Synthesis of observers for infinite-dimensional vibrating systems with application to a rotating body-beam system

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Outlines

1. **Luenberger observer : from finite dimension to infinite dimension**
2. **Exact observability and exponential convergence of observer**
3. **Application to a rotating body-beam system**
4. **Conclusions**

Finite dimensional Luenberger-like observer

Harmonic oscillator : $A^t = -A$ on the state space $\mathbb{R}^n$

$$\begin{cases} \dot{x} = Ax, \quad x(0) = x_0 \\ y = Cx \end{cases}$$

where (C, A) is observable.

Luenberger-like observer

$$\dot{\hat{x}} = A\hat{x} - \kappa C^T(C\hat{x} - y), \quad \hat{x}(0) = \hat{x}_0, \;\; \kappa > 0.$$

Error system of the observer

$$\dot{\varepsilon} = (A - \kappa C^T C)\varepsilon, \quad \varepsilon(0) = \varepsilon_0 \tag{1}$$

where $\varepsilon = \hat{x} - x$.

If (C, A) is observable, then the observer is convergent : $\hat{x}(t) \to x(t)$ as $t \to \infty \; \forall \; \kappa > 0$.

Linear system on an infinite-dimensional Hilbert space X

$$\begin{cases} \dot{w}(t) = Aw(t) + Bu(t), \\ w(0) = w_0, \ y(t) = Cw(t) \end{cases}$$

where $w(t) \in X$, $u(t) \in U$, $y(t) \in Y$, and A is the generator of a C_0 unitary group $\mathbb{T}_t := e^{At}$ on X.
To consider the boundary control and the boundary observation, we need two more auxiliary spaces X_1 ans X_{-1}. Take a $\beta \in \rho(A) \cap \mathbb{R}$. Then $X_1 = \mathcal{D}(A)$, $\|z\|_1 = \|(\beta I - A)z\|$, and X_{-1} is the completed space of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. Hence $X_1 \subset X \subset X_{-1}$.
Assume that $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$.

Infinite-dimensional Luenberger observer

Infinite-dimensional Luenberger-like observer *(formally)*

$$\begin{cases} \dot{\hat{w}}(t) = A\hat{w}(t) - \kappa\, C^* \left(C\hat{w}(t) - y(t) \right) + Bu(t), \quad \kappa > 0, \\ \hat{w}(0) = \hat{w}_0, \end{cases}$$

Error system of the observer :

$$\dot{\varepsilon} = (A - \kappa C^T C)\varepsilon := A^\kappa \varepsilon, \quad \varepsilon(0) = \varepsilon_0$$

Exact observability and regularity

Couple (A, B) is *admissible* if, for some $\tau > 0$,

$$\int_0^{\tau} \mathbb{T}_{\tau-t} Bu(t)dt \in X \quad \forall\, u \in L^2_{loc}(\mathbb{R}^+; U).$$

Couple (C, A) is *admissible* if, for some $\tau > 0$, $\exists\ K_\tau > 0$ s.t.

$$\int_0^{\tau} \| C\mathbb{T}_t w_0\|^2_Y dt \leq K_\tau \|w_0\|^2_X, \quad \forall\, \underline{w_0 \in \mathcal{D}(A)}.$$

Couple (C, A) is *exactly observable* over τ if it is admissible and there exists some positive constants $\tau > 0$ and $k_\tau > 0$ such that

$$k_\tau \|w_0\|^2_X \leq \int_0^{\tau} \|C\mathbb{T}_t w_0\|^2_Y dt, \quad \forall\, w_0 \in \mathcal{D}(A).$$

The *state* of the system is uniquely determined from the observation of the *input* and *output* on a sufficiently long time interval $[0, \tau]$.

Exact observability and regularity

The linear system is *regular* if

i) (A, B) and (C, A) are admissible ;

ii) the system is *well-posed* such that there exists some $D \in \mathcal{L}(U, Y)$ satisfying the following

$$\lim_{s \to +\infty} \mathbb{G}(s)u = Du \ \ \forall \ u \in U$$

where $\mathbb{G}(s)$ is the transfer function analytic and bounded in the right-half complex plane $\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \Re e(s) > \alpha\}$ for some $\alpha \in \mathbb{R}$, and C_Λ is the Λ-*extension* of C defined by

$$C_\Lambda w = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} w, \quad \forall w \in \mathcal{D}(C_\Lambda)$$

with $\mathcal{D}(C_\Lambda)$ consisting of all $w \in X$ for which the limit exists.

Exponential convergence of the observer

If the system is regular, then $C_\Lambda(sI - A)^{-1}B$ makes sense $\forall\ s \in \rho(A)$, and

$$\mathbb{G}(s)u = C_\Lambda(sI - A)^{-1}Bu + Du.$$

Assume that $D = 0$.

Exponential convergence of the observer

Theorem

Let A be the generator of a C_0 unitary group on X. Let both (A, C^*, C) and (A, B, C) be *regular* such that (C, A) is *exactly observable*. Then there exist positive constants $K_{\min}$ and $K_{\max}$ with $0 \le K_{\min} \le K_{\max}$ such that

- the observer **converges** *exponentially* if $0 < \kappa < 1/K_{\max}$
- the observer **diverges** *exponentially* for $\kappa > 1/K_{\min}$ being admissible, where

$$\begin{aligned} K_{\max} &= \sup_{\substack{f \in \mathrm{Ran}(C_\Lambda) \\ |f|=1}} \lim_{\substack{\beta \in \mathbb{R}^+ \\ \beta \to +\infty}} \beta \|(\beta - A)^{-1} C^* f\|_X^2, \\ K_{\min} &= \inf_{\substack{f \in \mathrm{Ran}(C_\Lambda) \\ |f|=1}} \lim_{\substack{\beta \in \mathbb{R}^+ \\ \beta \to +\infty}} \beta \|(\beta - A)^{-1} C^* f\|_X^2. \end{aligned}$$

Transport equation

$$\begin{cases} w_t(x,t) = w_x(x,t), \\ w(0,t) = w(1,t), \\ y(t) = w(0,t) \end{cases}$$

state space $X = L^2(0,1)$, observation space $Y = \mathbb{R}$.

Luenberger observer

$$\begin{cases} \hat{w}_t(x,t) = \hat{w}_x(x,t), \\ \hat{w}(0,t) = \hat{w}(1,t) + \kappa\left(\hat{w}(0,t) - y(t)\right). \end{cases}$$

- $K_{\max} = K_{\min} = 1/2$: the observer is *exponentially* **convergent** for $0 < \kappa < 2$, and *exponentially* **divergent** for $\kappa > 2$.
- If the state space has *finite* dimension, it is always true that $K_{\max} = K_{\min} = 0$.

Exponential convergence of the observer

Lemma

$$\langle A^\kappa \varepsilon, \varepsilon\rangle_X \leq -\kappa(1-\kappa K_{\max})|C_\Lambda \varepsilon|_Y^2, \qquad \forall \varepsilon \in \mathcal{D}(A^\kappa),\ \forall 0<\kappa<1/K_{\max}$$

$$\langle A^\kappa \varepsilon, \varepsilon\rangle_X \geq \kappa(\kappa K_{\min}-1)|C_\Lambda \varepsilon|_Y^2, \qquad \forall \varepsilon \in \mathcal{D}(A^\kappa),\ \forall \kappa>1/K_{\min}.$$

Proof : For each $\varepsilon \in \mathcal{D}(A^\kappa)$,

$$\langle A^\kappa \varepsilon, \varepsilon\rangle_X = -\kappa |C_\Lambda \varepsilon|_Y^2 + \lim_{\beta\to\infty} \kappa^2 \beta\, |R(\beta, A) C^* C_\Lambda \varepsilon|_X^2 \,.$$

Proof of the main Theorem : $0<\kappa<1/K_{\max}$,

$$\frac{d|\varepsilon|_X^2}{dt} = 2\,\langle A^\kappa \varepsilon, \varepsilon\rangle_X \,,$$

$$|\varepsilon(T)|_X^2 \leq |\varepsilon_0|_X^2 - 2\kappa(1-\kappa K_{\max}) \int_0^T |C_\Lambda \varepsilon(\tau)|_Y^2 d\tau$$

exact observability $\int_0^T |C_\Lambda \varepsilon(t)|_Y^2 dt \geq K|\varepsilon_0|_X^2 \Rightarrow |\varepsilon(t)|_X^2 \leq Me^{-rt}|\varepsilon_0|_X^2.$

Exponential convergence of the observer

$\kappa > 1/K_{\min}$,

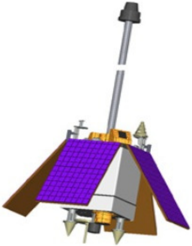
$$|\varepsilon(T)|_X^2 \geq |\varepsilon_0|_X^2 + 2\kappa(\kappa K_{\min} - 1)\int_0^T |C_\Lambda \varepsilon(\tau)|_Y^2 d\tau,$$

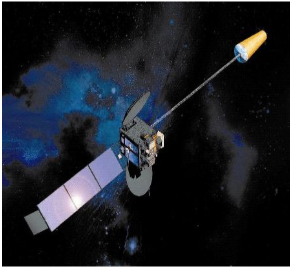
exact observability $\Rightarrow |\varepsilon(t)|_X^2 \geq \bar{M}e^{-\bar{r}t}|\varepsilon_0|_X^2$.

Motivation

Physical models

satellites with flexible appendixes



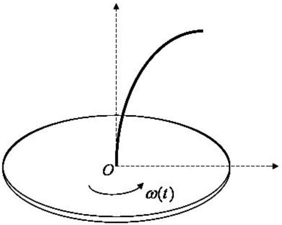


- THEMIS *(1250 kg)* - NASA, '07 (Geoscience)
- MirsSat 1 *(100 kg)* - Egypt, '07 (Scientific observation)
- MTSAT 2 *(126 kg)* - Japan, '06 (Meteorology)

Rotating body-beam system

simplified model

- rotating disc
- flexible Euler-Bernoulli beam



stabilization + observability :

interesting info for control of mechanical systems

(*example : satellites with solar panels, robot with flexible joint...*)

Rotating body-beam system

System governed by PDE and ODE

- Lagrangian formulation

$$\begin{cases} \rho w_{tt}(x,t) + EIw_{xxxx}(x,t) + \rho B w_t(x,t) = \rho\omega^2(t) w(x,t), \\ \Gamma(t) = \dfrac{d}{dt}\left(\omega(t)(I_d + \displaystyle\int_0^L w^2(x,t)dx)\right) \quad \textit{(Newton's law)} \end{cases}$$

- BC : $w(0,t) = w_x(0,t) = w_{xx}(L,t) = w_{xxx}(L,t) = 0$

$w(x,t)$ = *deformation of the beam*
$\Gamma(t)$ = *torque control*
B = *friction operator*
L = *length of the beam* $[m]$
ρ = *mass density of the beam* $[kg/m]$
E = *elastic module* $[N/m^2]$, I = *moment of inertia* $[m^4]$
I_d = *moment of inertia of the disc* $[kg \cdot m^2]$
$\omega(t)$ = *angular velocity of the disc* $[rad/s]$

stabilization : find $\Gamma(t)$ such that $(w(\cdot,t),\omega(t)) \xrightarrow[t\to\infty]{} (0,\omega_*)$?

- with friction $B \neq 0$, stabilization control law by Xu & Baillieul 1993 ;
- without friction $(B = 0)$ + boundary control Xu & Sallet 1994 ;
- without boundary control and without friction state feedback control law by Coron & d'Andéa-Novel 1998

$$\begin{aligned}\dot{\omega} = \gamma &= \mathcal{F}(\omega, w, w_t) \\ &= -\left(\omega + \omega_* - \sigma\left(\int_0^1 w w_t dx\right)\right) \int_0^1 w w_t dx - C_2\left(\omega - \omega_*\right) \\ &\quad -\sigma'\left(\int_0^1 w w_t\right) \int_0^1 \left(w_t^2 - w_{xx}^2 + \omega^2 w^2\right) dx\end{aligned}$$

with $C_2 > 0$ and $\sigma \in \mathcal{C}^1(\mathbb{R}) : (2\omega_* - \sigma(s))s\sigma(s) > 0 \ \forall\, s \in \mathbb{R}$.

The closed-loop system is *strongly GAS* around $(0, \omega_*)$ with $0 < \omega_* < \omega_{crit}$.

The points of Coron & d'Andréa-Novel's control law :

the state space has *infinite* dimension ;
torque control applied on the rigid body $\Rightarrow$ easy to be applied in practice ;
the infinite-dimensional state needs to be accessible, while the observation space is of *finite* dimension.

solution : estimate online the state through an observer

$$\hat{w}(x,t) \xrightarrow[t\to\infty]{} w(x,t) \ \forall \ \hat{w}_0$$

and replace $\Gamma(w, w_t, \omega)$ by $\Gamma(\hat{w}, \hat{w}_t, \omega)$.

Observer and convergence

simplified model

$$
\begin{cases}
w_{tt}(x,t) + w_{xxxx}(x,t) = \omega_*^2 w(x,t), & \textit{(second order)} \\
w(0,t) = w_x(0,t) = 0, \\
w_{xx}(1,t) = w_{xxx}(1,t) = 0, \\
w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \\
y(t) = w_{xx}(0,t). \qquad \textit{(single boundary measurement)}
\end{cases}
$$

state space $X = H_L^2 \times L^2(0,1)$, $H_L^2 = \{f \in H^2(0,1); f(0) = f_x(0) = 0\}$

$$\langle f, g\rangle_X = \int_0^1 f_{1xx}g_{1xx} + f_2 g_2 - \omega_*^2 f_1 g_1 \, dx$$

Luenberger-like observer *(first order, explicit) :*

$$\begin{cases} \hat{w}_{1t}(x,t) = \hat{w}_2(x,t) - \kappa F(x)\left(\hat{w}_{1xx}(0,t) - y(t)\right), \\ \hat{w}_{2t}(x,t) = -\hat{w}_{1xxxx}(x,t) + \omega_*^2 \hat{w}_1(x,t), \\ \hat{w}_1(0,t) = \hat{w}_{1x}(0,t) = 0, \\ \hat{w}_{1xx}(1,t) = \hat{w}_{1xxx}(1,t) = 0, \\ \hat{w}(x,0) = \hat{w}_0(x),\ \hat{w}_t(x,0) = \hat{w}_1(x) \end{cases}$$

with $\kappa > 0$ and F the unique solution of

$$\begin{cases} F^{''''}(x) - \omega_*^2 F(x) = 0, \\ F(0) = F^{''}(1) = F^{'''}(1) = 0, \\ F^{'}(0) = 1. \end{cases}$$

$$\mathcal{D}(A^\kappa) = \{(f_1\ f_2)^T \in (H^4(0,1)\cap H_L^2)\times H^2(0,1);\ f_{2x}(0) = \kappa f_{1xx}(0),\ f_2(0) = f_{1xx}(1) = f_{1xxx}(1) = 0\}$$

$$A^\kappa \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\kappa F(x)\Psi & I \\ -\partial_x^4 + \omega_*^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A^\kappa)$$

where $\Psi f = f_{xx}(0)$.

Error system

$$\begin{cases} \dot{\varepsilon}(t) = A^\kappa \varepsilon(t) \\ \varepsilon(0) = \varepsilon_0. \end{cases}$$

- *Lyapunov* stable if $|\omega_*| < \omega_{crit}$!
- *exponential* stability ?

Exponential convergence

Theorem

Assume that $|\omega_*| < \omega_{crit}$. The observer is *exponentially convergent* for *every positive gain* of correction $\kappa > 0$, i.e., given $\kappa > 0$ there exist $M > 0$, $\alpha > 0$ such that

$$\left\| \begin{pmatrix} \hat{w}_1(\cdot, t) \\ \hat{w}_2(\cdot, t) \end{pmatrix} - \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} \right\|_X \leq M e^{-\alpha t} \left\| \begin{pmatrix} \hat{w}_0 \\ \hat{w}_1 \end{pmatrix} - \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right\|_X$$

Let $A^\kappa = A - \kappa C_\Lambda^* C$ that has compact resolvents.

Let $\alpha = \sup_{\lambda \in \sigma_p(A^\kappa)} Re(\lambda)$.

Proof :

- $\forall \kappa \geq 0$, A^κ is dissipative and the generator of C_0 semi-group of contractions on X : Lumer-Phillips
- exact observability of the observation system (A, C) on X

Lemma

Let $y = w_{1xx}(0, t)$. Then the observation system is *exactly observable* $\forall\, \omega_* \in \mathbb{R}$, i.e., $\exists T > 0, K > 0$ such that

$$K\|w_0\|_X^2 \leq \int_0^T w_{1xx}(0, t)dt \leq K^{-1}\|w_0\|_X^2 \qquad \forall\, w_0 \in X.$$

Decay rate assignment

Theorem

The eigenvalues $(\lambda_n)_{n\geq 0}$ of the operator A^κ are eventually with algebraic multiplicity 1 and the real part tends to -2κ when $n \to \infty$. The sequence of the corresponding generalized eigenfunctions forms a Riesz basis on X. Moreover the exponential decay rate of the C_0-semigroup is determined by the spectre of its generator.

Accelerate the convergence rate of the observer

pole placement or spectrum assignment for the error system

second step in observer design : assign the convergence rate to a fixed value

Decay rate assignment

Corollary

On replacing $\kappa[F(x)\ 0]^T$ by $\kappa[F(x)\ 0]^T + B(x)$ with some appropriate $\kappa > 0$ and $B(x)$, the convergence rate of the observer is tuned to as quick as we like.

example

1 eigenvalue :

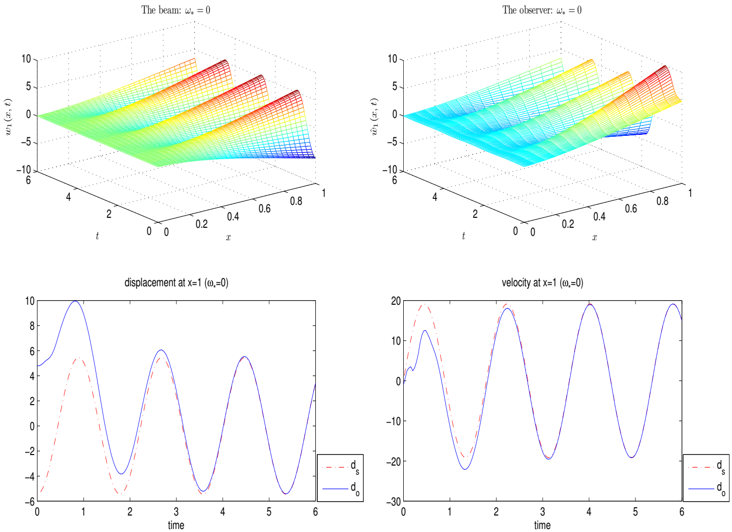
$$B = \frac{\tilde{\lambda}_1 - \lambda_1}{u_{1xx}(0)} e_1$$

allows to shift λ_1 to $\tilde{\lambda}_1$

where $e_1 = (u_1\ v_1)^T$ is the eigenvector associated to λ_1.

Simulation

$\omega = 3$ [rad/s]



Simulation

$\omega(t) = 3 \cdot sin^2(t)$ [rad/s]

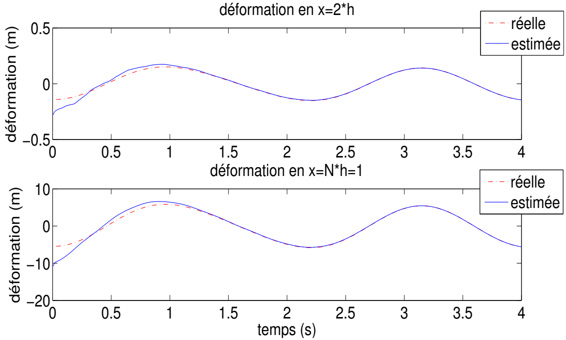


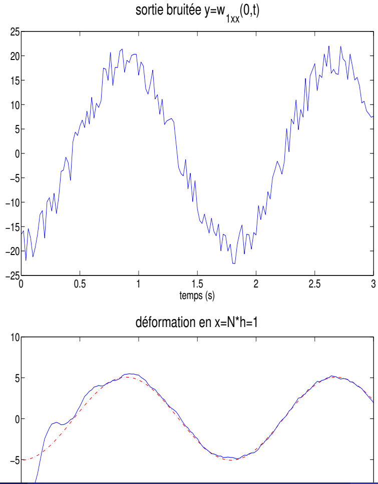
Figure – deformation history at $x = 0.1L$ (upper) and $x = L$ (lower) for the observation system (dashed) and the observer (solid). $\kappa = 1$.

Simulation

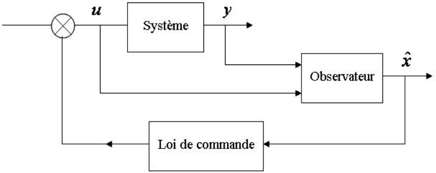
robustness

perturbation : $\tilde{y}(t) = w_{1xx}(0,t) + b(t),\ b = $ white noise

(amplitude $=$ 20% w.r.t the measurement, mean$=$0)



- non-linear system described by *(PDE-ODE)*
- unbounded observation operator
- non local control



Conjecture

Let $\bar{\omega} \in]-\sqrt{l_1}, \sqrt{l_1}[\backslash\{0\}$. $(0, 0, \bar{\omega})$ (*resp.* $(0, 0, 0)$) is an equilibrium *locally asymptotically* stable on $X \times X \times \mathbb{R}$ for closed-loop system coupled with the feedback γ (*resp.* $\tilde{\gamma}$).

Conclusions

- **Summary**
  - exponentially convergent Luenberger-like observers
  - application to a rotating body-beam system
  - decay rate assignment
- **Perspectives**
  - global stabilization

References I

J.M. Coron & B. d'Andrea-Novel.
Stabilization of a rotating body-beam without damping.
IEEE Trans. Auto. Control, 43 :608-618, 1998.

J.P. Gauthier & I. Kupka.
A separation principle for bilinear systems with dissipative drifts.
IEEE Trans. Auto. Control, 37 :1970-1974, 1992.

C.Z. Xu & J. Baillieul.
Stabilizability and stabilization of a rotating body beam system with torque control.
IEEE Trans. Auto. Control, 38 :1754-1765, 1993.

Thank you for your attention