\leq n}$

$$t_{k+j/p},\ j=0,\dots,p-1,\ k=0,\dots,n$$



Polynomial approximation of m_1 and m_2 on each interval $[t_{k+j/p}, t_{k+(j+1)/p}]$.

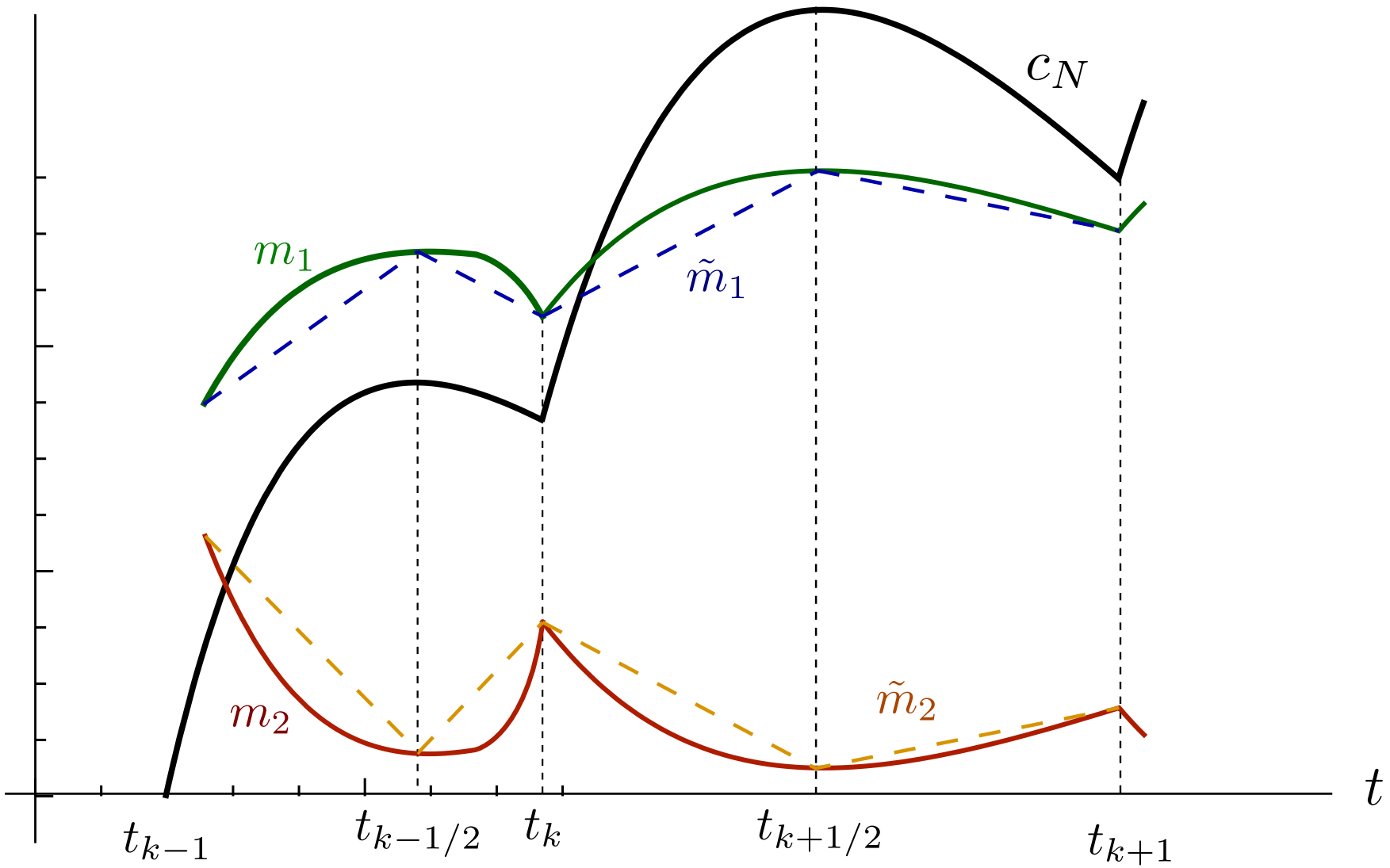
Example

Take $p=2$, $t_{k+1/2} = \underset{t\in[t_k,t_{k+1}]}{\text{argmax}}\ c_N(t)$, $k=0,\dots,n$ *and for* $i=1,2$, $j=0,1$ *:*

$$\tilde{m}_i(t) = a_{ij,k}(t-t_{k+j/2}) + b_{ij,k},\ k=0,\dots,n.$$

Imposing $\tilde{m}_i(t_{k+j/2}) = m_i(t_{k+j/2})$ *and* $\tilde{m}_i(t_{k+(j+1)/2}) = m_i(t_{k+(j+1)/2})$, *we get :*

$$a_{ij,k} = \frac{m_i(t_{k+(j+1)/2}) - m_i(t_{k+j/2})}{t_{k+(j+1)/2} - t_{k+j/2}},\ b_{ij,k} = m_i(t_{k+j/2}).$$


c_N
m_1
$\tilde{m}_1$
m_2
$\tilde{m}_2$
t
t_{k-1}
$t_{k-1/2}$
t_k
$t_{k+1/2}$
t_{k+1}

Definition

As an approximation of F for the Ding et al model without fatigue, take

$$\tilde{F}(t) = A \int_0^t \tilde{M}(t)\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,\mathrm{d}s,$$

where $\tilde{M}(t) = \exp\left(-\int_0^t \tilde{m}_2(s)\,\mathrm{d}s\right)$.

Decomposing the sum over each interval $[t_{k_t+j_t/p}, t_{k_t+(j_t+1)/p}]$, $k_t = 0,\ldots,n$, $j_t = 0,\ldots,p-1$, we have :

$$\begin{aligned}\tilde{F}(t)/A = &\sum_{i=0}^{k_t-1}\sum_{j=0}^{p-1}\int_{t_{i+j/p}}^{t_{i+(j+1)/p}} \tilde{M}(t)\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,\mathrm{d}s \\ &+ \sum_{j=0}^{j_t-1}\int_{t_{k_t+j/p}}^{t_{k_t+(j+1)/p}} \tilde{M}(t)\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,\mathrm{d}s \\ &+ \int_{t_{k_t+j_t/p}}^{t} \tilde{M}(t)\tilde{M}^{-1}(s)\,\tilde{m}_1(s)\,\mathrm{d}s, \qquad (2)\end{aligned}$$

Application

Take

$$\tilde{m}_1(t) = \begin{cases} m_1(t_{k+1/2}) & \text{if } t \in [t_k, t_{k+1/2}] \\ a_{1j,k}(t - t_{k+1}) + b_{1j,k}, & \text{if } t \in [t_{k+1/2}, t_{k+1}] \end{cases},$$

$$\tilde{m}_2(t) = \begin{cases} \frac{m_2(t_k)+m_2(t_{k+1/2})}{2} & \text{if } t \in [t_k, t_{k+1/2}] \\ \frac{m_2(t_{k+1/2})+m_2(t_{k+1})}{2}, & \text{if } t \in [t_{k+1/2}, t_{k+1}] \end{cases},$$

where $t_{k+1/2} = \underset{t \in [t_k, t_{k+1}]}{\text{argmax}}\, c_N(t)$, $a_{1j,k} = (m_1(t_{k+1}) - m_1(t_{k+1/2}))/(t_{k+1} - t_{k+1/2})$ and $b_{1j,k} = m_1(t_{k+1})$.

Then $\tilde{F}$ has a closed form and its expression can be generated *offline* with **formal computation**.

$$\sigma = (t_1, \ldots, t_n, \eta_1, \ldots, \eta_n)$$

Finite dimensional optimization problem

$$\begin{aligned} \min_{\sigma} \quad & \Theta(\sigma) \\ & \Im(\sigma) \leq 0, \end{aligned}$$

where $\Im(\sigma) = (\Xi_1(\sigma), \ldots, \Xi_{3n+5}(\sigma))$ with :

$$\begin{aligned} &\Xi_i(\sigma^*) = t^*_{i-1} - t^*_i + I_{\min},\ i = 1, \ldots n, \\ &\Xi_{n+1}(\sigma^*) = t^*_n - T, \\ &\Xi_{n+2+i}(\sigma^*) = -\eta^*_i,\ i = 0, \ldots n+1,\ \Xi_{2n+4+i}(\sigma^*) = \eta^*_i - 1,\ i = 0, \ldots n+1. \end{aligned}$$

Algorithm : Primal-dual interior point method allows real time computation (standard computer).

$\Theta(\sigma) := -\tilde{F}(T)$

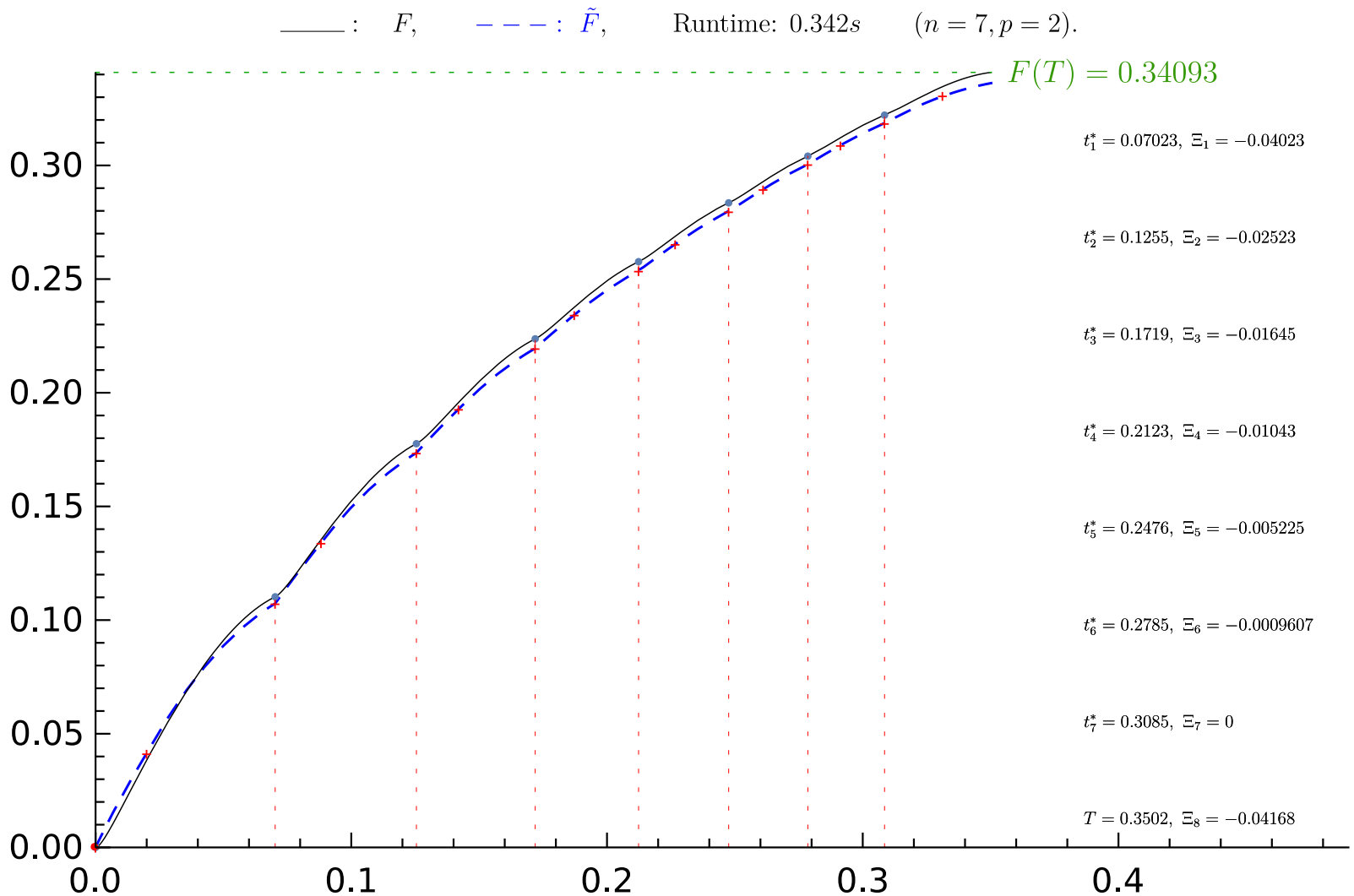


FIGURE – Optimal solution computed with the approximation. Integration of the true F with respect to this solution

$$\Theta(\sigma) = \sum_{k=0}^{n} \left(\tilde{F}(t_{k+1}) - F_{ref}\right)^2 (t_{k+1} - t_k).$$

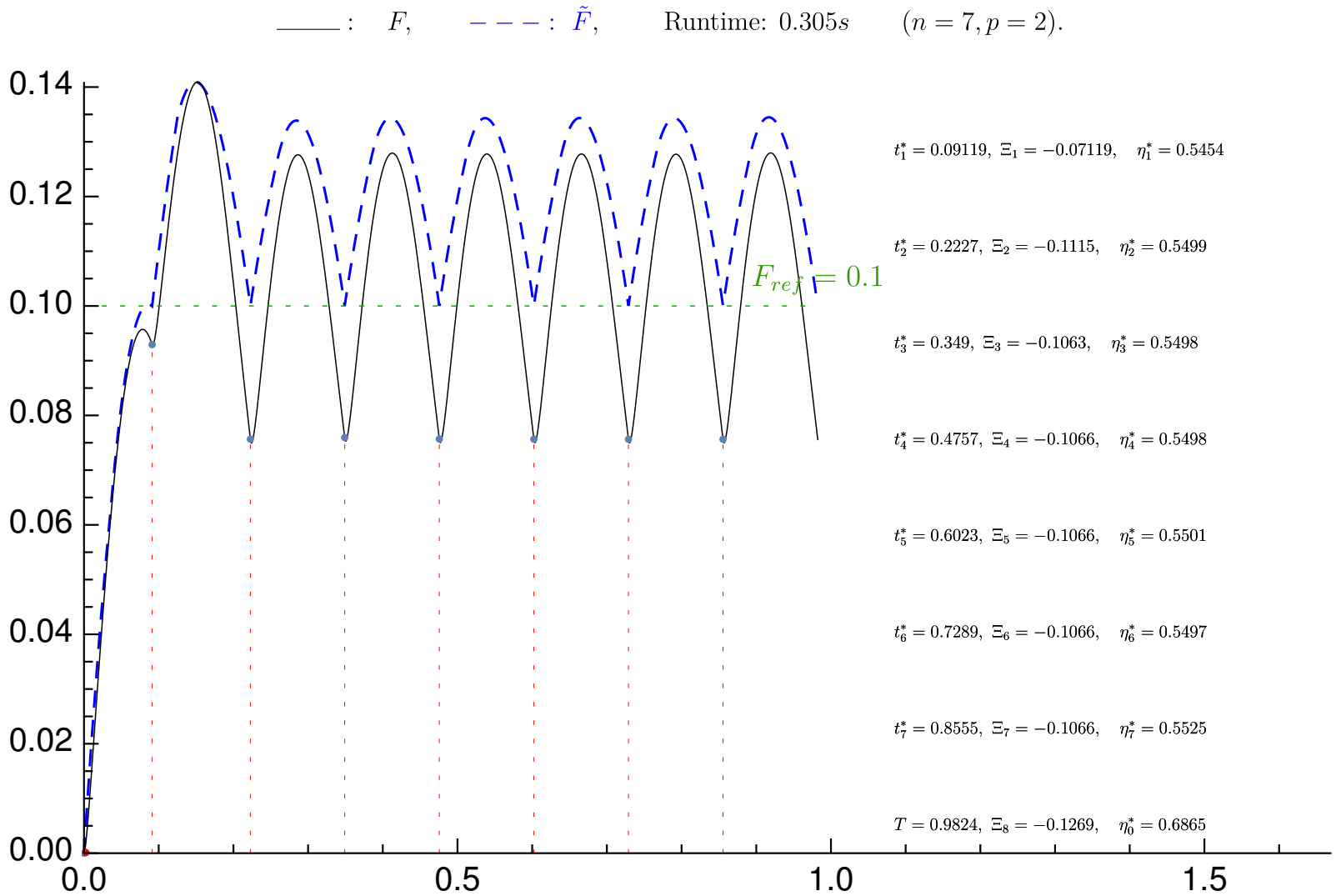


FIGURE – Optimal solution computed with the approximation. Integration of the true F with respect to this solution

Remarks

- Mathematical structure of m_1, m_2.
- We can tune $\tilde{m}_1, \tilde{m}_2$ to obtain upper (or lower) approximation of F
- Using naive discretization and offline computation of $\tilde{F}$ leads to inefficient method.
- Error estimate $k = 0, \ldots, n$:
  $|F(t_k) - \tilde{F}(t_k)| / A \leq \int_0^{t_k} |m_1(s) - \tilde{m}_1(s)| \, \mathrm{d}s + t_k \int_0^{t_k} |m_2(s) - \tilde{m}_2(s)| \, \mathrm{d}s.$

Conclusion

- Compare with fast MPC [8]
- Non smooth integrator (“event driven methods”)
- Fatigue model
- Nonisometric case

8. Y. Wang and S. Boyd. Fast Model Predictive Control using Online Optimization. Control Systems Technology, IEEE Transactions on, 18(2) :267–278, 2010.

With explicit Euler scheme :

$$\tilde{F}(t_{i+(j+1)/p}) = \tilde{F}(t_{i+j/p})\, c_{i,j} + A\, d_{i,j},$$

where $c_{ij} = (1 - h_{i,j}\, m_2(t_{i+j/p}))$, $h_{i,j} = t_{i+(j+1)/p} - t_{i+j/p}$ and $d_{ij} = m_1(t_{i+j/p})$ for $i = 0, \ldots, n$ and $j = 0, \ldots, p-1$.

Then

$$\tilde{F}(t_{k_t+j_t/p})/A = \sum_{j=0}^{j_t-1} h_{k_t,j}\, d_{k_t,j} \prod_{j'=j+1}^{j_t-1} c_{k_t,j'} + \tag{3}$$
$$\sum_{i=0}^{k_t-1} \sum_{j=0}^{p-1} h_{i,j}\, d_{i,j} \left(\prod_{j'=0}^{p-1} \prod_{i'=i+1}^{k_t-1} c_{i',j'} \prod_{j'=j+1}^{p-1} c_{i,j'} \prod_{j'=0}^{j_t-1} c_{k_t,j'} \right).$$