Sparse Optimization

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CIMI-ANITI School on Optimisation, Toulouse, 2021

(Based on other tutorials with Stephen Wright, University of Wisconsin, USA)

Outline

- **Part 1:** Sparse Optimization and Applications
- **Part 2:** First-Order Methods
- **Part 3:** Augmented Lagrangian Methods

Part 1: Sparse Optimization and Applications

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Inference via Optimization

Many **inference** problems are formulated as **optimization** problems:

- image reconstruction, restoration, denoising, segmentation, ...
- machine learning
  - ✓ supervised learning
  - ✓ unsupervised learning
  - ✓ many other tasks
- statistical inference
- decision theory
- system identification
- ...

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Standard formulation:

- observed data: y
- unknown mathematical object (signal, image, vector, matrix,...): x
- inference criterion:

$$\widehat{x} \in \arg\min_{x} g(x, y)$$

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Question 1: how to build g? Where does it come from?

Answer: from the application domain (machine learning, signal processing, inverse problems, system identification, statistics, computer vision, bioinformatics,...);
... examples ahead.

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Question 2: how to solve the optimization problem?

Answer: the focus of this tutorial.

Regularized Optimization

Inference criterion: $\widehat{x} \in \arg\min_{x} g(x, y)$

Typical structure of g: $g(x, y) = h(x, y) + \tau\psi(x)$

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- $\tau \geq 0$: the **regularization parameter/constant**.

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- $\tau \geq 0$: the **regularization parameter/constant**.
- Since y is fixed, we often write simply $f(x) = h(x, y)$,

$$\min_x f(x) + \tau\psi(x)$$

Probabilistic/Bayesian Interpretations

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- Log-posterior: $\log\ p(x|y) = K(y) - h(x, y) - \tau\psi(x) = K(y) - g(x, y)$
- $\widehat{x}$ is a maximum a posteriori (MAP) estimate.

Regularizers

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Typically, the unknown is a vector $x \in \mathbb{R}^n$

or a matrix $x \in \mathbb{R}^{n \times m}$

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Common regularizers impose/encourage one (or a combination of) the following characteristics:

- small norm (vector or matrix)
- sparsity (few nonzeros)
- specific nonzero patterns (*e.g.*, group/tree structure)
- low-rank (matrix)
- smoothness or piece-wise smoothness
- ...

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Under mild conditions, these are all *equivalent* (in a precise sense).

Which one is more convenient is problem-dependent.

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- Non-trivial cases: resort to optimization and regularization.
- Quadratic (Euclidean norm) losses and regularizers have a long, rich history: Gauss, Legendre, Wiener, Moore-Penrose, Tikhonov, ...

Norms: A Quick Review

Consider some real vector space $\mathcal{V}$, for example, $\mathbb{R}^n$ or $\mathbb{R}^{n \times n}$, ...

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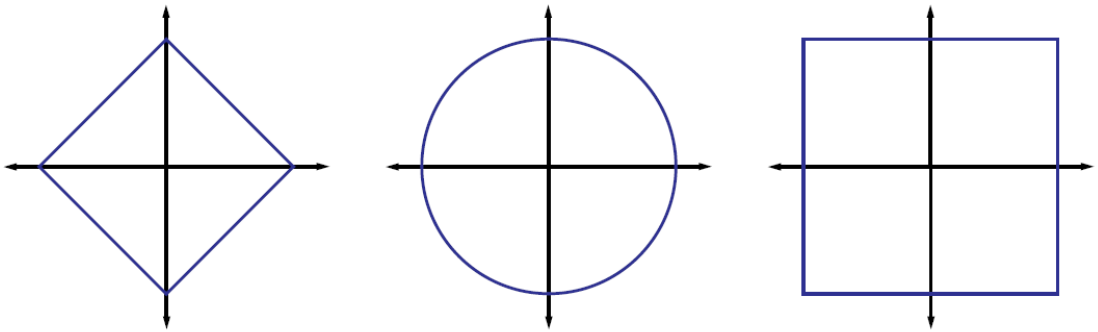
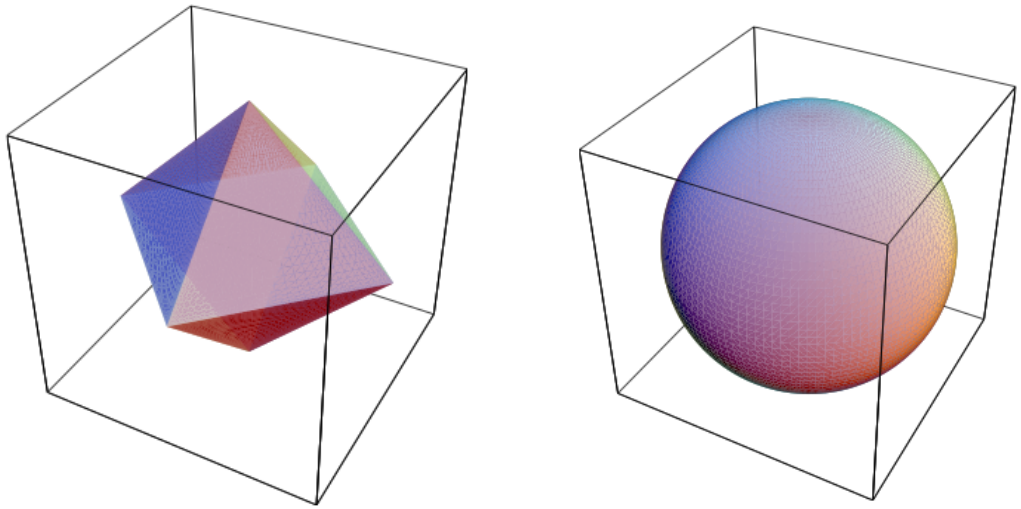
Also important (but not a norm): $\|x\|_0 = \lim_{p\to 0} \|x\|_p^p = |\{i : x_i \neq 0\}|$

Norm balls

Radius r ball in ℓ_p norm: $B_p(r) = \{x \in \mathbb{R}^n : \|x\|_p \leq r\}$





$p = 1$ $p = 2$

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- Can we hope to recover x? Yes! ...if x is sparse enough ($\|x\|_0 < k$) and A satisfies some **conditions**, using

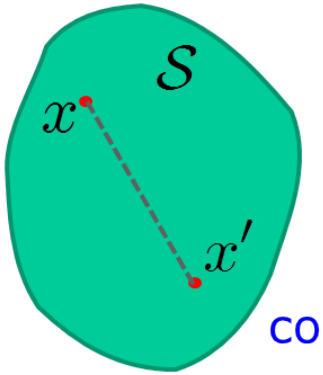
$$\begin{array}{rl} \widehat{x} = & \arg\min\limits_{x} \|x\|_0 \\ & \text{s.t. } Ax = y \end{array}$$

Several proofs, under different conditions (more later).

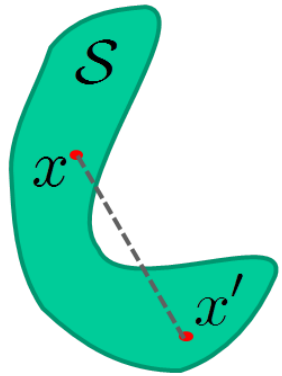
But, this is a hard problem! ℓ_0 "norm" is not **convex**.

Convex and strictly convex sets

$\mathcal{S}$ is convex if $x, x' \in \mathcal{S} \Rightarrow \forall \lambda \in [0,1], \quad \lambda x + (1-\lambda)x' \in \mathcal{S}$

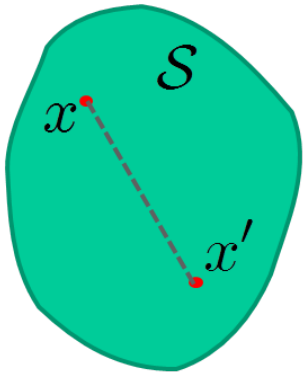


convex

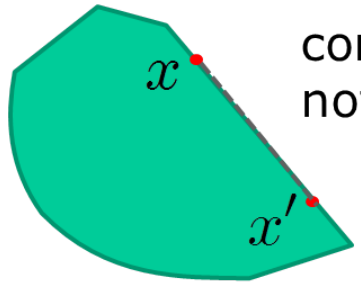


non-convex

$\mathcal{S}$ is strictly convex if $x, x' \in \mathcal{S} \Rightarrow \forall \lambda \in (0,1), \quad \lambda x + (1-\lambda)x' \in \text{int}(\mathcal{S})$



strictly convex



convex, but not strictly

Review of Basics: Convex Functions

Extended real valued function: $f : \mathbb{R}^N \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

Domain: $\mathrm{dom}(f) = \{x : f(x) \neq +\infty\}$

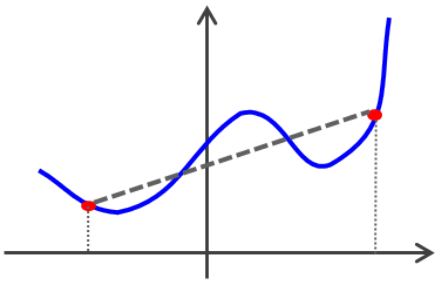
f is proper if $\mathrm{dom}(f) \neq \emptyset$

f is convex if

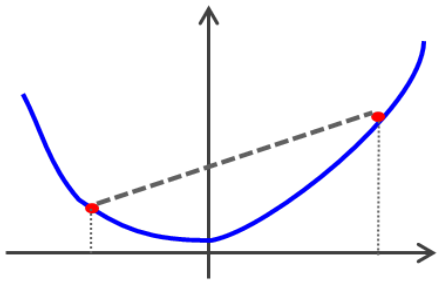
$$\forall \lambda \in [0,1], x, x' \in \mathrm{dom}(f) \quad f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$

f is strictly convex if

$$\forall \lambda \in (0,1), x, x' \in \mathrm{dom}(f) \quad f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$$



non-convex



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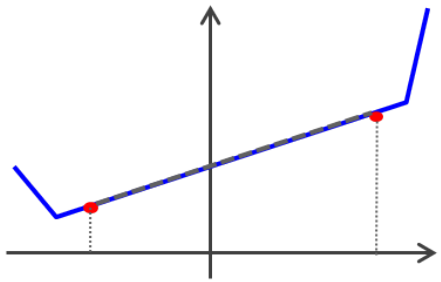


convex, not strictly

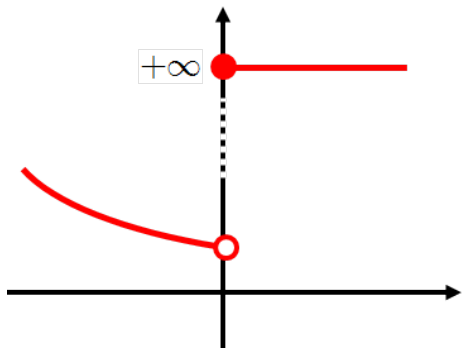
Lower Semi-Continuity: Why Is It Important?

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is lower semi-continuous (l.s.c.) if

$$\liminf_{x \to x_0} \geq f(x_0), \quad \text{for any } x_0 \in \text{dom}(f)$$

or, equivalently, $\{x : f(x) \leq \alpha\}$ is a closed set, for any $\alpha \in \mathbb{R}$

$$f(x) = \begin{cases} e^{-x}, & \text{if } x < 0 \\ +\infty, & \text{if } x \geq 0 \end{cases}$$



$\text{dom}(f) =]-\infty, 0[, \quad \arg\min_x f(x) = \emptyset$

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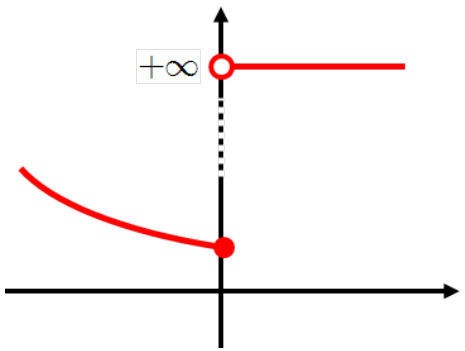

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Unless stated otherwise, we only consider l.s.c. functions.

Coercivity, Convexity, and Minima

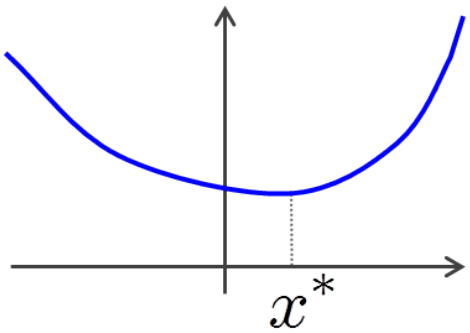
$$f : \mathbb{R}^N \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$$

f is coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$

if f is coercive, then $G \equiv \arg\min_x f(x)$ is a non-empty set

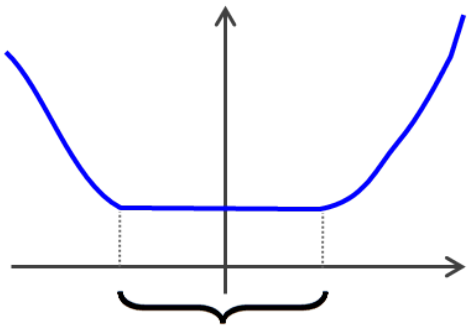
if f is strictly convex, then G has at most one element

coercive and strictly convex



$G = \{x^*\}$

coercive, not strictly convex



G

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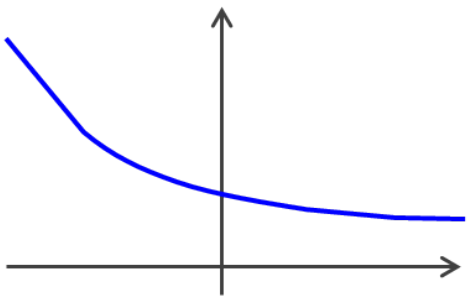
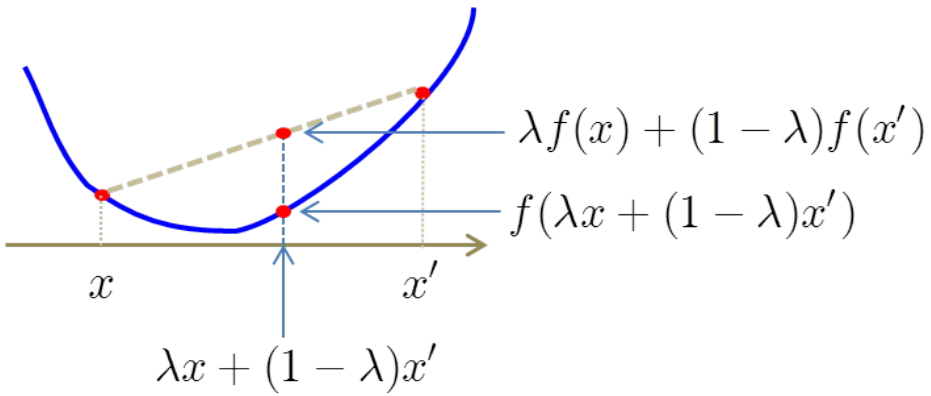


$G = \emptyset$

Another Important Concept: Strong Convexity

Recall the definition of convex function: $\forall \lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$



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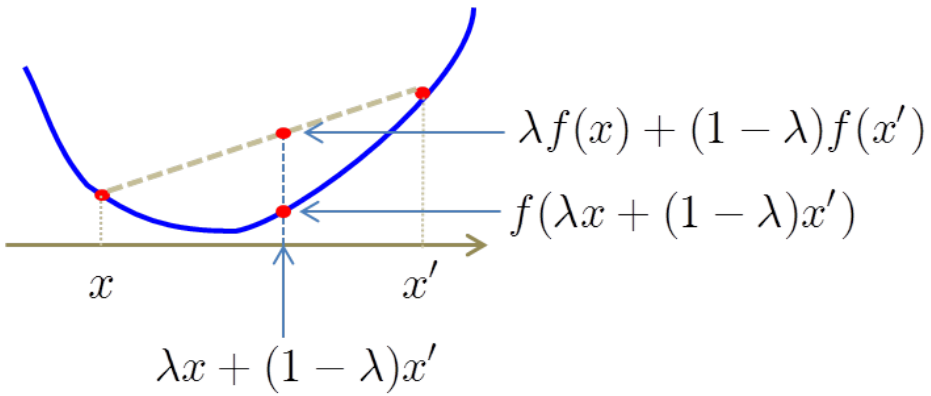

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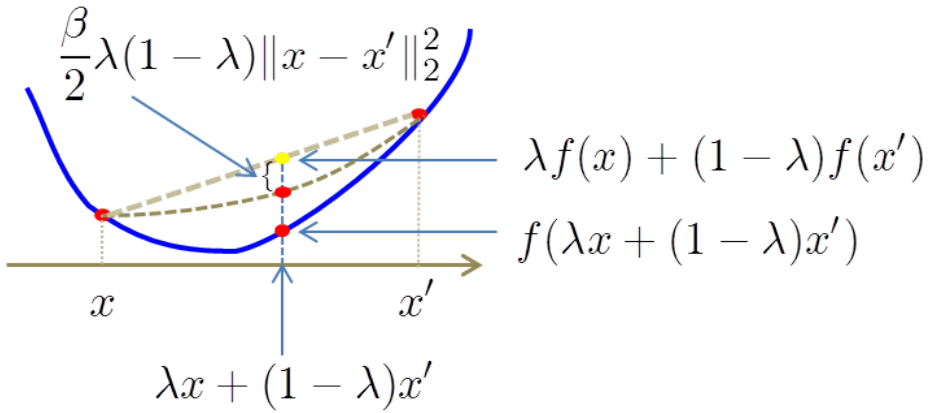

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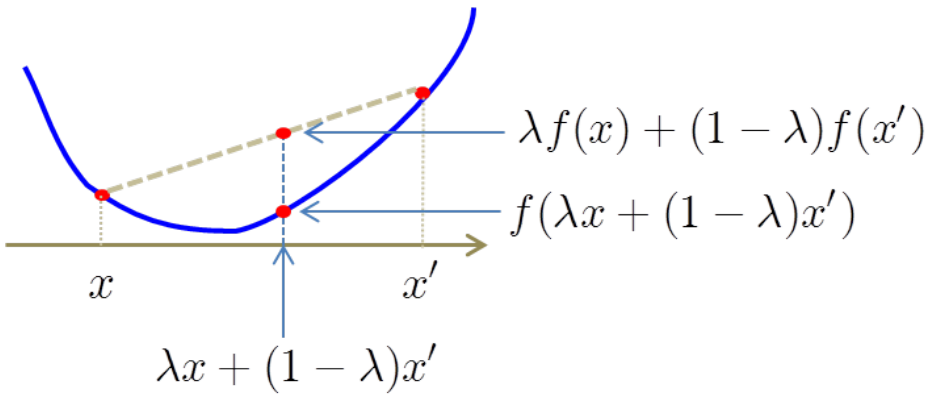

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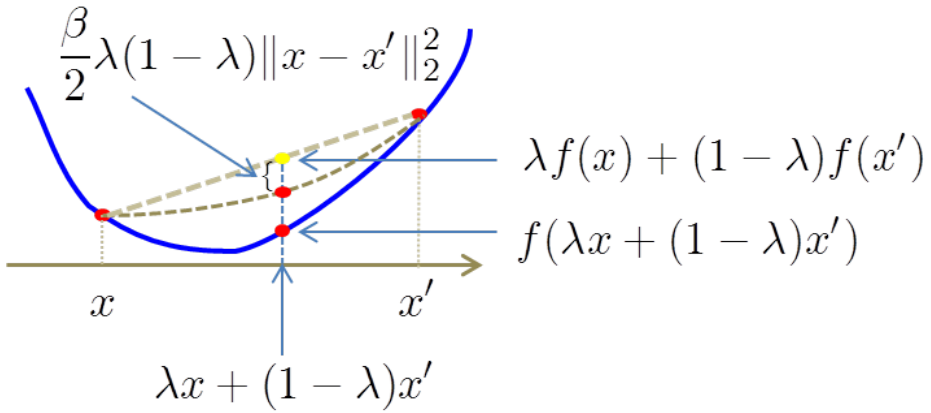

strong convexity

Strong convexity $\Rightarrow$ strict convexity.

$\not\Leftarrow$

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An important function: the indicator of a set $C \subset \mathbb{R}^n$,

$$\iota_C : \mathbb{R}^n \to \bar{\mathbb{R}}, \quad \iota_C(x) = \begin{cases} 0 & \Leftarrow \quad x \in C \\ +\infty & \Leftarrow \quad x \notin C \end{cases}$$

If C is a closed convex set, ι_C is a l.s.c. convex function.

The Case of Differentiable Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and consider its Hessian matrix at x, denoted $\nabla^2 f(x)$ (or $Hf(x)$):

$$\left(\nabla^2 f(x)\right)_{ij} = \frac{\partial f}{\partial x_i \partial x_j}, \quad \text{for } i, j = 1, ..., n.$$

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- f is β-strongly convex $\Leftrightarrow$ its Hessian $\nabla^2 f(x) \succeq \beta I$, with $\beta > 0$, $\forall_x$.

More on the Relationship Between ℓ_1 and ℓ_0

Finding the sparsest solution is NP-hard (Muthukrishnan, 2005).

$$\begin{aligned} \widehat{w} &= \arg\min_{w} \|w\|_0 \\ &\quad \text{s. t. } \|Aw - y\|_2^2 \leq \delta \end{aligned}$$

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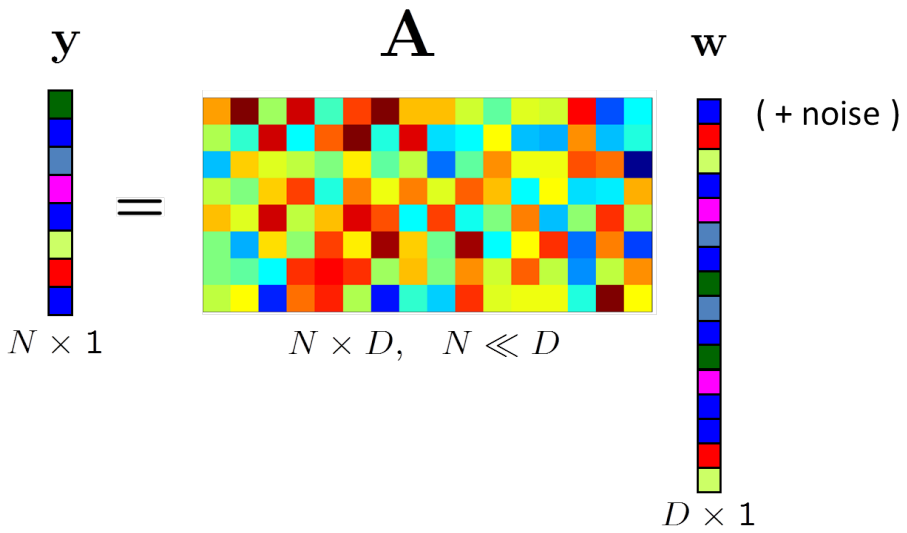
Under conditions, replacing ℓ_0 with ℓ_1 yields "similar" results: central issue in compressive sensing (CS) (Candès et al., 2006a; Donoho, 2006)

Compressive Sensing in a Nutshell



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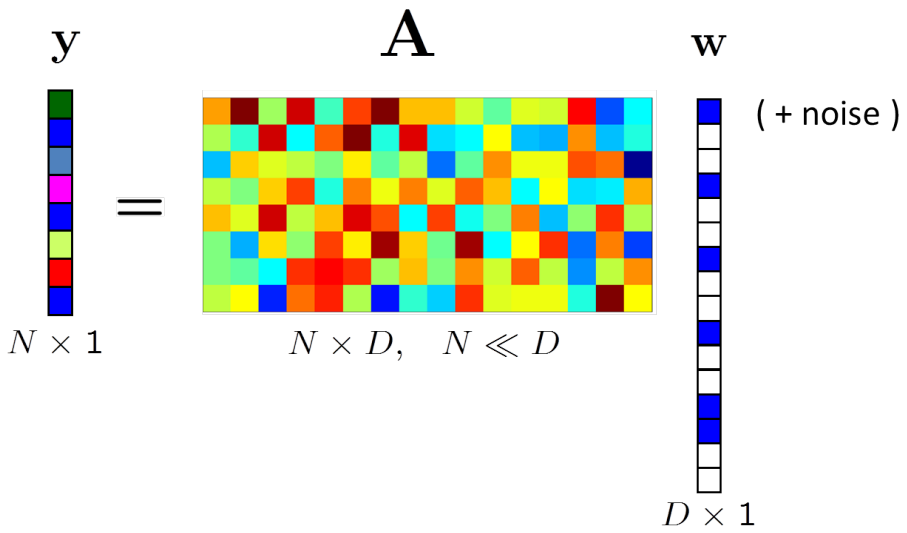

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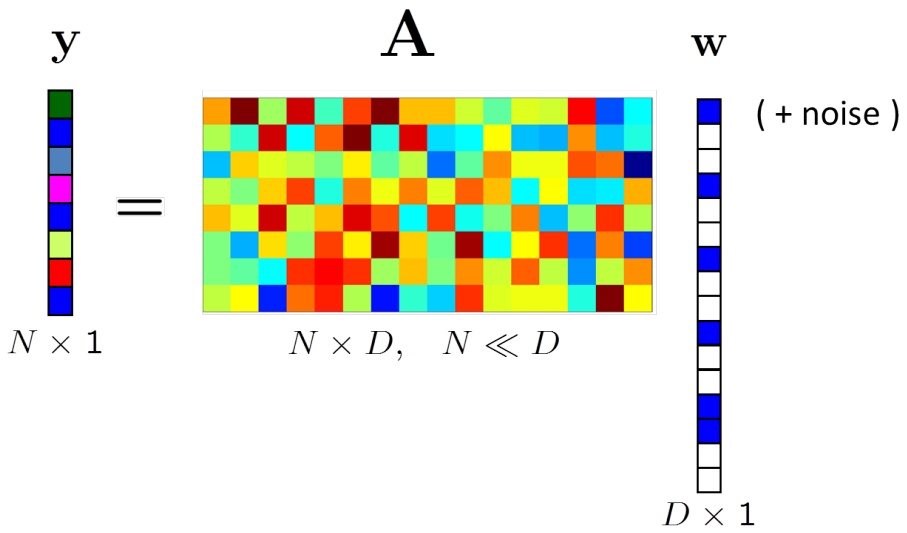

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If w is sparse enough and A has certain properties, then w is stably recovered via (Haupt and Nowak, 2006)

$$\begin{aligned} \widehat{w} \quad = \quad & \arg\min_{w} \|w\|_0 \\ & \text{s. t. } \|Aw - y\| \leq \delta \end{aligned}$$

NP-hard!

Compressive Sensing in a Nutshell

Under some conditions on A (e.g., the restricted isometry property (RIP)), ℓ_0 can be replaced with ℓ_1 (Candès et al., 2006b):

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Matrix A satisfies the RIP of order k, with constant $\delta_k \in (0, 1)$, if

$$\|w\|_0 \leq k \;\Rightarrow\; (1 - \delta_k)\|w\|_2^2 \leq \|Aw\|_2^2 \leq (1 + \delta_k)\|w\|_2^2$$

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Other properties (spark and null space property (NSP)) can be used; caveat: checking RIP, NSP, spark is **NP-hard** (Tillmann and Pfetsch, 2012).

Examples: Back to Under-Constrained Systems

Let $\bar{x}$ be the sparsest solution of $Ax = y$, where $A \in \mathbb{R}^{m \times n}$ and $m < n$.

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Next: elementary analysis by Yin and Zhang (2008), based on work by Kashin (1977) and Garnaev and Gluskin (1984).

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&= \|\bar{x}\|_1 + \|v\|_1 - 2\|v_S\|_1 \\
&\geq \|\bar{x}\|_1 + \|v\|_1 - 2\sqrt{k}\|v\|_2. && (\|a\|_1 \leq \sqrt{n}\,\|a\|_2)
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&= \|\bar{x}\|_1 + \|v\|_1 - 2\|v_S\|_1 \\
&\geq \|\bar{x}\|_1 + \|v\|_1 - 2\sqrt{k}\|v\|_2. && (\|a\|_1 \leq \sqrt{n}\,\|a\|_2)
\end{aligned}
$$

Hence, $\bar{x} \in G$, if $\frac{1}{2}\frac{\|v\|_1}{\|v\|_2} \geq \sqrt{k}, \ \forall v \in \ker(A)$

...but, in general, we have only: $1 \leq \frac{\|v\|_1}{\|v\|_2} \leq \sqrt{n}$

Equivalence Between ℓ_1 and ℓ_0 Optimization

- Minimum ℓ_0 (sparsest) solution: $\bar{x} \in \arg\min \|x\|_0$ s.t. $Ax = y$.
- Minimum ℓ_1 solution(s): $G = \arg\min \|x\|_1$ s.t. $Ax = y$.
- $\bar{x} \in G$, if $\|\bar{x} + v\|_1 \geq \|\bar{x}\|_1, \ \forall v \in \ker(A)$
- Let $S = \{i : \bar{x}_i \neq 0\}$ and $Z = \{1, ..., n\} \setminus S$

$$
\begin{aligned}
\|\bar{x} + v\|_1 &= \|\bar{x}_S + v_S\|_1 + \|v_Z\|_1 \\
&\geq \|\bar{x}_S\|_1 + \|v_Z\|_1 - \|v_S\|_1 && (\|a+b\| \geq \|a\| - \|b\|) \\
&= \|\bar{x}\|_1 + \|v\|_1 - 2\|v_S\|_1 \\
&\geq \|\bar{x}\|_1 + \|v\|_1 - 2\sqrt{k}\|v\|_2. && (\|a\|_1 \leq \sqrt{n}\,\|a\|_2)
\end{aligned}
$$

Hence, $\bar{x} \in G$, if $\frac{1}{2}\frac{\|v\|_1}{\|v\|_2} \geq \sqrt{k}, \ \forall v \in \ker(A)$

...but, in general, we have only: $1 \leq \frac{\|v\|_1}{\|v\|_2} \leq \sqrt{n}$

However, we may have $\frac{\|v\|_1}{\|v\|_2} \gg 1$, if v is restricted to a random subspace.

Bounding the ℓ_1/ℓ_2 Ratio in Random Matrices

If the elements of $A \in \mathbb{R}^{m \times n}$ are sampled i.i.d. from $\mathcal{N}(0,1)$ (zero mean, unit variance Gaussian), then, with high probability,

$$\frac{\|v\|_1}{\|v\|_2} \geq \frac{C\sqrt{m}}{\sqrt{\log(n/m)}}, \quad \text{for all } v \in \ker(A),$$

for some constant C (based on concentration of measure phenomena).

Thus, with high probability, $\bar{x} \in G$, if

$$m \geq \frac{4}{C^2} k \log n$$

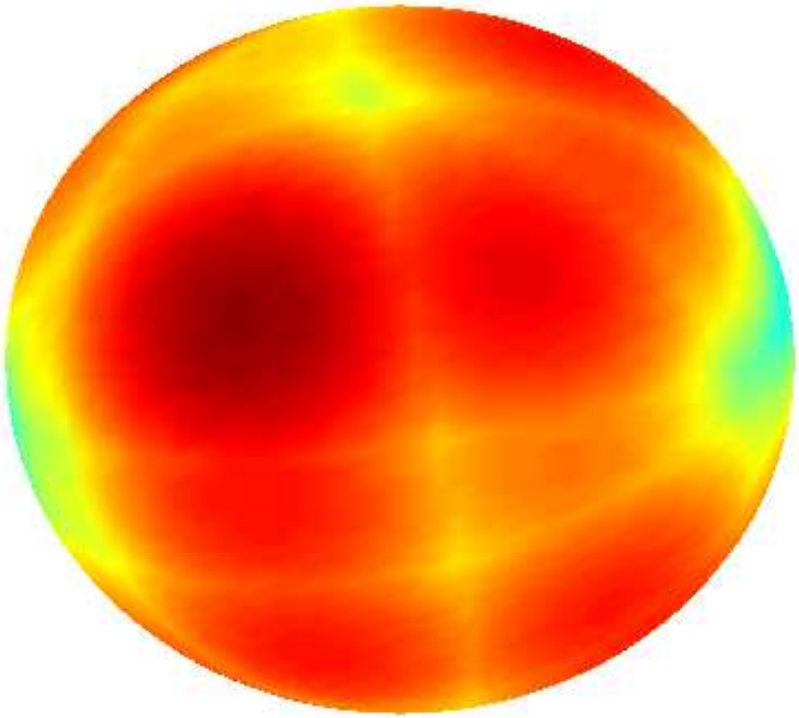
Conclusion: Can solve under-determined system, where A has i.i.d. $\mathcal{N}(0,1)$ elements, by solving

$$\min_x \|x\|_1 \;\; s.t. \;\; Ax = b,$$

(a convex problem), if the solution is sparse enough.

Ratio $\|v\|_1/\|v\|_2$ on Random Null Spaces

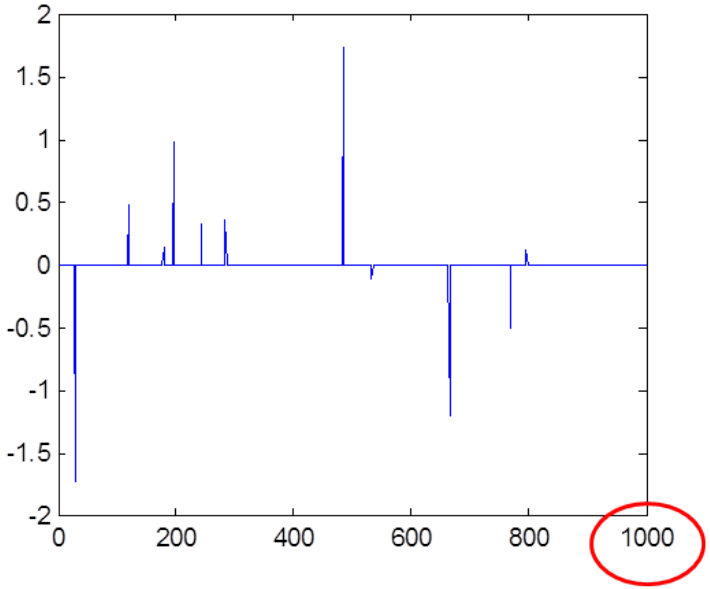
Random $A \in \mathbb{R}^{4\times 7}$, showing ratio $\|v\|_1$ for $v \in \ker(A)$ with $\|v\|_2 = 1$



Blue: $\|v\|_1 \approx 1$. Red: ratio $\approx \sqrt{7}$. Note that $\|v\|_1$ is well away from the lower bound of 1 over the whole nullspace.

When Data is Noisy

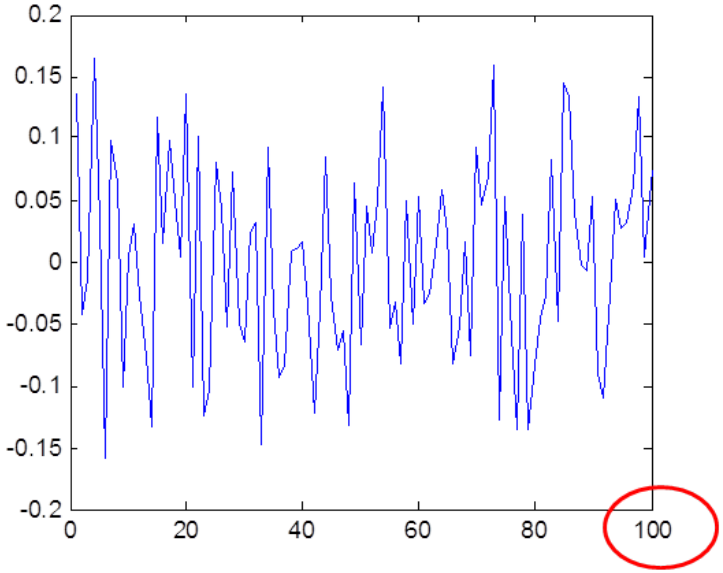
Sparse vector $\mathbf{x}$



$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

Random matrix

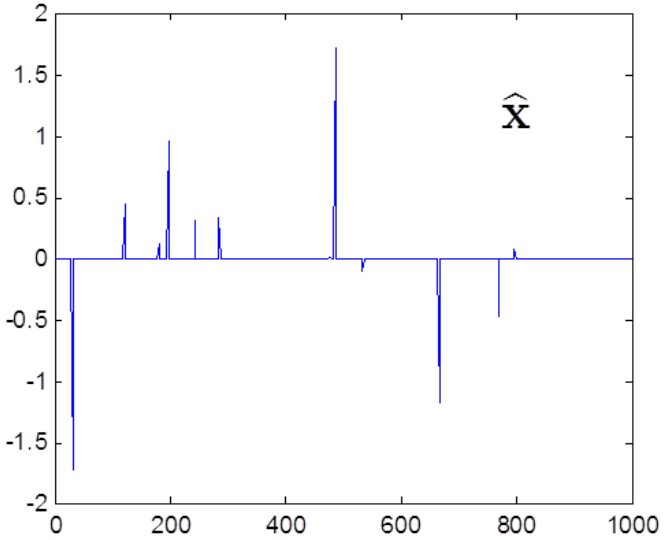
Observed data $\mathbf{y}$



Under certain conditions, "perfect" recovery is possible

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + 2\,\lambda\,\|\mathbf{x}\|_1 \right\}$$

[Candès, Romberg, Tao, 2004 – 2006]
[Donoho, 2006]



The Ubiquitous ℓ_1 Norm

- Lasso (*least absolute shrinkage and selection operator*) (Tibshirani, 1996) a.k.a. *basis pursuit denoising* (Chen et al., 1995):

$$\min_x \frac{1}{2}\|Ax - y\|_2^2 + \tau\|x\|_1 \quad \text{or} \quad \min_x \|Ax - y\|_2^2 \ \text{ s.t. } \ \|x\|_1 \leq \delta$$

or, more generally,

$$\min_x f(x) + \lambda\|x\|_1 \quad \text{or} \quad \min_x f(x) \ \text{ s.t. } \ \|x\|_1 \leq \delta$$

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- Geology/geophysics
  - Claerbout and Muir (1973)
  - Taylor et al. (1979)
  - Levy and Fullager (1981)
  - Oldenburg et al. (1983)
  - Santosa and Symes (1988)
- Radio astronomy
  - Högbom (1974)
  - Schwarz (1978)
- Fourier transform spectroscopy
  - Kawata et al. (1983)
  - Mammone (1983)
  - Minami et al. (1985)
- NMR spectroscopy
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  - Newman (1988)
- Medical ultrasound
  - Papoulis and Chamzas (1979)

from (Goyal et al, 2010)

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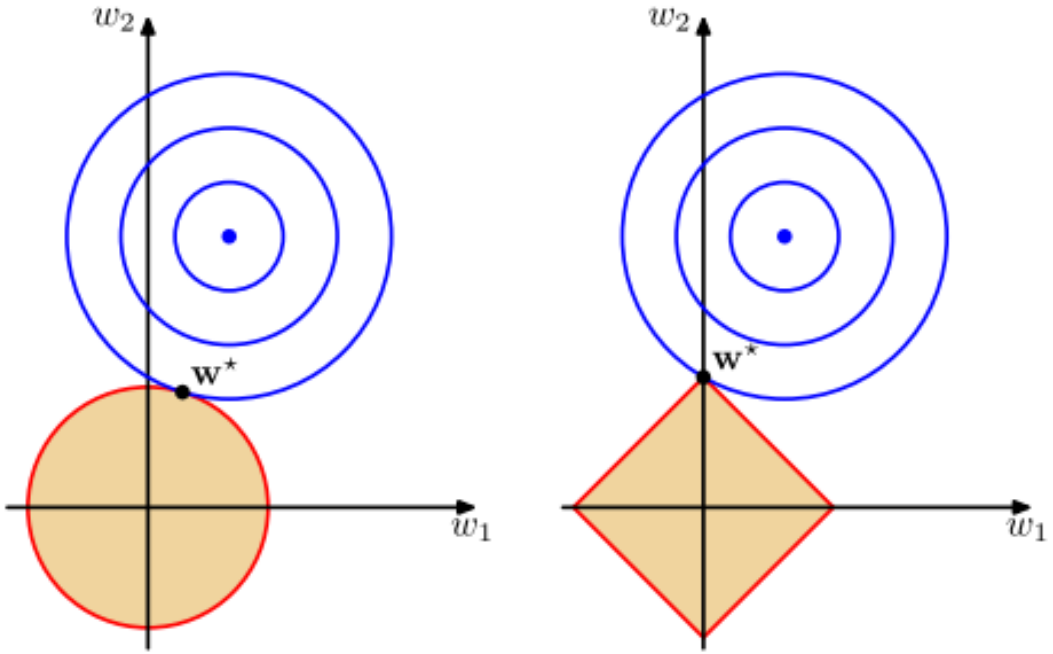
- Why does ℓ_1 yield sparse solutions? (next slides)
- How to solve these problems? (this tutorial)

Why ℓ_1 Yields Sparse Solution

$$w^* = \begin{array}{ll} \arg\min_w & \|Aw - y\|_2^2 \\ \text{s.t.} & \|w\|_2 \leq \delta \end{array} \quad \text{vs} \quad w^* = \begin{array}{ll} \arg\min_w & \|Aw - y\|_2^2 \\ \text{s.t.} & \|w\|_1 \leq \delta \end{array}$$



Why ℓ_1 Yields Sparse Solution

The simplest problem with ℓ_1 regularization

$$\widehat{w} = \arg\min_w \frac{1}{2}(w-y)^2 + \lambda|w| = \text{soft}(y,\lambda) = \begin{cases} y-\lambda & \Leftarrow \quad y > \lambda \\ 0 & \Leftarrow \quad |y| \leq \lambda \\ y+\lambda & \Leftarrow \quad y < -\lambda \end{cases}$$

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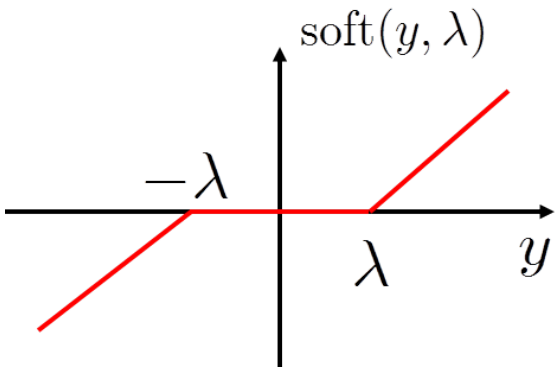

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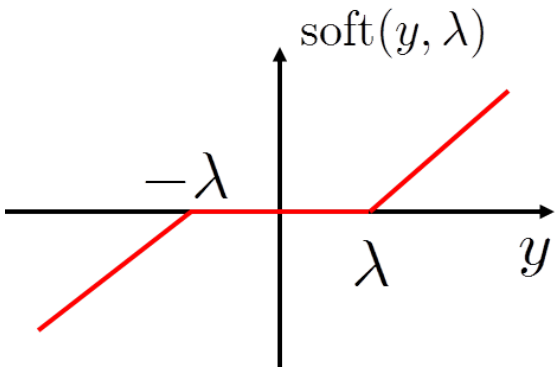

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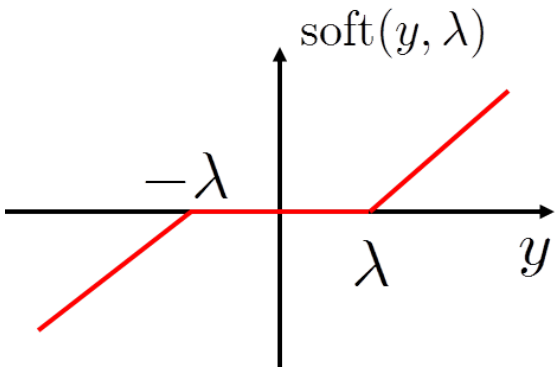

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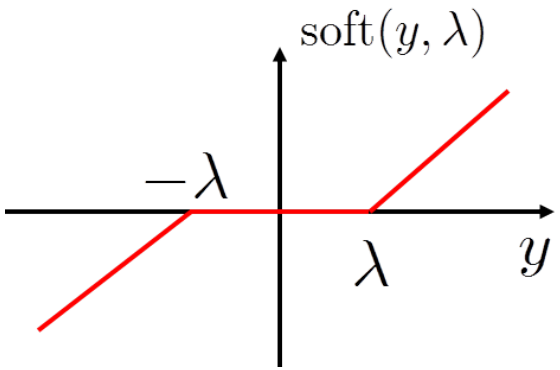

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More on the Relationship Between ℓ_1 and ℓ_0

The ℓ_0 "norm" (number of non-zeros): $\|w\|_0 = |\{i : w_i \neq 0\}|$.
Not a norm, not convex, but in the simple case...

$$\widehat{w} = \arg\min_w \frac{1}{2}(w-y)^2 + \lambda|w|_0 = \text{hard}(y, \sqrt{2\lambda}) = \begin{cases} y & \Leftarrow \quad |y| > \sqrt{2\lambda} \\ 0 & \Leftarrow \quad |y| \leq \sqrt{2\lambda} \end{cases}$$

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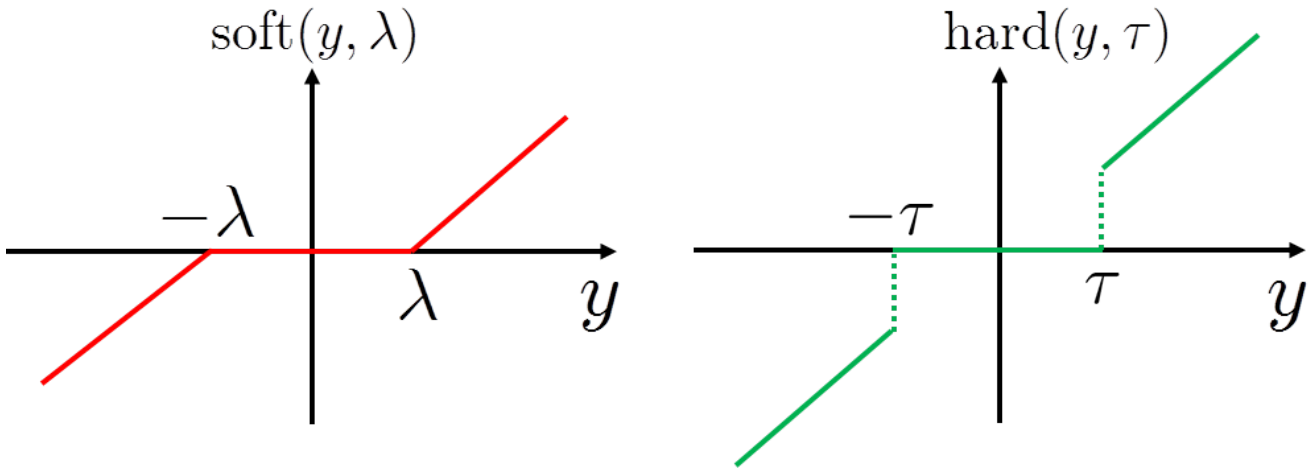

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Natural images are well represented by a few coefficients in some bases.

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Original 1000×1000 image $x \in \mathbb{R}^{10^6}$

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Original 1000×1000 image $x \in \mathbb{R}^{10^6}$

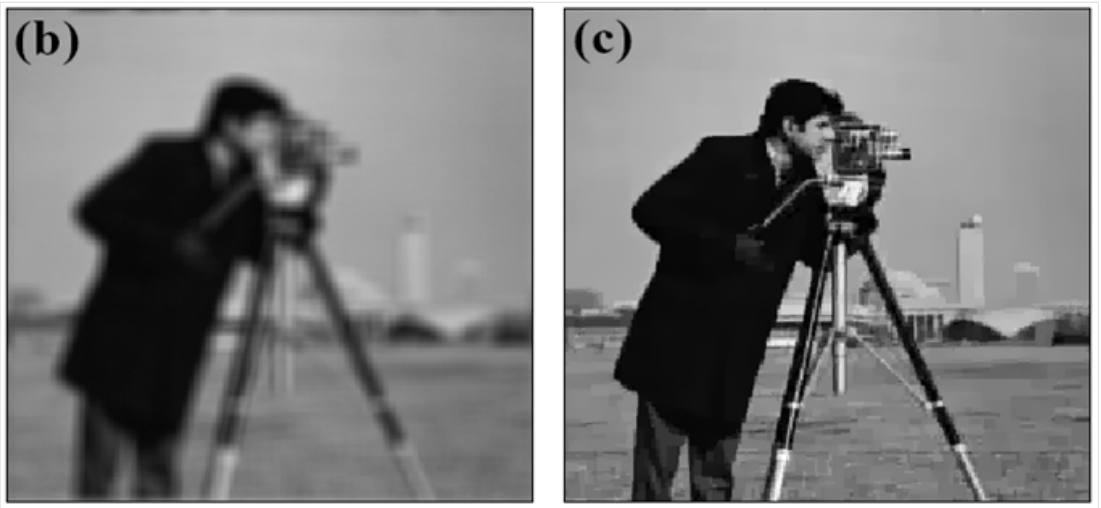
...only its 25000 largest coefficients.

- Also (even more) true with an over-complete tight frame; W is "fat" (more columns than rows) and $WW^T = I$, but $W^T W \neq I$.

Application to Image Deblurring/Deconvolution

blurred

restored



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$$\mathbf{A} = \mathbf{B}\mathbf{W}$$

convolution (blur)

wavelet basis (or tight frame)

Application to Magnetic Resonance Imaging

$$\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \tau \|\mathbf{x}\|_1$$

$$\mathbf{A} = \mathbf{M}\mathbf{U}\mathbf{W}$$

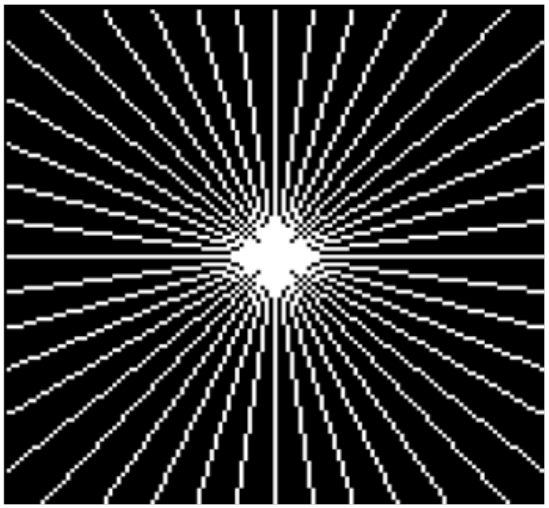
binary mask

discrete Fourier transform

wavelet basis (or tight frame)

original

acquired slices in DFT domain



reconstruction $\mathbf{W}\widehat{\mathbf{x}}$

Machine/Statistical Learning: Linear Regression

Data N pairs $(x_1, y_1), ..., (x_N, y_N)$, where $x_i \in \mathbb{R}^d$ (feature/variable vectors) and $y_i \in \mathbb{R}$ (outputs).

Goal: find "good" linear function: $\widehat{y} = \sum_{j=1}^{d} w_j x_j + w_{d+1} = [x^T 1] w$

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Mean squared error: $\min_w \mathbb{E}\big(Y - [X^T 1]w\big)^2$ impossible! $P_{X,Y}$ unknown

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design matrix: $A_{ij} = (x_i)_j$ (j-th component of i-th sample, $A_{i(d+1)} = 1$)

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Regularization: $\min_w \|y - Aw\|_2^2 + \tau\psi(w)$

Machine/Statistical Learning: Linear Classification

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Convexification: EE neither convex nor differentiable (NP-hard problem).
Solution: replace $h : \mathbb{R} \to \{0, 1\}$ with convex loss $L : \mathbb{R} \to \mathbb{R}_+$.

Machine/Statistical Learning: Linear Classification

Criterion: $$\min_{w} \underbrace{\sum_{i=1}^{N} L\big(\underbrace{y_i \, (w^T x_i + b)}_{\text{margin}} \big)}_{f(w)} + \tau \psi(w)$$

Convex loss: $L : \mathbb{R} \to \mathbb{R}_+$ is a (preferably convex) loss function.

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Regularizer: $\psi = \ell_1 \quad \Rightarrow \quad$ encourage sparseness $\quad \Rightarrow$ feature selection

Machine/Statistical Learning: Linear Classification

Criterion: $$\min_w \underbrace{\sum_{i=1}^{N} L\big(\underbrace{y_i\,(w^T x_i + b)}_{\text{margin}}\big) + \tau\psi(w)}_{f(w)}$$

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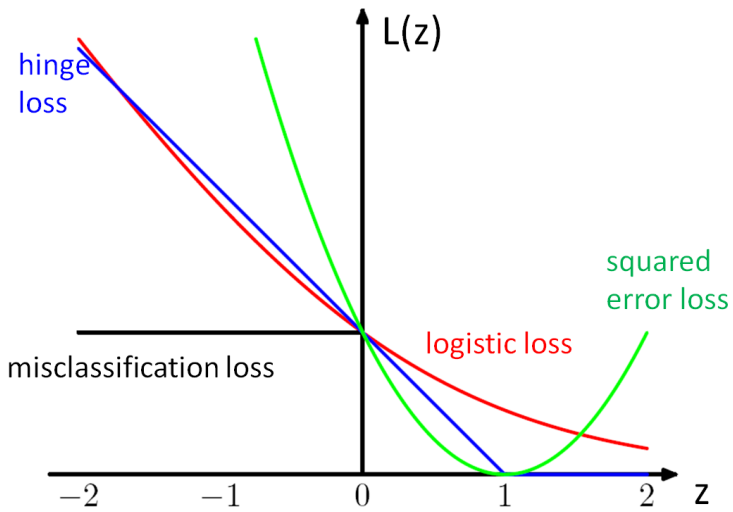
- Misclassification loss: $L(z) = 1_{z<0}$
- Hinge loss: $L(z) = \max\{1 - z, 0\}$
- Logistic loss: $L(z) = \dfrac{\log\big(1+\exp(-z)\big)}{\log 2}$
- Squared loss: $L(z) = (z-1)^2$



Machine/Statistical Learning: Classification

This formulation covers a wide range of linear ML methods:

$$\min_w \underbrace{\sum_{i=1}^{N} L\left(y_i \left([x_i^T \; 1]w\right)\right)}_{f(w)} + \tau\psi(w)$$

- Least squares regression: $L(z) = (z-1)^2, \;\; \psi(w) = 0.$
- Ridge regression: $L(z) = (z-1)^2, \;\; \psi(w) = \|w\|_2^2.$
- Lasso regression: $L(z) = (z-1)^2, \;\; \psi(w) = \|w\|_1$
- Logistic regression: $L(z) = \log(1+\exp(-z))$ (ridge, if $\psi(w) = \|w\|_2^2$
- Sparse logistic regression: $L(z) = \log(1+\exp(-z)), \;\; \psi(w) = \|w\|_1$
- Support vector machines: $L(z) = \max\{1-z, 0\}, \;\; \psi(w) = \|w\|_2^2$
- Boosting: $L(z) = \exp(-z),$
- ...

Machine/Statistical Learning: Nonlinear Problems

What about non-linear functions?

Simply use $\widehat{y} = \phi(x, w) = \sum_{j=1}^{D} w_j \, \phi_j(x)$, where $\phi_j : \mathbb{R}^d \to \mathbb{R}$

Essentially, nothing changes; computationally, a lot may change!

$$\min_w \underbrace{\sum_{i=1}^{N} L\big(y_i \; \phi(x, w)\big)}_{f(w)} + \tau\psi(w)$$

Key feature: $\phi(x, w)$ is still linear with respect to w, thus f inherits the convexity of L.

Examples: polynomials, radial basis functions, wavelets, splines, kernels,...

Recover the linear case, letting $D = d + 1$, $f_j(x) = x_j$, and $f_{d+1} = 1$.

Structured Sparsity

ℓ_1 regularization promotes sparsity

A very simple sparsity pattern: prefer models with small cardinality

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Can we promote less trivial sparsity patterns? How?

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Group/structured regularization.

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Main goal: to promote structural patterns, not just penalize cardinality

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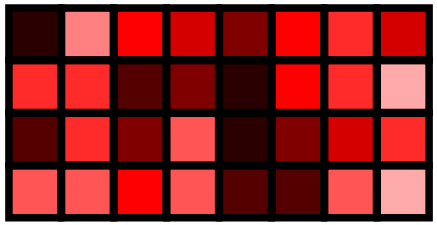
Many applications:

- feature template selection (Martins et al., 2011)
- multi-task learning (Caruana, 1997; Obozinski et al., 2010)
- learning the structure of graphical models (Schmidt and Murphy, 2010)

"Grid" Sparsity

For feature spaces that can be arranged as a grid (examples next)



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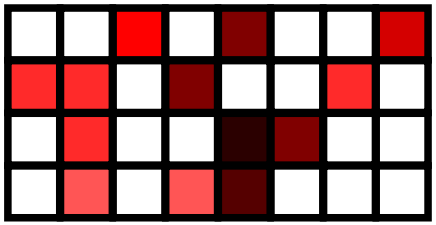

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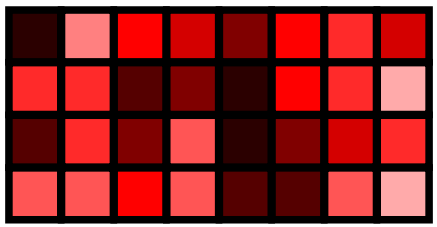

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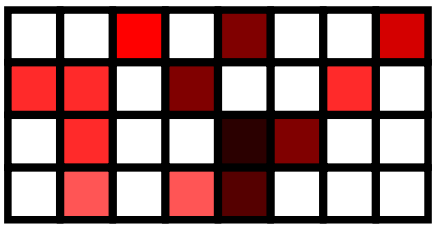

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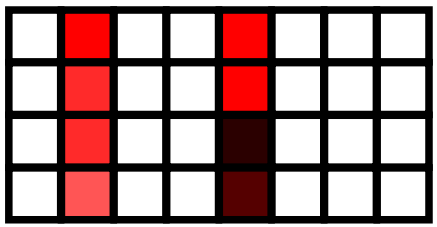

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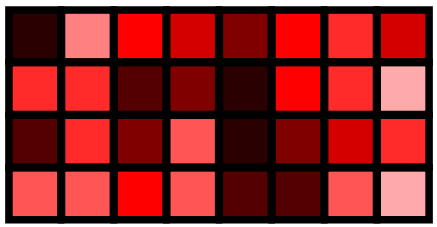

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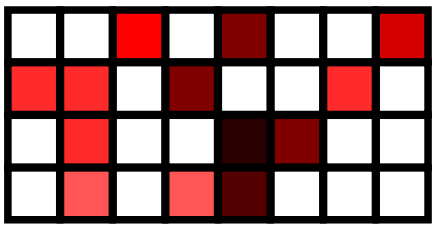

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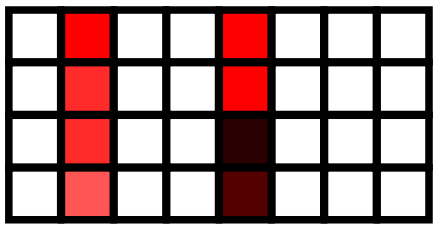

group sparse

Goal: push *entire columns* to have zero weights

The groups are the columns of the grid

Example: Sparsity with Multiple Classes

In multi-class (more than just 2 classes) classification, a common formulation is

$$\widehat{y} = \arg \max_{y \in \{1,...,K\}} x^T w_y$$

Weight vector $w = (w_1, ..., w_K) \in \mathbb{R}^{Kd}$ has a natural group/grid organization:

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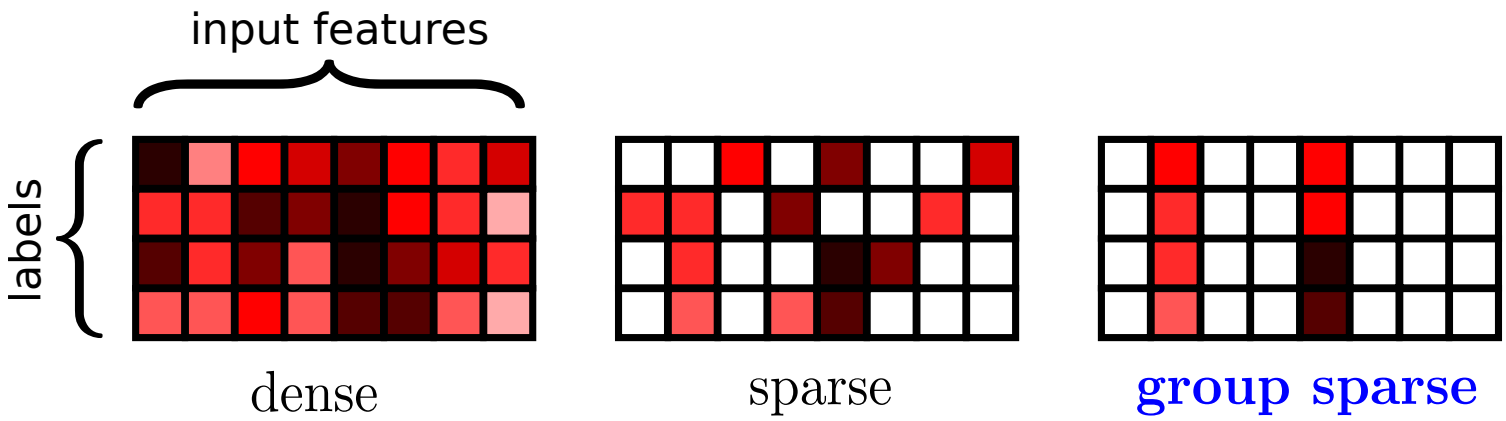


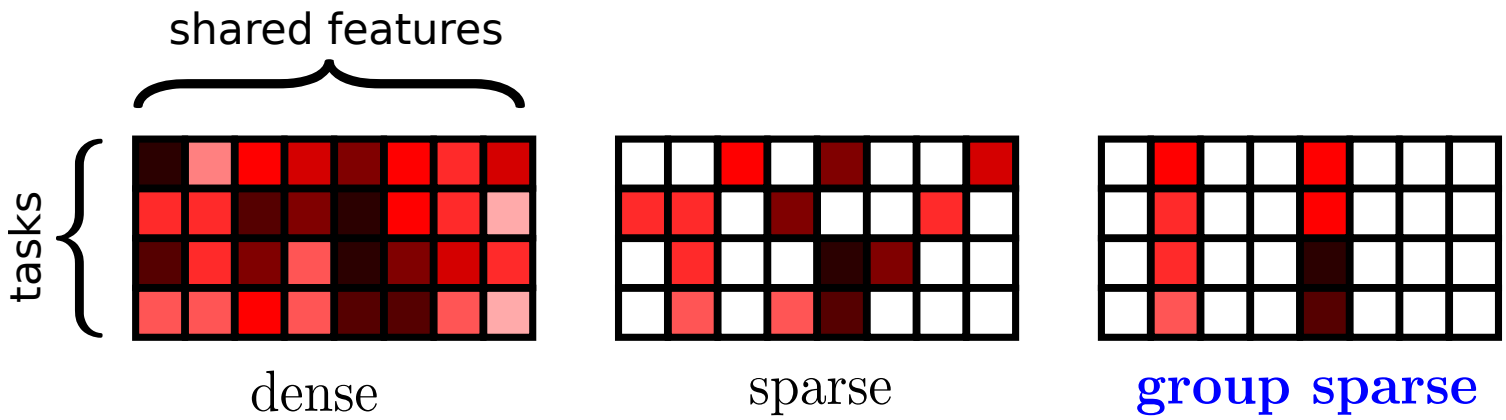
Simple sparsity is wasteful: may still need to keep all the features

Structured sparsity: discard some input features (feature selection)

Example: Multi-Task Learning

Same thing, except now rows are tasks and columns are **features**

Example: simultaneous regression (seek function into $\mathbb{R}^d \to \mathbb{R}^b$)



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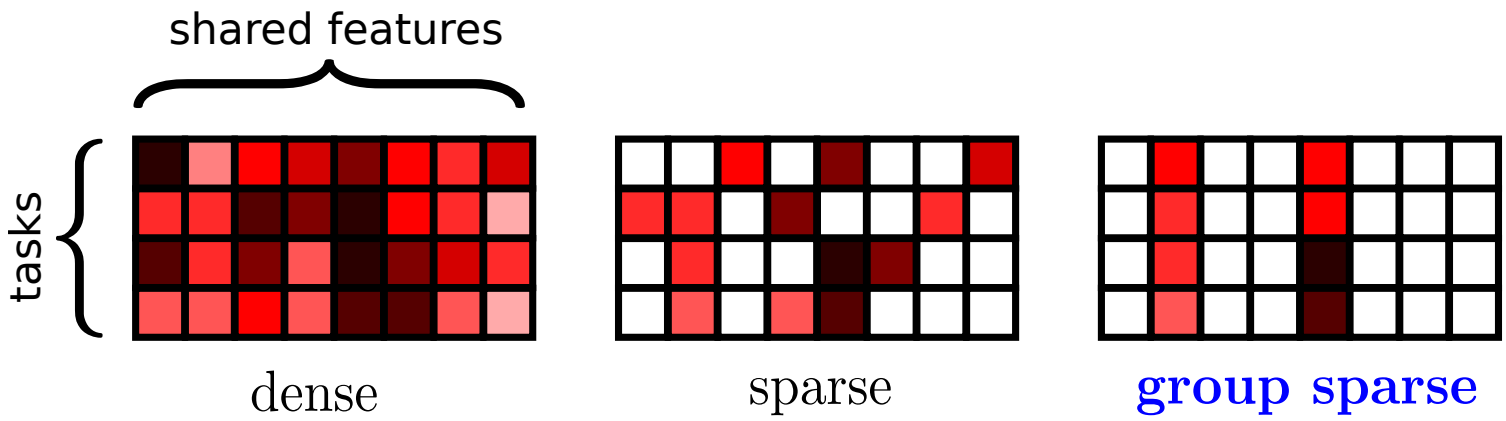
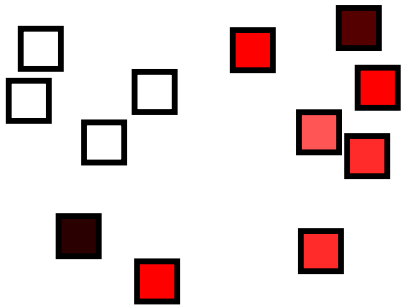


Goal: discard features that are irrelevant for *all* tasks

Approach: one group per feature (Caruana, 1997; Obozinski et al., 2010)

Group Sparsity



- D features

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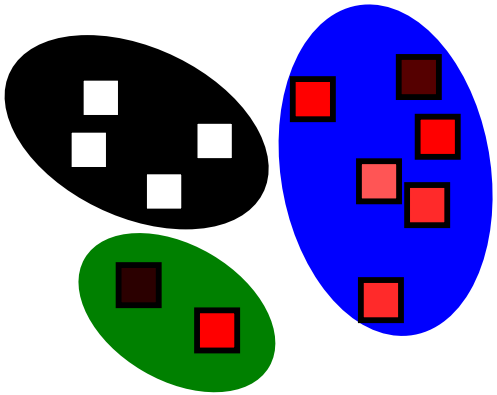

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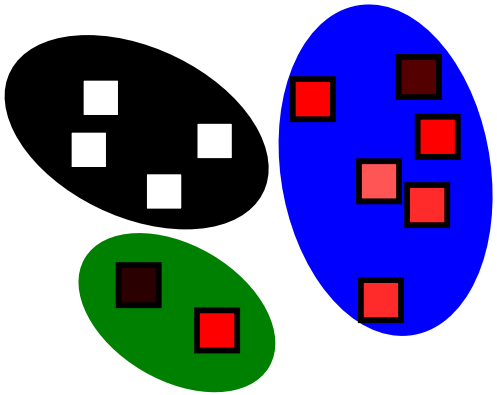

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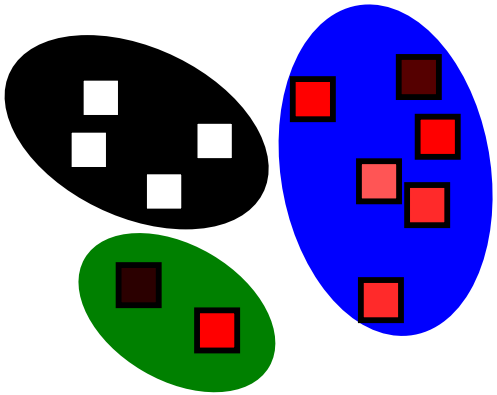

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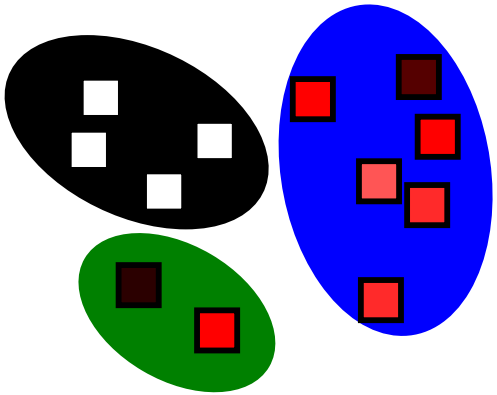

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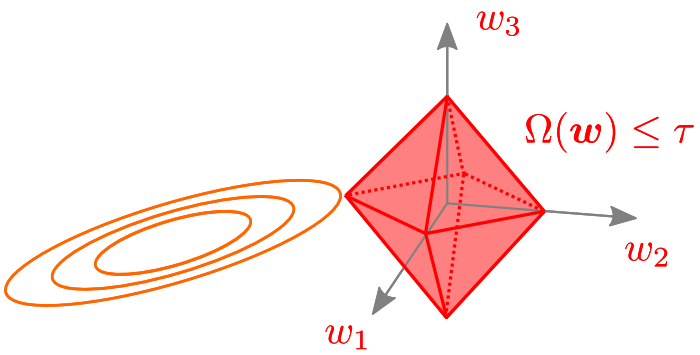
- Weighted version: λ_m are prior weights for groups (groups may have different sizes)

Lasso versus group-Lasso



$$\Omega(\boldsymbol{w}) = |w_1| + |w_2| + |w_3|$$

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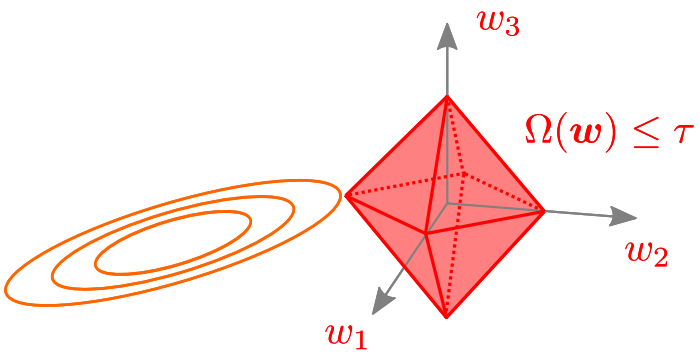

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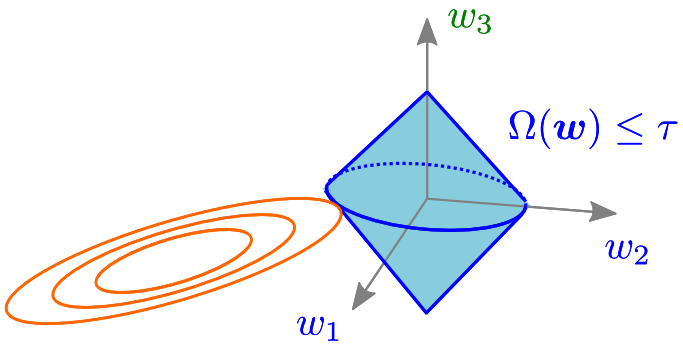
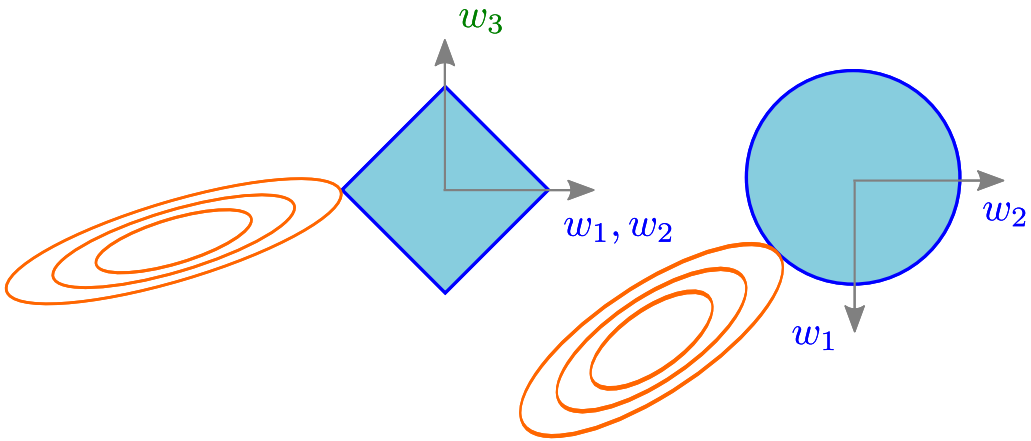


$$\Omega(\boldsymbol{w}) = \sqrt{w_1^2 + w_2^2} + |w_3|$$



Composite Absolute Penalties (Zhao et al., 2009)

A mixed-norm regularization:

$$\psi(x) = \left(\sum_{m=1}^{M} \|x_m\|_q^r \right)^{1/r}$$

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- The most common choice: $\ell_{2,1}$ norm
- Another frequent choice: $\ell_{\infty,1}$ norm (Quattoni et al., 2009; Graça et al., 2009; Eisenstein et al., 2011; Wright et al., 2009)

Three Scenarios

- Non-overlapping Groups
- Tree-structured Groups
- Graph-structured Groups

Non-overlapping Groups

Assume that $G_1, \ldots, G_M$ (where $G_m \subset \{1, ..., d\}$) constitute a partition:

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Examples of non-trivial groups:

- label-based groups
- task-based groups

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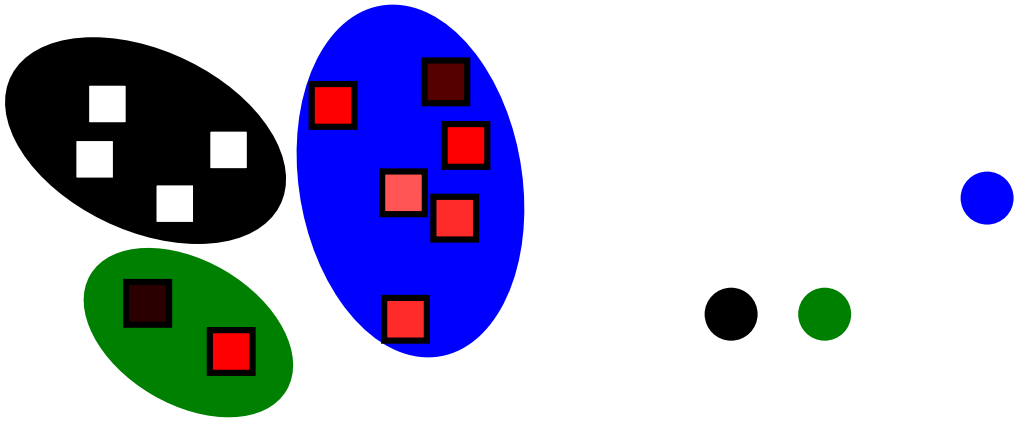

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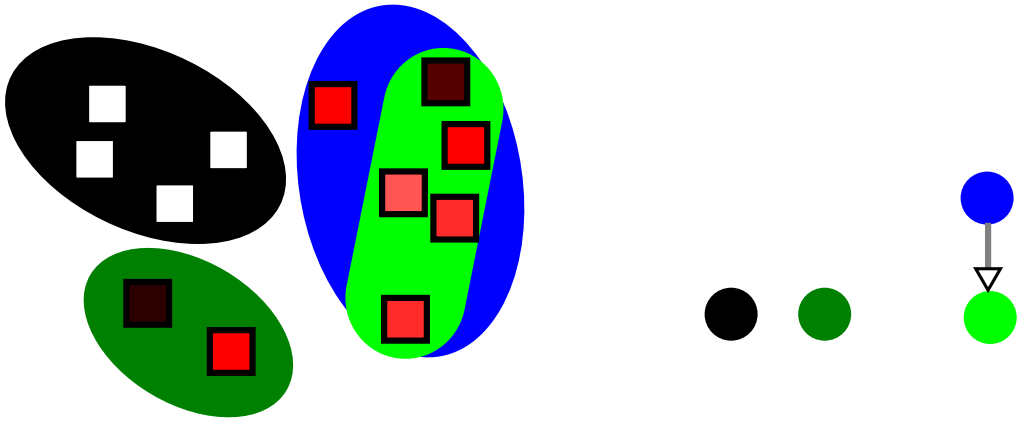

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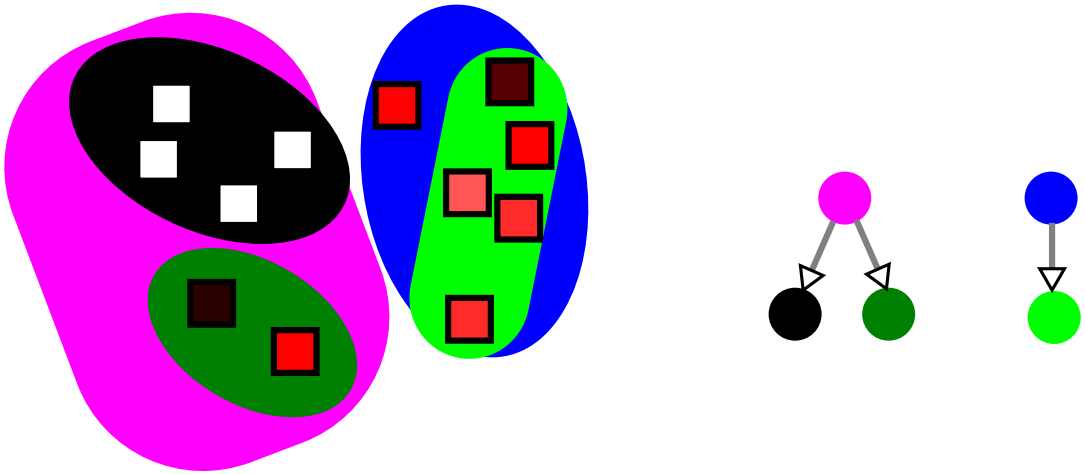

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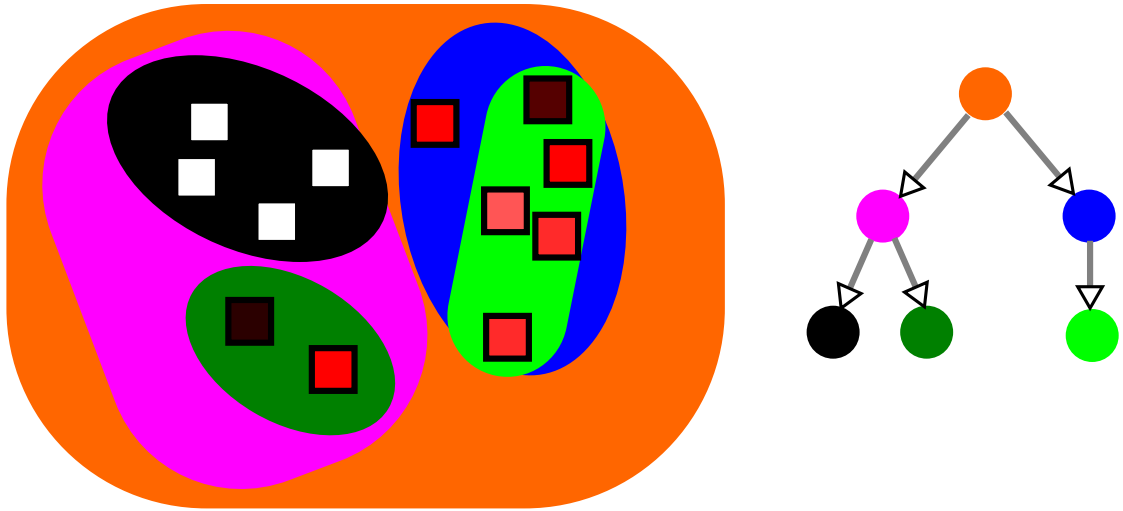

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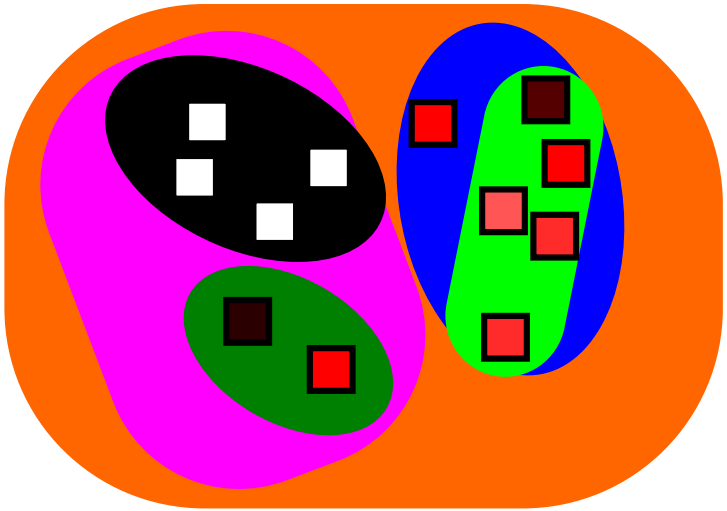

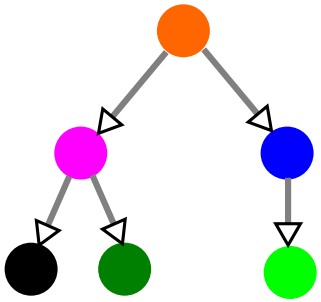

- What is the **sparsity pattern**?

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Assumption: if two groups overlap, one is contained in the other
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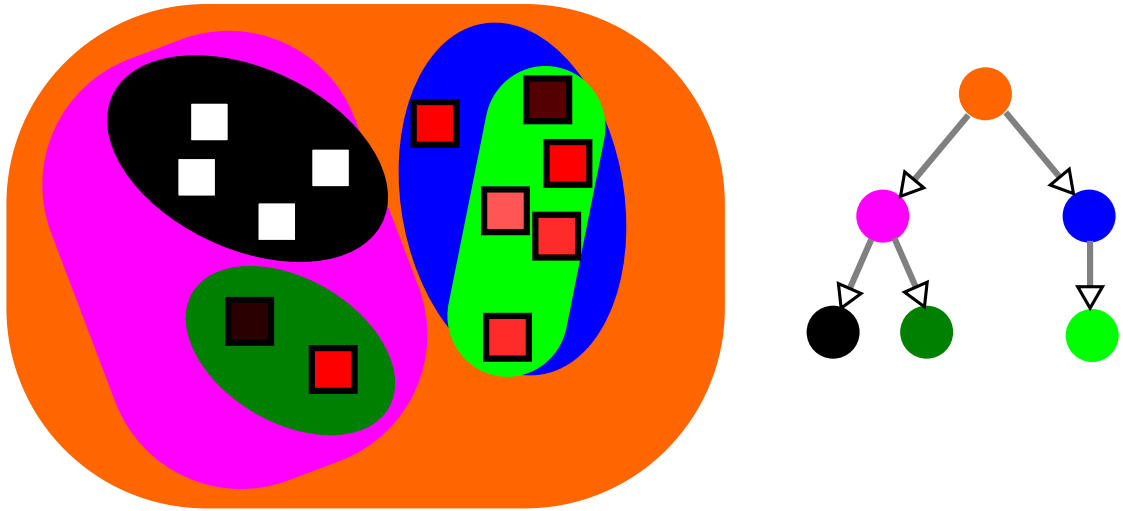

- What is the **sparsity pattern**?
- If a group is discarded, all its descendants are also discarded

Matrix Inference Problems

Sparsest solution:

- From $Bx = b \in \mathbb{R}^p$, find $x \in \mathbb{R}^n$ $(p < n)$.
- $\min_x \|x\|_0$ s.t. $Bx = b$
- Yields exact solution, under some conditions.

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Lowest rank solution:

- From $\mathcal{B}(X) = b \in \mathbb{R}^p$, find $X \in \mathbb{R}^{m \times n}$ $(p < m\,n)$.
- $\min_X \text{rank}(X)$ s.t. $\mathcal{B}(X) = b$
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Both $NP-$hard (in general); the same is true of noisy versions:

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Under some conditions, the same solution is obtained by replacing $\text{rank}(X)$ by the nuclear norm $\|X\|_*$ (as any norm, it is convex) (Recht et al., 2010)

Matrix Nuclear Norm (and Other Norms)

- Also known as trace norm; the ℓ_1-type norm for matrices $X \in \mathbb{R}^{m \times n}$
- Definition: $\|X\|_* = \text{trace}\left(\sqrt{X^T X}\right) = \sum_{i=1}^{\min\{m,n\}} \sigma_i$,

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- Two other notable Schatten norms:
  - Frobenius norm: $\|X\|_2 = \|X\|_F = \sqrt{\sum_{i=1}^{\min\{m,n\}} (\sigma_i)^2} = \sqrt{\sum_{i,i} X_{i,j}^2}$
  - Spectral norm: $\|X\|_\infty = \max\left\{\sigma_1, ..., \sigma_{\min\{m,n\}}\right\}$

Nuclear Norm Regularization

Tikhonov formulation: $\min_X \underbrace{\|\mathcal{B}(X) - b\|_2^2}_{f(X)} + \underbrace{\tau \|X\|_*}_{\tau\psi(X)}$

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Linear observations: $\mathcal{B} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$, $\big(\mathcal{B}(X)\big)_i = \langle B_{(i)}, X\rangle$,

$$B_{(i)} \in \mathbb{R}^{m\times n}, \text{ and } \langle B, X\rangle = \sum_{jk} B_{jk}X_{jk} = \text{trace}(B^T X)$$

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Why does the nuclear norm favor low rank solutions? Let $Y = U\Lambda V^T$ be the singular value decomposition, where $\Lambda = \text{diag}\big(\sigma_1, ..., \sigma_{\min\{m,n\}}\big)$; then

$$\arg\min_X \frac{1}{2}\|Y - X\|_F^2 + \tau\|X\|_* = U \underbrace{\text{soft}(\Lambda, \tau)}_{\text{may yield zeros}} V^T$$

...singular value thresholding (Ma et al., 2011; Cai et al., 2010)

Another Matrix Inference Problem: Inverse Covariance

Consider n samples $y_1, ..., y_n \in \mathbb{R}^d$ of a Gaussian r.v. $Y \sim \mathcal{N}(\mu, C)$; the log-likelihood is

$$L(P) = \log p(y_1, ..., y_n | P) = \log \det(P) - \text{trace}(SP) + \text{constant}$$

where $S = \frac{1}{n} \sum_{i=1}^{n} (y_i - \mu)(y_i - \mu)^T$ and $P = C^{-1}$ (inverse covariance).

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Zeros in P reveal conditional independencies between components of Y:

$$P_{ij} = 0 \quad \Leftrightarrow \quad Y_i \perp\!\!\!\perp Y_j | \{Y_k,\ k \neq i, j\}$$

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Sparsity (presence of zeros) in P is encouraged by solving

$$\min_{P \succ 0} \underbrace{-\log \det(P) + \text{trace}(SP)}_{f(P)} + \tau \underbrace{\|\text{vect}(P)\|_1}_{\psi(P)}$$

where $\text{vect}(P) = [P_{1,1}, ..., P_{d,d}]^T$.

Atomic-Norm Regularization

Key concept in sparse modeling: synthesize "object" using a few atoms:

$$x = \sum_{i=1}^{|\mathcal{A}|} c_i \, a_i$$

- $\mathcal{A}$ is the set of atoms (the atomic set), or building blocks.
- $c_i \geq 0$ are weights; x is simple/sparse object $\Rightarrow \quad \|c\|_0 \ll |\mathcal{A}|$
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The (Minkowski) gauge of $\mathcal{A}$ is:

$$\|x\|_{\mathcal{A}} = \inf\left\{t > 0 : \; x \in t \, \mathrm{conv}(\mathcal{A})\right\}$$

Assuming that $\mathcal{A}$ centrally symmetry about the origin ($a \in \mathcal{A} \;\Rightarrow\; -a \in \mathcal{A}$), $\|\cdot\|_{\mathcal{A}}$ is a norm, called the atomic norm (Chandrasekaran et al., 2012).

Atomic-Norm Regularization

The atomic norm

$$\|x\|_{\mathcal{A}} = \inf\Big\{t > 0 : \; x \in t\,\mathrm{conv}(\mathcal{A})\Big\}$$

$$= \inf\Big\{\sum_{i=1}^{|\mathcal{A}|} c_i : \; x = \sum_{i=1}^{|\mathcal{A}|} c_i\, a_i, \; c_i \geq 0\Big\}$$

...assuming that the centroid of $\mathcal{A}$ is at the origin.

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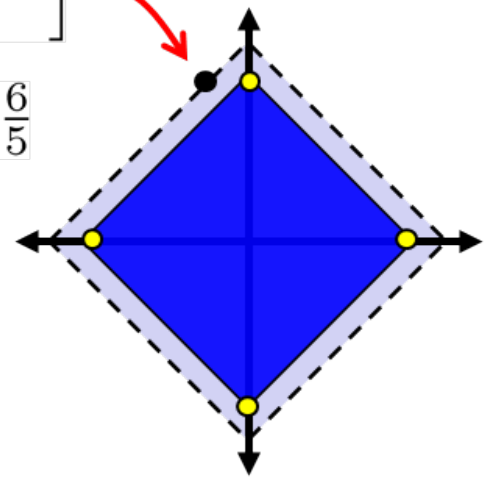
Example: the ℓ_1 norm as an atomic norm

- $\mathcal{A} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$
- $\mathrm{conv}(\mathcal{A}) = B_1(1)$ (ℓ_1 unit ball).
- $\|x\|_{\mathcal{A}} = \inf\left\{t > 0 : \; x \in t\, B_1(1)\right\}$
  $= \|x\|_1$

$$x = \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}$$

$$\|x\|_{\mathcal{A}} = \tfrac{6}{5}$$



Atomic Norms: More Examples

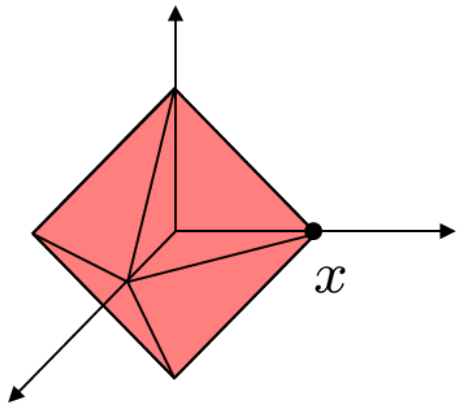
Examples with easy forms:

- *sparse vectors*

$$\mathcal{A} = \{\pm e_i\}_{i=1}^N$$

$$\text{conv}(\mathcal{A}) = \text{cross-polytope}$$

$$\|x\|_{\mathcal{A}} = \|x\|_1$$

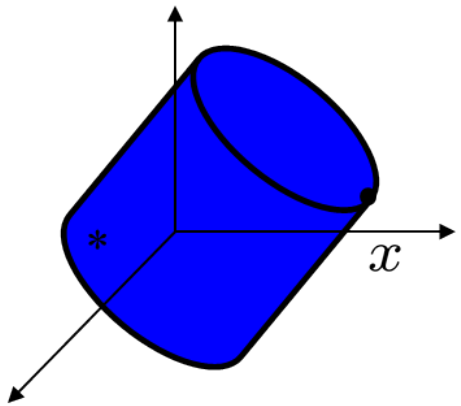


- *low-rank matrices*

$$\mathcal{A} = \{A : \text{rank}(A) = 1, \|A\|_F = 1\}$$

$$\text{conv}(\mathcal{A}) = \text{nuclear norm ball}$$

$$\|x\|_{\mathcal{A}} = \|x\|_\star$$



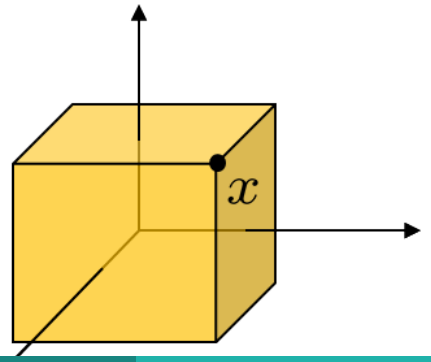
*symmetric matrices

- *binary vectors*

$$\mathcal{A} = \{\pm 1\}^N$$

$$\text{conv}(\mathcal{A}) = \text{hypercube}$$

$$\|x\|_{\mathcal{A}} = \|x\|_\infty$$



Atomic-Norm Regularization

Given an atomic set $\mathcal{A}$, we can adopt an Ivanov formulation

$$\min f(x) \text{ s.t. } \|x\|_{\mathcal{A}} \leq \delta$$

(for some $\delta > 0$) tends to recover x with sparse atomic representation.

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Yes! The conditional gradient (more later.)

Summary

- Many inference, learning, signal/image processing problems can be formulated as optimization problems.
- Sparsity-inducing regularizers play an important role in these problems
- There are several way to induce sparsity
- It is possible to formulate structured sparsity
- It is possible to extend the sparsity rationale to other objects, namely matrices
- Atomic norms provide a unified framework for sparsity/simplicity regularization

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Part 2: First-Order Methods

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CIMI-ANITI School on Optimisation, 2021

Focus (Initially) on Smooth Convex Functions

Consider $\min_{x \in \mathbb{R}^n} f(x)$, with f smooth and convex.

Usually assume $\mu I \preceq \nabla^2 f(x) \preceq LI, \ \forall_x$, with $0 \leq \mu \leq L$
(L is a Lipschitz constant of ∇f).

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If $\mu > 0$, then f is μ-strongly convex (as seen in Part 1) and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|_2^2.$$

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We are often interested in convex quadratics:

$$f(x) = \frac{1}{2} x^T A x, \qquad \mu I \preceq A \preceq LI \ \text{ or}$$

$$f(x) = \frac{1}{2} \|Bx - b\|_2^2, \quad \mu I \preceq B^T B \preceq LI$$

What's the Setup?

We consider iterative algorithms

$$x_{k+1} = \Phi(x_k), \quad \text{or} \quad x_{k+1} = \Phi(x_k, x_{k-1})$$

Assume we can evaluate $f(x_t)$ and $\nabla f(x_t)$ at each iteration.

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Later, we look at broader classes of problems:

- nonsmooth regularization; *i.e.*, instead of just $f(x)$, minimize $f(x) + \tau\psi(x)$;
- nonsmooth f;
- f not available (or too expensive to evaluate exactly);
- only an *estimate* of the gradient is available;
- a constraint $x \in \Omega$, usually for a simple Ω (e.g. ball, box, simplex).

We focus on algorithms that can be adapted to those scenarios.

Steepest Descent

Steepest descent (a.k.a. gradient descent):

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k), \qquad \text{for some } \alpha_k > 0.$$

Different ways to select an appropriate α_k.

1. Hard: interpolating scheme with safeguarding to identify an approximate minimizing α_k.

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2. Easy: backtracking. $\bar{\alpha}$, $\frac{1}{2}\bar{\alpha}$, $\frac{1}{4}\bar{\alpha}$, $\frac{1}{8}\bar{\alpha}$, ... until sufficient decrease in f is obtained.
3. Trivial: don't test for function decrease; use rules based on L and μ.

Analysis of 1 and 2 yields global convergence at unspecified rate, but not applicable to non-smooth problems.

Analysis of 3 focuses on convergence rate, and leads to accelerated multi-step methods.

Constant (Short) Steplength

By elementary use of Taylor's theorem, and since $\nabla^2 f(x) \preceq LI$,

$$f(x_{k+1}) \leq f(x_k) - \alpha_k \left(1 - \frac{\alpha_k}{2} L\right) \|\nabla f(x_k)\|_2^2$$

For $\alpha_k \equiv 1/L$, $\quad f(x_{k+1}) \leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|_2^2$,

thus $\quad \|\nabla f(x_k)\|^2 \leq 2L[f(x_k) - f(x_{k+1})]$

Summing for $k = 0, 1, \ldots, N$, and telescoping the sum,

$$\sum_{k=0}^{N} \|\nabla f(x_k)\|^2 \leq 2L[f(x_0) - f(x_{N+1})].$$

It follows that $\nabla f(x_k) \to 0$ if f is bounded below.

Rate Analysis

Suppose that the minimizer x^* is unique.

Another elementary use of Taylor's theorem shows that

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \alpha_k \left(\frac{2}{L} - \alpha_k\right) \|\nabla f(x_k)\|^2,$$

so that $\{\|x_k - x^*\|\}$ is decreasing.

Define for convenience: $\Delta_k := f(x_k) - f(x^*)$. By convexity, have

$$\Delta_k \leq \nabla f(x_k)^T (x_k - x^*) \leq \|\nabla f(x_k)\| \, \|x_k - x^*\| \leq \|\nabla f(x_k)\| \, \|x_0 - x^*\|.$$

From previous page (subtracting $f(x^*)$ from both sides of the inequality), and using the inequality above, we have

$$\Delta_{k+1} \leq \Delta_k - (1/2L)\|\nabla f(x_k)\|^2 \leq \Delta_k - \frac{1}{2L\|x_0 - x^*\|^2} \Delta_k^2.$$

Weakly convex: $1/k$ sublinear; Strongly convex: linear

Take reciprocal of both sides and manipulate (using $(1-\epsilon)^{-1} \geq 1+\epsilon$):

$$\frac{1}{\Delta_{k+1}} \geq \frac{1}{\Delta_k} + \frac{1}{2L\|x_0 - x^*\|^2} \geq \frac{1}{\Delta_0} + \frac{k+1}{2L\|x_0 - x^*\|^2},$$

which yields

$$f(x_{k+1}) - f(x^*) \leq \frac{2L\|x_0 - x\|^2}{k+1}.$$

The classic $1/k$ convergence rate!

By assuming $\mu > 0$, can set $\alpha_k \equiv 2/(\mu + L)$ and get a linear (geometric) rate: Much better than sublinear, in the long run

$$\|x_k - x^*\|^2 \leq \left(\frac{L-\mu}{L+\mu}\right)^{2k} \|x_0 - x^*\|^2 = \left(1 - \frac{2}{\kappa+1}\right)^{2k} \|x_0 - x^*\|^2.$$

Weakly convex: $1/k$ sublinear; Strongly convex: linear

Since by Taylor's theorem we have

$$\Delta_k = f(x_k) - f(x^*) \leq (L/2)\|x_k - x^*\|^2,$$

it follows immediately that

$$f(x_k) - f(x^*) \leq \frac{L}{2}\left(1 - \frac{2}{\kappa + 1}\right)^{2k} \|x_0 - x^*\|^2.$$

Note: A geometric/linear rate is generally better than almost any sublinear ($1/k$ or $1/k^2$) rate.

Detour: Convergence Rates

Sequence t_k, for $k = 1, 2, \ldots$, that converges to zero, $\lim_{k \to \infty} t_k = 0$

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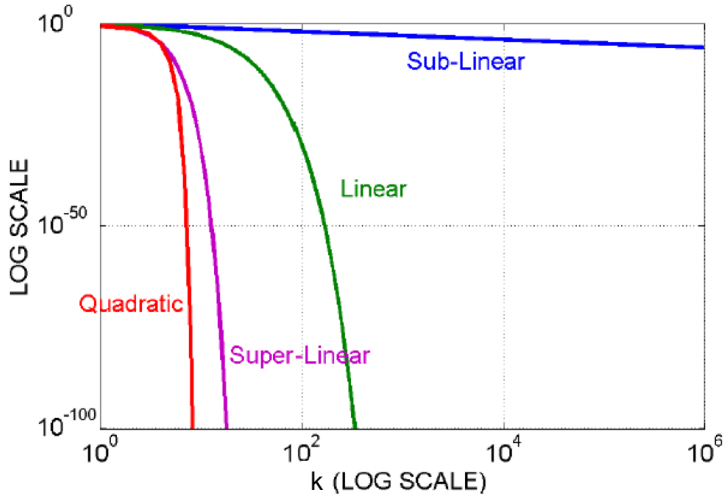
The above sequences with $\gamma = 0.5$



Multistep Methods: The Heavy-Ball

Enhanced search direction with a contribution from the previous step.
(known as heavy ball, momentum, or two-step)

Consider first a constant step length α, and a second parameter β for the "momentum" term:

$$x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta(x_k - x_{k-1})$$

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Analyze by defining a composite iterate vector:

$$w_k := \begin{bmatrix} x_k - x^* \\ x_{k-1} - x^* \end{bmatrix}.$$

Thus

$$w_{k+1} = B w_k + o(\|w_k\|), \qquad B := \begin{bmatrix} -\alpha \nabla^2 f(x^*) + (1+\beta) I & -\beta I \\ I & 0 \end{bmatrix}.$$

Multistep Methods: The Heavy-Ball

Matrix B has same eigenvalues as

$$\begin{bmatrix} -\alpha\Lambda + (1+\beta)I & -\beta I \\ I & 0 \end{bmatrix}, \qquad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where λ_i are the eigenvalues of $\nabla^2 f(x^*)$.

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Choose α, β to explicitly minimize the max eigenvalue of B, obtain

$$\alpha = \frac{4}{L}\frac{1}{(1+1/\sqrt{\kappa})^2}, \quad \beta = \left(1 - \frac{2}{\sqrt{\kappa}+1}\right)^2.$$

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Leads to linear convergence for $\|x_k - x^*\|$ with rate approximately

$$\left(1 - \frac{2}{\sqrt{\kappa}+1}\right).$$

Summary: Linear Convergence, Strictly Convex f

- Steepest descent: Linear rate approx $\left(1 - \frac{2}{\kappa}\right)$;
- Heavy-ball: Linear rate approx $\left(1 - \frac{2}{\sqrt{\kappa}}\right)$.

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Big difference! To reduce $\|x_k - x^*\|$ by a factor ϵ, need k large enough that

$$\left(1 - \frac{2}{\kappa}\right)^k \leq \epsilon \quad \Leftarrow \quad k \geq \frac{\kappa}{2}|\log \epsilon| \quad \text{(steepest descent)}$$

$$\left(1 - \frac{2}{\sqrt{\kappa}}\right)^k \leq \epsilon \quad \Leftarrow \quad k \geq \frac{\sqrt{\kappa}}{2}|\log \epsilon| \quad \text{(heavy-ball)}$$

A factor of $\sqrt{\kappa}$ difference; e.g. if $\kappa = 1000$ (not at all uncommon in inverse problems), need ~ 30 times fewer steps.

Accelerated First-Order Methods

Accelerate the rate to $1/k^2$ for weakly convex, while retaining the linear rate (related to $\sqrt{\kappa}$) for strongly convex case.

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Nesterov (1983) describes a method that requires κ.

Initialize: Choose x_0, $\alpha_0 \in (0,1)$; set $y_0 \leftarrow x_0$.

Iterate: $x_{k+1} \leftarrow y_k - \frac{1}{L}\nabla f(y_k)$; (*short-step*)

find $\alpha_{k+1} \in (0,1)$: $\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + \frac{\alpha_{k+1}}{\kappa}$;

set $\beta_k = \dfrac{\alpha_k(1-\alpha_k)}{\alpha_k^2 + \alpha_{k+1}}$;

set $y_{k+1} \leftarrow x_{k+1} + \beta_k(x_{k+1} - x_k)$.

Still works for weakly convex ($\kappa = \infty$).

Convergence Results: Nesterov

If $\alpha_0 \geq 1/\sqrt{\kappa}$, have

$$f(x_k) - f(x^*) \leq c_1 \min\left(\left(1 - \frac{1}{\sqrt{\kappa}}\right)^k, \frac{4L}{(\sqrt{L} + c_2 k)^2}\right),$$

where constants c_1 and c_2 depend on x_0, α_0, L.

- Linear convergence "heavy-ball" rate for strongly convex f;
- $1/k^2$ sublinear rate otherwise.

In the special case of $\alpha_0 = 1/\sqrt{\kappa}$, this scheme yields

$$\alpha_k \equiv \frac{1}{\sqrt{\kappa}}, \qquad \beta_k \equiv 1 - \frac{2}{\sqrt{\kappa} + 1}.$$

Beck and Teboulle

Beck and Teboulle (2009) propose a similar algorithm.

Initialize: Choose x_0; set $y_1 = x_0$, $t_1 = 1$;

Iterate:
$$x_k \leftarrow y_k - \tfrac{1}{L}\nabla f(y_k);$$
$$t_{k+1} \leftarrow \tfrac{1}{2}\left(1 + \sqrt{1 + 4t_k^2}\right);$$
$$y_{k+1} \leftarrow x_k + \frac{t_k - 1}{t_{k+1}}(x_k - x_{k-1}).$$

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For (weakly) convex f, converges with $f(x_k) - f(x^*) \sim 1/k^2$.

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Beck and Teboulle (2009) do the convergence analysis in 2-3 pages; elementary, but not intuitive.

A Non-Monotone Gradient Method: Barzilai-Borwein

Barzilai and Borwein (1988) (BB) proposed an unusual choice of α_k.
Allows f to increase (sometimes a lot) on some steps: non-monotone.

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k), \qquad \alpha_k := \arg\min_{\alpha} \|s_k - \alpha z_k\|^2,$$

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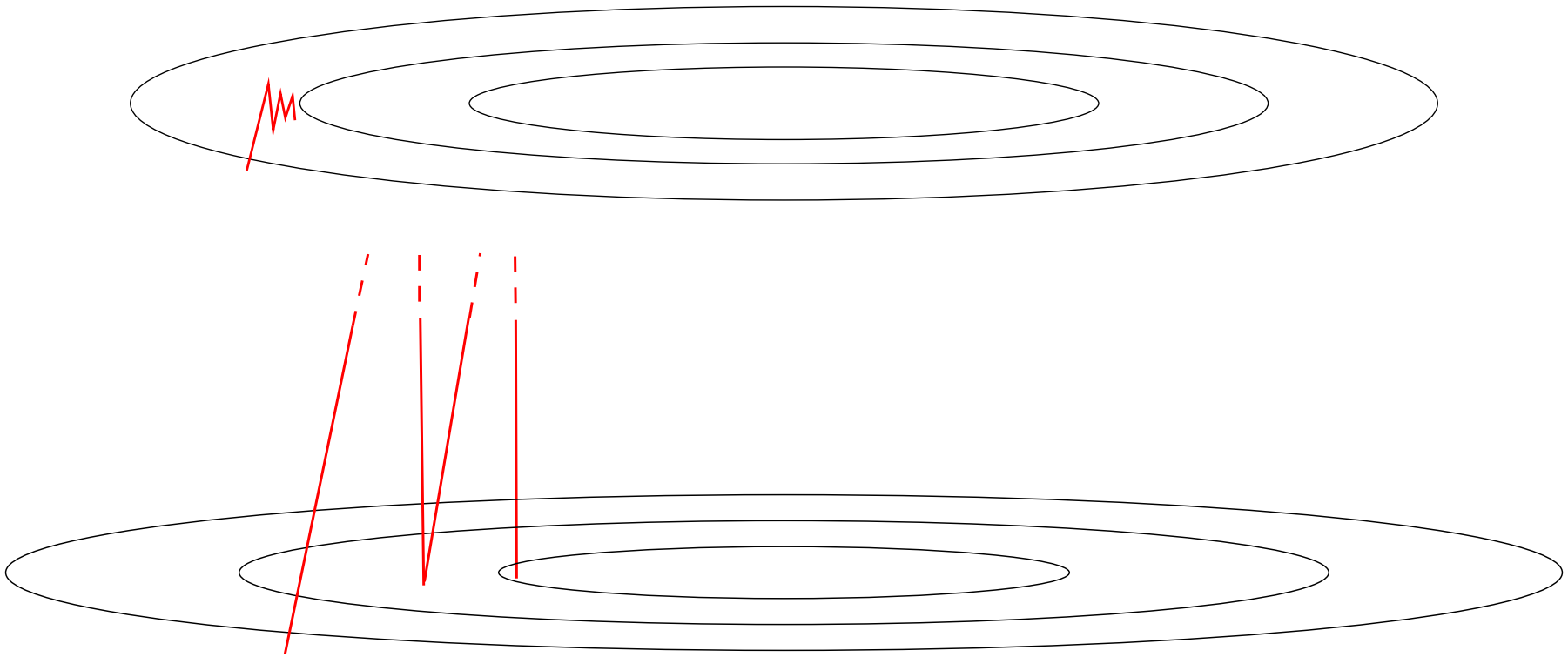
BB can be seen as a quasi-Newton method, with Hessian $\simeq \alpha_k^{-1} I$.

Comparison: BB vs Greedy Steepest Descent



Extending to the Constrained Case: $x \in \Omega$

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Example: Nesterov's constant step scheme requires just one calculation to be changed from the unconstrained version.

Initialize: Choose x_0, $\alpha_0 \in (0,1)$; set $y_0 \leftarrow x_0$.

Iterate: $x_{k+1} \leftarrow \arg\min_{y \in \Omega} \frac{1}{2}\|y - [y_k - \frac{1}{L}\nabla f(y_k)]\|_2^2$;

find $\alpha_{k+1} \in (0,1)$: $\alpha_{k+1}^2 = (1-\alpha_{k+1})\alpha_k^2 + \frac{\alpha_{k+1}}{\kappa}$;

set $\beta_k = \frac{\alpha_k(1-\alpha_k)}{\alpha_k^2+\alpha_{k+1}}$;

set $y_{k+1} \leftarrow x_{k+1} + \beta_k(x_{k+1} - x_k)$.

Convergence theory is unchanged.

Regularized Optimization

How to change these methods to handle regularizers?

$$\min_x \ f(x) + \tau\psi(x),$$

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$$x_{k+1} = \arg\min_x \|x - \Phi(x_k)\|_2^2 + \lambda\psi(x).$$

where $\Phi(x_k)$ is gradient descent step, or something more complicated (such as heavy ball, with $\Phi(x_k, x_{k-1})$, or some other accelerated method).

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This is the shrinkage/tresholding step; how to solve it with a nonsmooth ψ? That is the topic of the following slides.

Handling Nonsmoothness (e.g. ℓ_1 Norm)

Convexity $\Rightarrow$ continuity (on the domain of the function).

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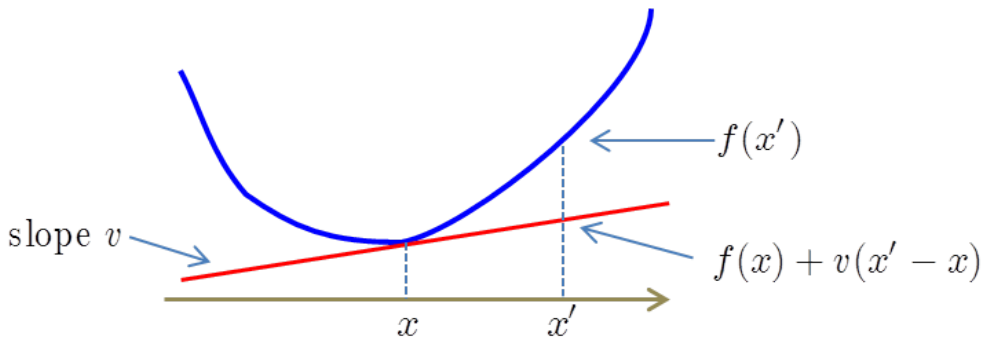
Subdifferential: $\partial f(x) = \{\text{all subgradients of } f \text{ at } x\}$



linear lower bound

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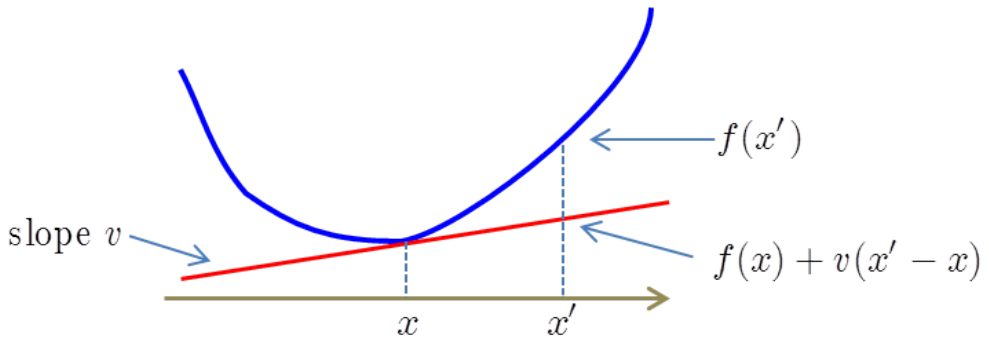

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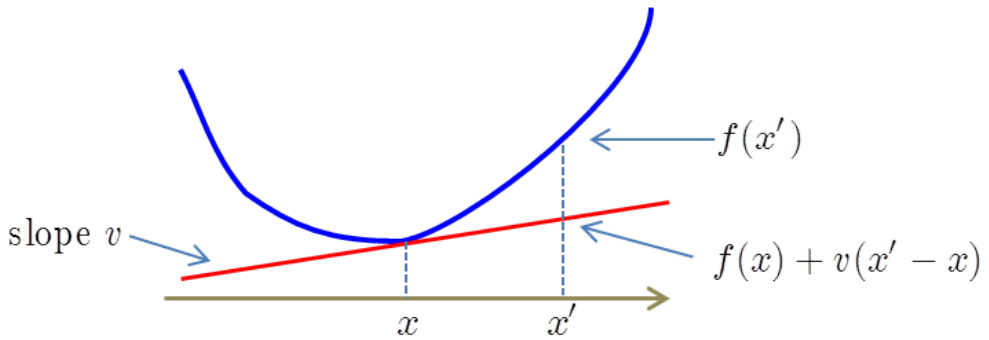

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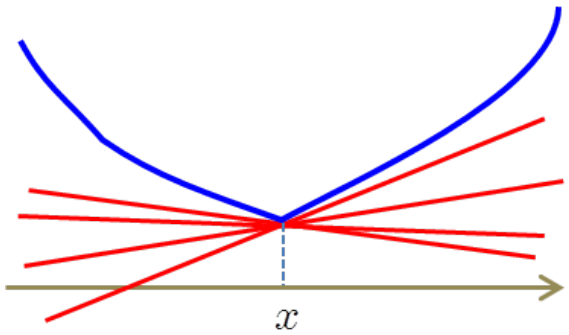

nondifferentiable case

More on Subgradients and Subdifferentials

The subdifferential is a set-valued function:

$$f : \mathbb{R}^d \to \mathbb{R} \quad \Rightarrow \quad \partial f : \mathbb{R}^d \to 2^{\mathbb{R}^d} \quad \text{(power set of } \mathbb{R}^d\text{)}$$

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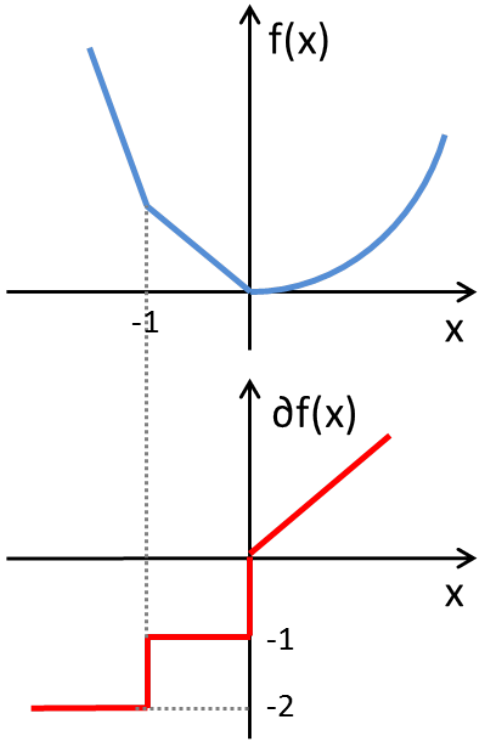
Example:

$$f(x) = \begin{cases} -2x - 1, & x \leq -1 \\ -x, & -1 < x \leq 0 \\ x^2/2, & x > 0 \end{cases}$$

$$\partial f(x) = \begin{cases} \{-2\}, & x < -1 \\ [-2, -1], & x = -1 \\ \{-1\}, & -1 < x < 0 \\ [-1, 0], & x = 0 \\ \{x\}, & x > 0 \end{cases}$$



More on Subgradients and Subdifferentials

The subdifferential is a set-valued function:

$$f : \mathbb{R}^d \to \mathbb{R} \quad \Rightarrow \quad \partial f : \mathbb{R}^d \to 2^{\mathbb{R}^d} \quad \text{(power set of } \mathbb{R}^d\text{)}$$

Example:

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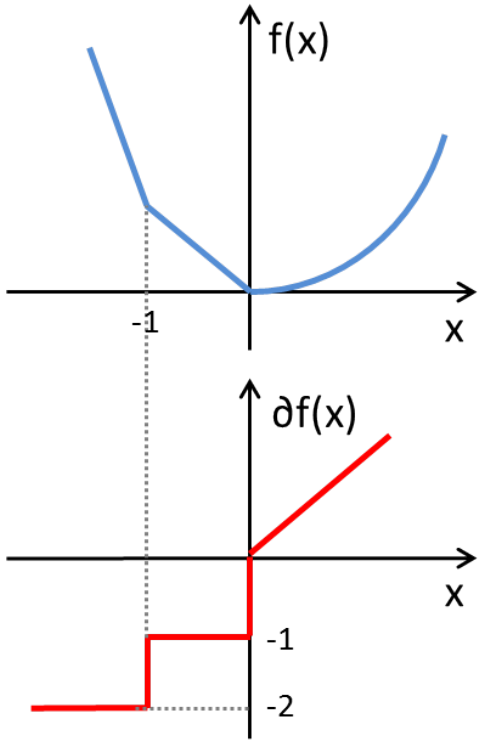

Fermat's Rule: $x \in \arg\min_x f(x) \Leftrightarrow 0 \in \partial f(x)$

A Key Tool: Moreau's Proximity Operators

Moreau (1962) proximity operator

$$\boxed{\widehat{x} \in \arg\min_x \frac{1}{2}\|x - y\|_2^2 + \psi(x) =: \text{prox}_\psi(y)}$$

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Block separability: $x = (x_1, ..., x_N)$ (a partition of the components of x)

$$\psi(x) = \sum_i \psi_i(x_i) \;\Rightarrow\; (\text{prox}_\psi(y))_i = \text{prox}_{\psi_i}(y_i)$$

Relationship with subdifferential: $z = \text{prox}_\psi(y) \;\Leftrightarrow\; z - y \in \partial\psi(z)$

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Resolvent: $z = \text{prox}_\psi(y) \;\Leftrightarrow\; 0 \in \partial\psi(z) + (z-y) \;\Leftrightarrow\; y \in (\partial\psi + I)z$

$$\text{prox}_\psi(y) = (\partial\psi + I)^{-1} y$$

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- Euclidean norm (not separable, nonsmooth):

$$\text{prox}_{\tau\|\cdot\|_2}(y) = \begin{cases} \frac{x}{\|x\|_2}(\|x\|_2 - \tau), & \text{if } \|x\|_2 > \tau \\ 0 & \text{if } \|x\|_2 \leq \tau \end{cases}$$

More Proximity Operators

	$\phi(x)$	$\operatorname{prox}_{\phi} x$
i	$\iota_{[\underline{\omega},\overline{\omega}]}(x)$	$P_{[\underline{\omega},\overline{\omega}]}\, x$
ii	$\sigma_{[\underline{\omega},\overline{\omega}]}(x) = \begin{cases} \underline{\omega} x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \overline{\omega} x & \text{otherwise} \end{cases}$	$\operatorname{soft}_{[\underline{\omega},\overline{\omega}]}(x) = \begin{cases} x - \underline{\omega} & \text{if } x < \underline{\omega} \\ 0 & \text{if } x \in [\underline{\omega},\overline{\omega}] \\ x - \overline{\omega} & \text{if } x > \overline{\omega} \end{cases}$
iii	$\psi(x) + \sigma_{[\underline{\omega},\overline{\omega}]}(x)$ $\psi \in \Gamma_0(\mathbb{R})$ differentiable at 0 $\psi'(0) = 0$	$\operatorname{prox}_{\psi}(\operatorname{soft}_{[\underline{\omega},\overline{\omega}]}(x))$
iv	$\max\{\lvert x\rvert - \omega, 0\}$	$\begin{cases} x & \text{if } \lvert x\rvert < \omega \\ \operatorname{sign}(x)\omega & \text{if } \omega \leq \lvert x\rvert \leq 2\omega \\ \operatorname{sign}(x)(\lvert x\rvert - \omega) & \text{if } \lvert x\rvert > 2\omega \end{cases}$
v	$\kappa\lvert x\rvert^q$	$\operatorname{sign}(x)p$, where $p \geq 0$ and $p + q\kappa p^{q-1} = \lvert x\rvert$
vi	$\begin{cases} \kappa x^2 & \text{if } \lvert x\rvert \leq \omega/\sqrt{2\kappa} \\ \omega\sqrt{2\kappa}\lvert x\rvert - \omega^2/2 & \text{otherwise} \end{cases}$	$\begin{cases} x/(2\kappa+1) & \text{if } \lvert x\rvert \leq \omega(2\kappa+1)/\sqrt{2\kappa} \\ x - \omega\sqrt{2\kappa}\operatorname{sign}(x) & \text{otherwise} \end{cases}$
vii	$\omega\lvert x\rvert + \tau\lvert x\rvert^2 + \kappa\lvert x\rvert^q$	$\operatorname{sign}(x)\operatorname{prox}_{\kappa\lvert\cdot\rvert^q/(2\tau+1)}\dfrac{\max\{\lvert x\rvert - \omega, 0\}}{2\tau+1}$
viii	$\omega\lvert x\rvert - \ln(1+\omega\lvert x\rvert)$	$(2\omega)^{-1}\operatorname{sign}(x)\Big(\omega\lvert x\rvert - \omega^2 - 1 + \sqrt{\lvert\omega\lvert x\rvert - \omega^2 - 1\rvert^2 + 4\omega\lvert x\rvert}\Big)$
ix	$\begin{cases} \text{if } x \geq 0 \\ +\infty \quad \text{otherwise} \end{cases}$	$\begin{cases} x - \omega & \text{if } x \\ & \text{otherwise} \end{cases}$
x	$\begin{cases} & x \\ & \text{otherwise} \end{cases}$	p where p and p
xi	$\begin{cases} \omega x^{-q} & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$p > 0$ such that $p^{q+2} - xp^{q+1} = \omega q$
xii	$\begin{cases} x\ln(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ +\infty & \text{otherwise} \end{cases}$	$W(e^{x-1})$, where W is the Lambert W-function
xiii	$\begin{cases} -\ln(x-\underline{\omega}) + \ln(-\underline{\omega}) & \text{if } x \in]\underline{\omega}, 0] \\ -\ln(\overline{\omega} - x) + \ln(\overline{\omega}) & \text{if } x \in]0, \overline{\omega}[\\ +\infty & \text{otherwise} \end{cases}$ $\underline{\omega} < 0 < \overline{\omega}$	$\begin{cases} \frac{1}{2}\Big(x + \underline{\omega} + \sqrt{\lvert x-\underline{\omega}\rvert^2 + 4}\Big) & \text{if } x < 1/\underline{\omega} \\ \frac{1}{2}\Big(x + \overline{\omega} - \sqrt{\lvert x-\overline{\omega}\rvert^2 + 4}\Big) & \text{if } x > 1/\overline{\omega} \\ 0 & \text{otherwise} \end{cases}$ (see Figure 1)
xiv	$\begin{cases} -\kappa\ln(x) + \tau x^2/2 + \alpha x & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$\dfrac{1}{2(1+\tau)}\Big(x - \alpha + \sqrt{\lvert x-\alpha\rvert^2 + 4\kappa(1+\tau)}\Big)$
xv	$\begin{cases} -\kappa\ln(x) + \alpha x + \omega x^{-1} & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$p > 0$ such that $p^3 + (\alpha - x)p^2 - \kappa p = \omega$
xvi	$\begin{cases} -\kappa\ln(x) + \omega x^q & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$	$p > 0$ such that $q\omega p^q + p^2 - xp = \kappa$
xvii	$\begin{cases} -\underline{\kappa}\ln(x-\underline{\omega}) - \overline{\kappa}\ln(\overline{\omega} - x) & \\ \qquad \text{if } x \in]\underline{\omega}, \overline{\omega}[& \\ +\infty \quad \text{otherwise} & \end{cases}$	$p \in]\underline{\omega}, \overline{\omega}[$ such that $p^3 - (\underline{\omega} + \overline{\omega} + x)p^2 +$ $(\underline{\omega}\overline{\omega} - \underline{\kappa} - \overline{\kappa} + (\underline{\omega} + \overline{\omega})x)p = \underline{\omega}\overline{\omega}x - \underline{\omega}\overline{\kappa} - \overline{\omega}\underline{\kappa}$

Many others!

(Combettes and Pesquet, 2011)

Another Key Tool: Fenchel-Legendre Conjugates

The Fenchel-Legendre conjugate of a proper convex function f — denoted by $f^* : \mathbb{R}^n \to \bar{\mathbb{R}}$ — is defined by

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  ...meaning that, if you know prox_f, you know prox_{f^*}, and vice-versa.
- Conjugate of indicator: if $f(x) = \iota_C(x)$, where C is a convex set,

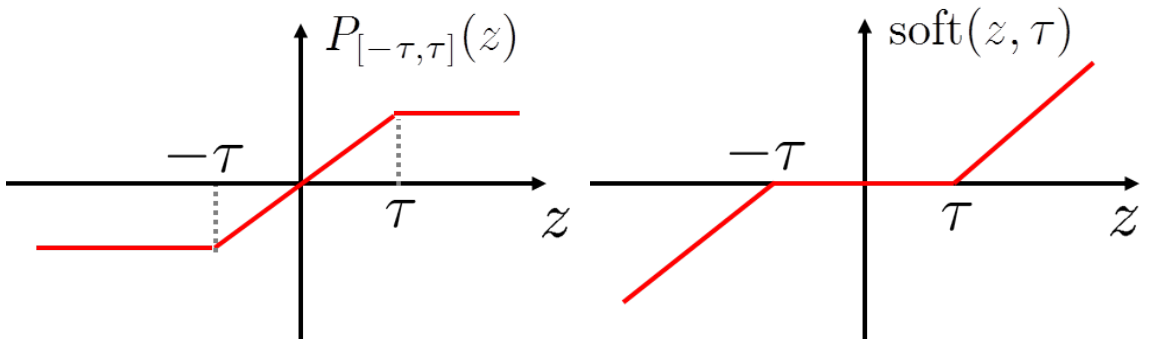
  $f^*(u) = \sup_x x^T u - \iota_C(x) = \sup_{x \in C} x^T u \equiv \sigma_C(u)$ (support function of C).

From Conjugates to Proximity Operators

Notice that $|u| = \sup_{x \in [-1,1]} x^T u = \sigma_{[-1,1]}(u)$, thus $|\cdot|^* = \iota_{[-1,1]}$.

Using Moreau's decomposition, we easily derive the soft-threshold:

$$\text{prox}_{\tau|\cdot|} = 1 - \text{prox}_{\iota_{[-\tau,\tau]}} = 1 - P_{[-\tau,\tau]} = \text{soft}(\cdot, \tau)$$



Conjugate of a norm: if $f(x) = \tau\|x\|_p$ then $f^* = \iota_{\{x:\|x\|_q \leq \tau\}}$,

where $\frac{1}{q} + \frac{1}{p} = 1$ (a Hölder pair, or Hölder conjugates).

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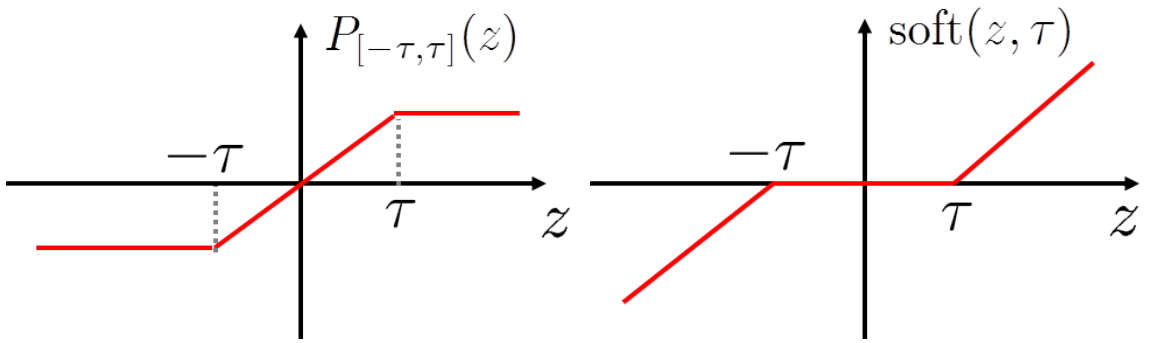

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That is, $\|\cdot\|_p$ and $\|\cdot\|_q$ are dual norms:

$$\|z\|_q = \sup\{x^T z : \|x\|_p \leq 1\} = \sup_{x \in B_p(1)} x^T z = \sigma_{B_p(1)}(z)$$

From Conjugates to Proximity Operators

- Proximity of norm:

$$\boxed{\text{prox}_{\tau\|\cdot\|_p} = I - P_{B_q(\tau)}}$$

where $B_q(\tau) = \{x : \ \|x\|_q \leq \tau\}$ and $\frac{1}{q} + \frac{1}{p} = 1$.

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- Example: computing $\text{prox}_{\|\cdot\|_\infty}$ (notice ℓ_∞ is not separable):

Since $\frac{1}{\infty} + \frac{1}{1} = 1$,

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... the proximity operator of ℓ_∞ norm is the residual of the projection on an ℓ_1 ball.

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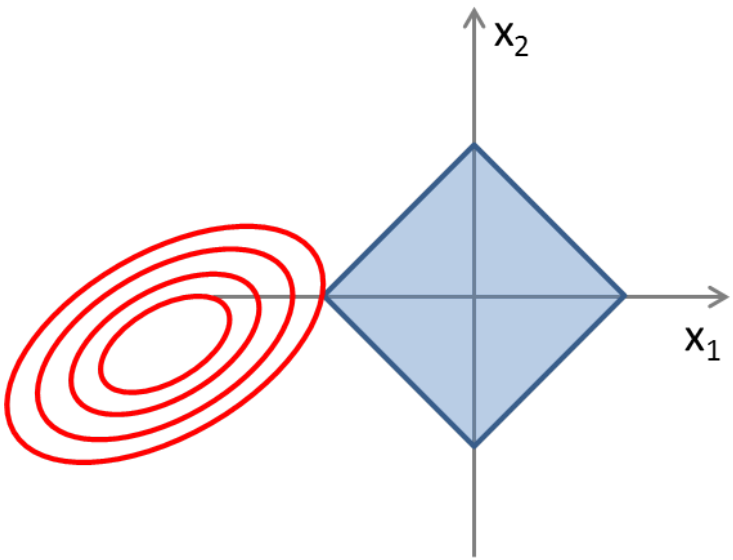
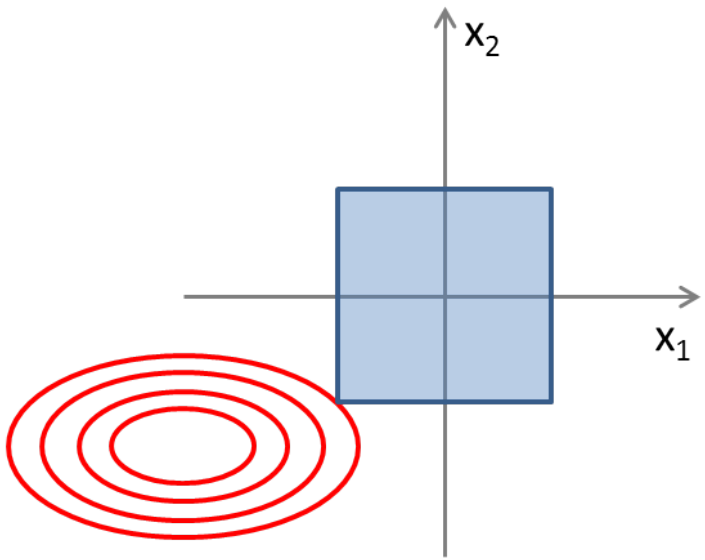
- Projection on ℓ_1 ball has no closed form, but there are efficient (linear cost) algorithms (Brucker, 1984), (Maculan and de Paula, 1989).

Geometry and Effect of $\text{prox}_{\ell_\infty}$

Whereas ℓ_1 promotes sparsity, ℓ_∞ promotes equality (in absolute value).

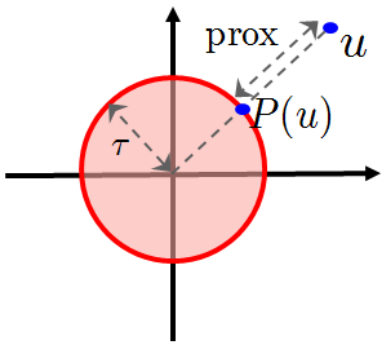




From Conjugates to Proximity Operators

The dual of the ℓ_2 norm is the ℓ_2 norm.

$$\text{prox}_{\tau\|\cdot\|_2}(u) = u - P_{\{x:\|x\|_2\leq\tau\}}(u)$$



$$= u - \begin{cases} u & \Leftarrow \quad \|u\|_2 \leq \tau \\ \tau\, u/\|u\|_2 & \Leftarrow \quad \|u\|_2 > \tau \end{cases}$$

$$= \frac{u}{\|u\|_2} \max\{0, \|u\|_2 - \tau\}$$

vector soft thresholding

Matrix Nuclear Norm and its Prox Operator

- Recall the trace/nuclear norm: $\|X\|_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i$.
- The dual of a Schatten p-norm is a Schatten q-norm, with $\frac{1}{q} + \frac{1}{p} = 1$. Thus, the dual of the nuclear norm is the spectral norm:

$$\|X\|_\infty = \max\left\{\sigma_1, ..., \sigma_{\min\{m,n\}}\right\}.$$

- If $Y = U\Lambda V^T$ is the SVD of Y, we have

$$\begin{aligned}\text{prox}_{\tau\|\cdot\|_*}(Y) &= U\Lambda V^T - P_{\{X:\max\{\sigma_1,...,\sigma_{\min\{m,n\}}\}\leq\tau\}}(U\Lambda V^T) \\ &= U\,\text{soft}\big(\Lambda,\tau\big)V^T.\end{aligned}$$

Atomic Norms: A Unified View

vectors			matrices		
norm	prox	atomic set	norm	prox	atomic set
ℓ_1 $\|x\|_1$	component soft thresholding	$\mathcal{A} = \{\pm e_i\}$ $\|\mathcal{A}\| = 2N$	nuclear $\|X\|_*$	singular value thresholding	$\mathcal{A}$ = set of all rank 1, norm 1 matrices
ℓ_∞ $\|x\|_\infty$	residual of projection on ℓ_1 ball	$\mathcal{A} = \{\pm 1\}^N$ $\|\mathcal{A}\| = 2^N$	spectral $\|X\|_2$	residual of s.v. proj. on ℓ_1 ball	$\mathcal{A}$ = set of all orthogonal matrices
ℓ_2 $\|x\|_2$	vector soft thresholding	$\mathcal{A}$ = set of all vectors with norm 1 $\|\mathcal{A}\| = \infty$	Frobenius $\|X\|_F$	matrix soft threshold.	$\mathcal{A}$ = all matrices of unit Frobenius norm.

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- ... often has the form: $\min_x g(Ax) + \psi(x)$

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$$\begin{aligned}
\min_x g(Ax) + \psi(x) &= \inf_x \sup_u u^T Ax - g^*(u) + \psi(x) \\
&= \sup_u (-g^*(u) + \inf_x u^T Ax + \psi(x)) \\
&= \sup_u (-g^*(u) - \underbrace{\sup_x -x^T A^T u - \psi(x)}_{\psi^*(-A^T u)}) \\
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- The problem $\inf_u g^*(u) + \psi^*(-A^T u)$ is sometimes easier to handle.

Basic Proximal-Gradient Algorithm

Use basic structure:

$$x_k = \arg\min_x \|x - \Phi(x_k)\|_2^2 + \psi(x).$$

with $\Phi(x_k)$ a simple gradient descent step, thus

$$\boxed{x_{k+1} = \text{prox}_{\alpha_k \psi}\big(x_k - \alpha_k \nabla f(x_k)\big)}$$

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Reinvented several times in different communities: optimization, PDEs, convex analysis, signal processing, machine learning.

Convergence of the Proximal-Gradient Algorithm

- Basic algorithm: $x_{k+1} = \text{prox}_{\alpha_k \psi}\big(x_k - \alpha_k \nabla f(x_k)\big)$

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- Generalized (possibly inexact) version:

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- Convergence is guaranteed (Combettes and Wajs, 2006) if
  - ✓ $0 < \inf \alpha_k \leq \sup \alpha_k < \frac{2}{L}$
  - ✓ $\lambda_k \in (0, 1]$, with $\inf \lambda_k > 0$
  - ✓ $\sum_k^\infty \|a_k\| < \infty$ and $\sum_k^\infty \|b_k\| < \infty$

Proximal-Gradient Algorithm: Quadratic Case

- Consider the quadratic case (of great interest): $f(x) = \frac{1}{2}\|Bx - b\|_2^2$.

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- Even more refined results are available if $\psi(x) = \tau\|x\|_1$

More on IST/FBS/PGA for the ℓ_2-ℓ_1 Case

- Problem: $\widehat{x} \in G = \arg\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Bx - b\|_2^2 + \tau\|x\|_1$ (recall $B^T B \preceq LI$)
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- The zero set: $\mathcal{Z} \subseteq \{1, ..., n\}$: $\widehat{x} \in G \Rightarrow \widehat{x}_{\mathcal{Z}} = 0$
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- After that, if $B_{\mathcal{Z}}^T B_{\mathcal{Z}} \succeq \mu I$, with $\mu > 0$ (thus $\kappa(B_{\mathcal{Z}}^T B_{\mathcal{Z}}) = L/\mu < \infty$), we have linear convergence

$$\|x_{k+1} - \widehat{x}\|_2 \leq \frac{1-\kappa}{1+\kappa}\|x_k - \widehat{x}\|_2$$

for the optimal choice $\alpha = 2/(L + \mu)$ (see unconstrained theory).

Heavy Ball Acceleration: FISTA

- FISTA (*fast iterative shrinkage-thresholding algorithm*) is heavy-ball-type acceleration of IST (based on Nesterov (1983)) (Beck and Teboulle, 2009).

Initialize: Choose $\alpha \leq 1/L$, x_0; set $y_1 = x_0$, $t_1 = 1$;

Iterate: $x_k \leftarrow \text{prox}_{\tau\alpha\psi}\big(y_k - \alpha\nabla f(y_k)\big)$;

$$t_{k+1} \leftarrow \tfrac{1}{2}\left(1 + \sqrt{1 + 4t_k^2}\right);$$

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- When L is not known, increase an estimate of L until it's big enough.

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$$\min_x \tfrac{1}{2}\|B\,x - b\|_2^2 + \tau\psi(x)$$

- Iterations (with $\alpha < 2/L$)

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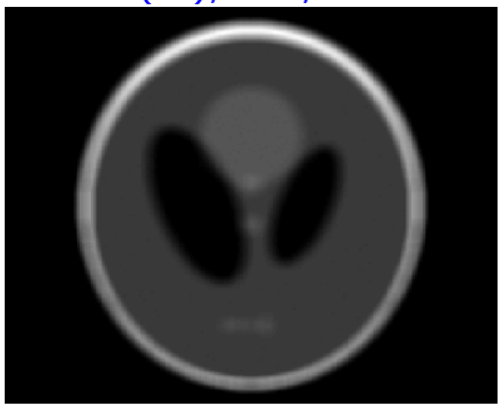
$$\|x_{k+1} - \widehat{x}\|_2 \leq \frac{1-\sqrt{\kappa}}{1+\sqrt{\kappa}} \|x_k - \widehat{x}\|_2 \quad \left(\text{versus } \tfrac{1-\kappa}{1+\kappa} \text{ for IST}\right)$$

Illustration of the TwIST Acceleration

original

Blurred (B), 9x9, 40db noise

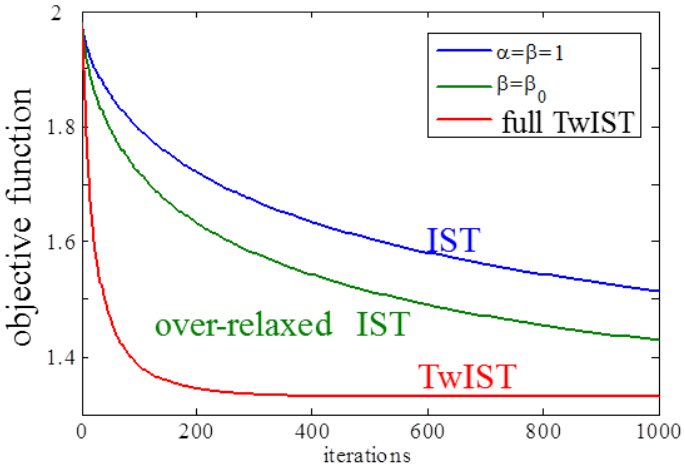


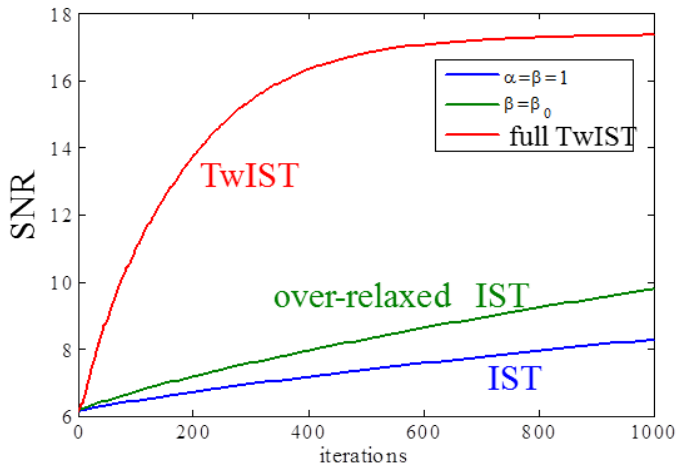
restored

$$\widehat{x} \quad \in \quad \arg\min_{x\in\mathbb{R}^n} \frac{1}{2}\|B\Psi x - u\|_2^2 + \tau\|x\|_1$$

representation coefficients

dictionary (e.g, wavelet basis, frame, ...)





Acceleration via Larger Steps: SpaRSA

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  - ✓ Barzilai-Borwein (see above), to mimic Newton steps — or at least get the scaling right.
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  - ✓ Barzilai-Borwein (see above), to mimic Newton steps — or at least get the scaling right.
  - ✓ keep increasing α_k until monotonicity is violated: backtrack.
- Convergence to critical points (minima in the convex case) is guaranteed for a safeguarded version: ensure sufficient decrease w.r.t. the worst value in previous M iterations.

Another Approach: Gradient Projection

- $\min_x \frac{1}{2}\|Bx - b\|_2^2 + \tau\|x\|_1$ can be written as a standard QP:

$$\min_{u,v} \frac{1}{2}\|B(u-v) - b\|_2^2 + \tau u^T 1 + \tau u^T 1 \quad \text{s.t.} \quad u \geq 0,\ v \geq 0,$$

where $u_i = \max\{0, x_i\}$ and $v_i = \max\{0, -x_i\}$.

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- With $z = \begin{bmatrix} u \\ v \end{bmatrix}$, problem can be written in canonical form

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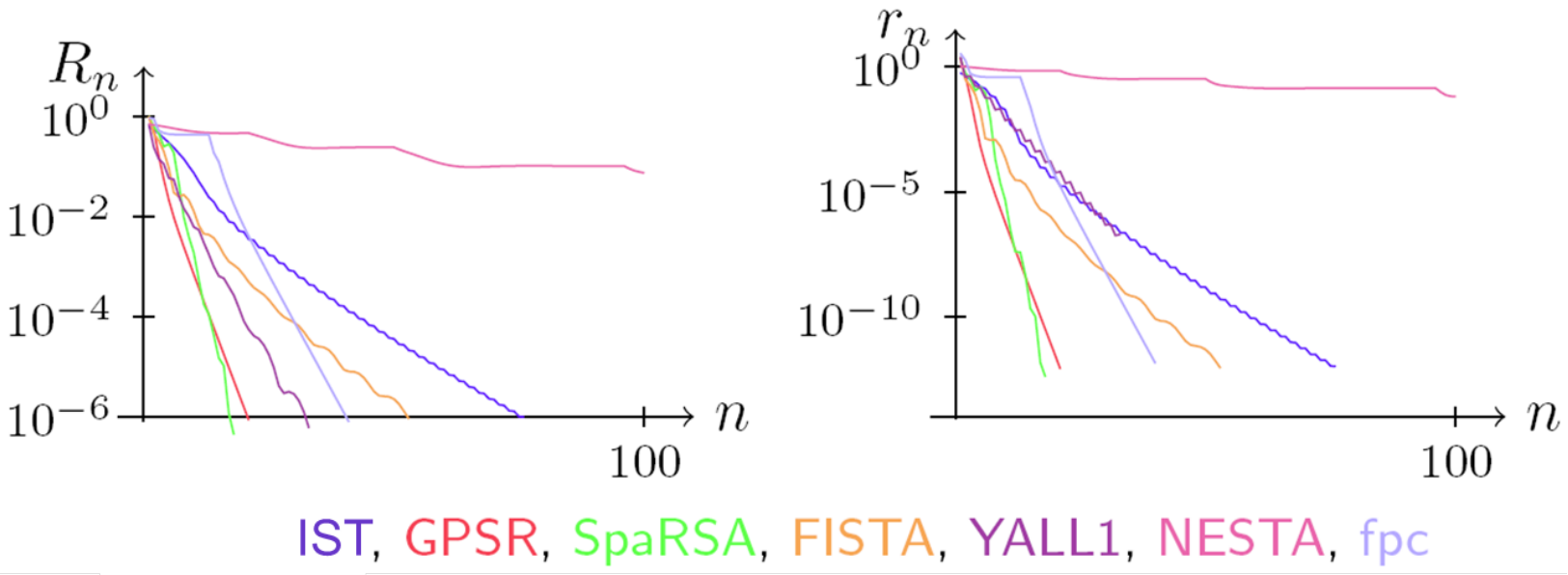
- Solving this problem with projected gradient using Barzilai-Borwein steps: GPSR (gradient projection for sparse reconstruction).

Speed Comparisons

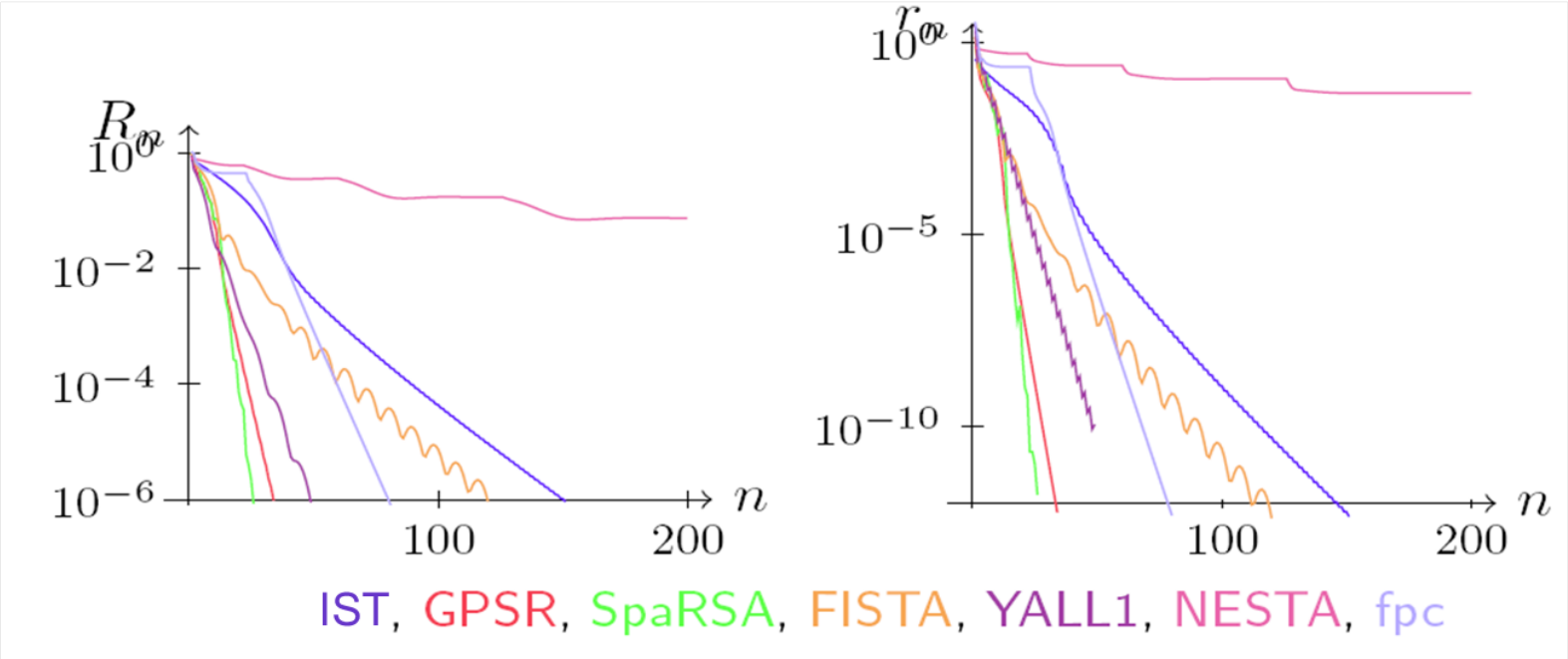
- Lorenz (2011) proposed a way of generating problem instances with known solution $\widehat{x}$: useful for speed comparison.
- Define: $R_k = \frac{\|x_k - \widehat{x}\|_2}{\|\widehat{x}\|_2}$ and $r_k = \frac{L(x_k) - L(\widehat{x})}{L(\widehat{x})}$ (where $L(x) = f(x) + \tau\psi(x)$).

Typical CS example: $\mathbf{A} = [\mathbf{I}\ \mathbf{U}]$ (512 x 1024), $\widehat{\mathbf{x}}$ has 80 non-zeros, $\tau = 0.1$



More Speed Comparisons

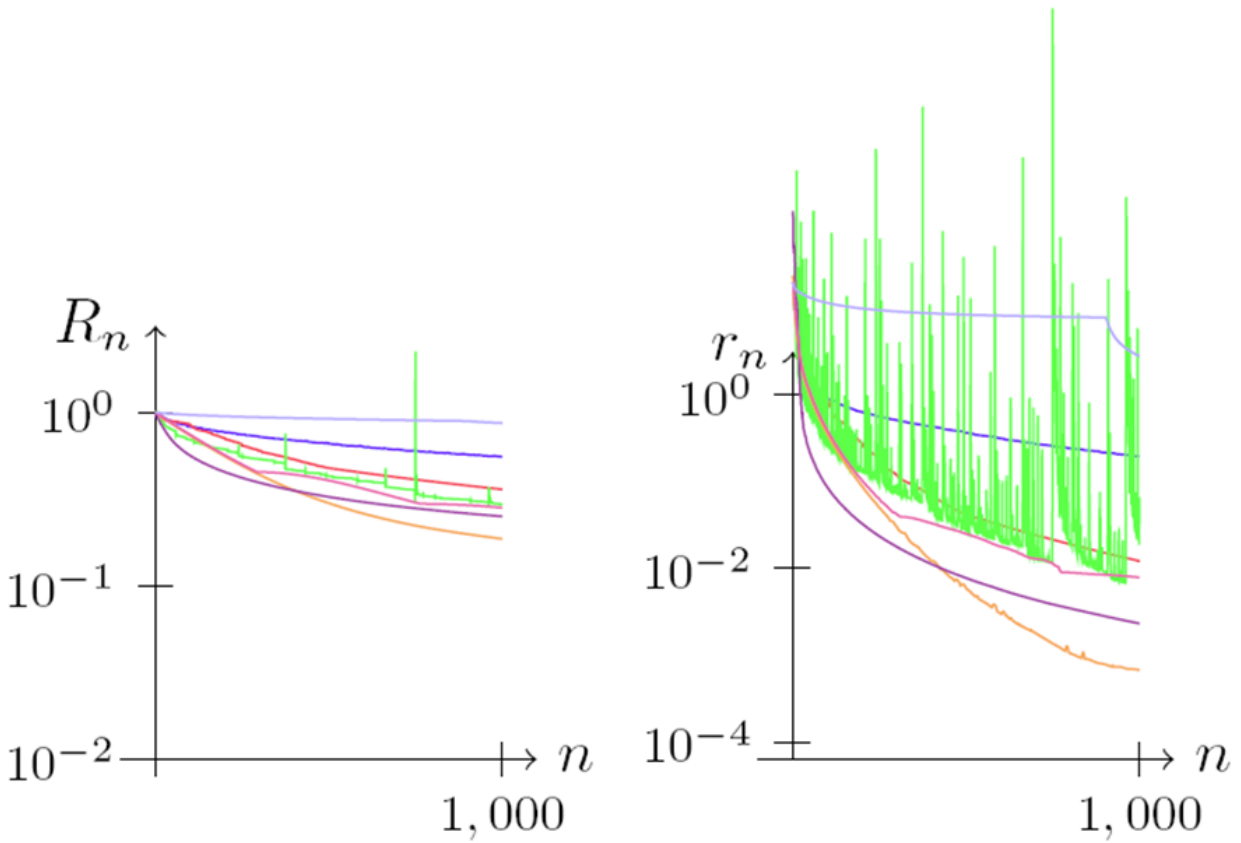
Typical CS example: $\mathbf{A} = [\mathbf{I}\ \mathbf{U}\ \mathbf{R}]$ (512 x 1536), $\widehat{\mathbf{x}}$ has 120 non-zeros, $\tau = 0.1$



Even More Speed Comparisons

A difficult problem: $\mathbf{A}$ is very coherent, τ is small $\tau = 10^{-3}$

All the solvers struggle…



IST, GPSR, SpaRSA, FISTA, YALL1, NESTA, fpc

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Initialization: Set $\tau_0 \gg \tau$, starting point $\bar{x}$, factor $\sigma \in (0, 1)$, and $k = 0$.

Iterations: Find approx solution $x(\tau_k)$ of $\min_x f(x) + \tau_k \psi(x)$, starting from $\bar{x}$;

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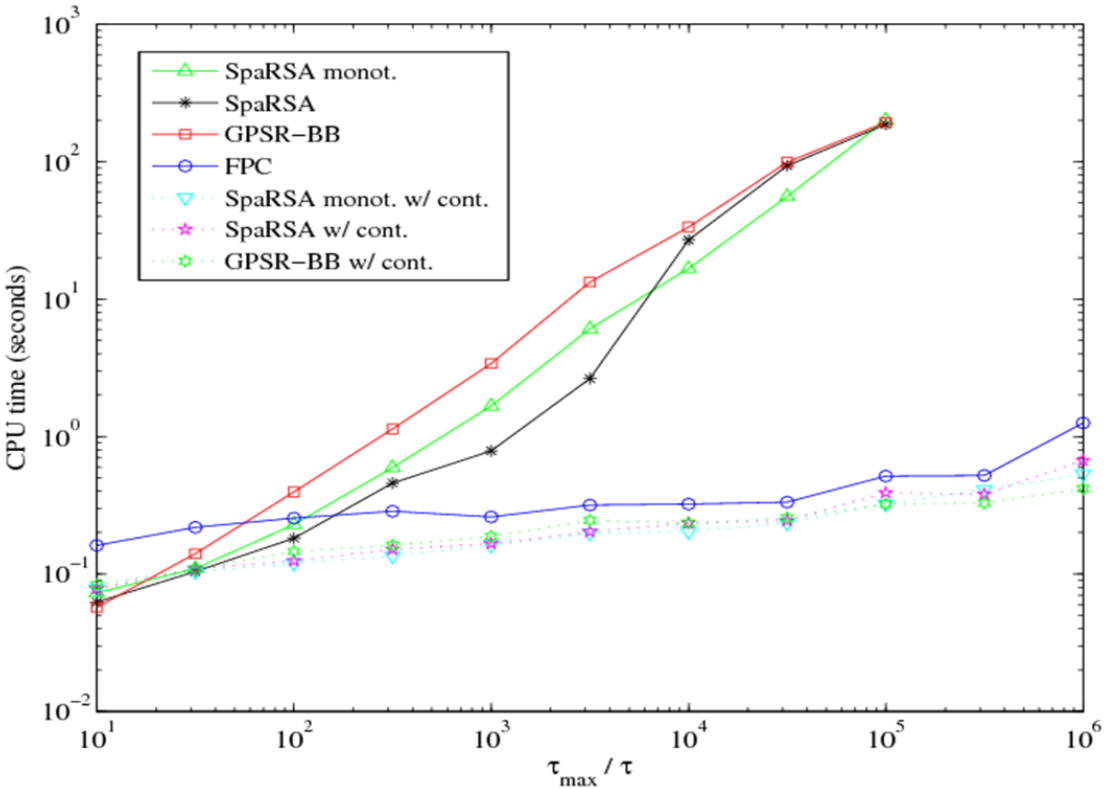
- Often the solution path $x(\tau)$, for a range of values of τ is desired, anyway (e.g., within an outer method to choose an optimal τ)
- Shown to be very effective in practice (Hale et al., 2008; Wright et al., 2009). Analyzed by Xiao and Zhang (2012).

Acceleration by Continuation: An Example

Classical sparse reconstruction problem (Wright et al., 2009)

$$\widehat{x} \in \arg\min_x \tfrac{1}{2}\|B\,x - b\|_2^2 + \tau\|x\|_1$$

with $B \in \mathbb{R}^{1024\times 4096}$ (thus $x \in \mathbb{R}^{4096}$ and $b \in \mathbb{R}^{1024}$).



A Final Touch: Debiasing

Consider problems of the form $\widehat{x} \in \arg\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Bx - b\|_2^2 + \tau\|x\|_1$

Often, the original goal was to minimize the quadratic term, after the support of x had been found. But the ℓ_1 shrinks the non-zero values.

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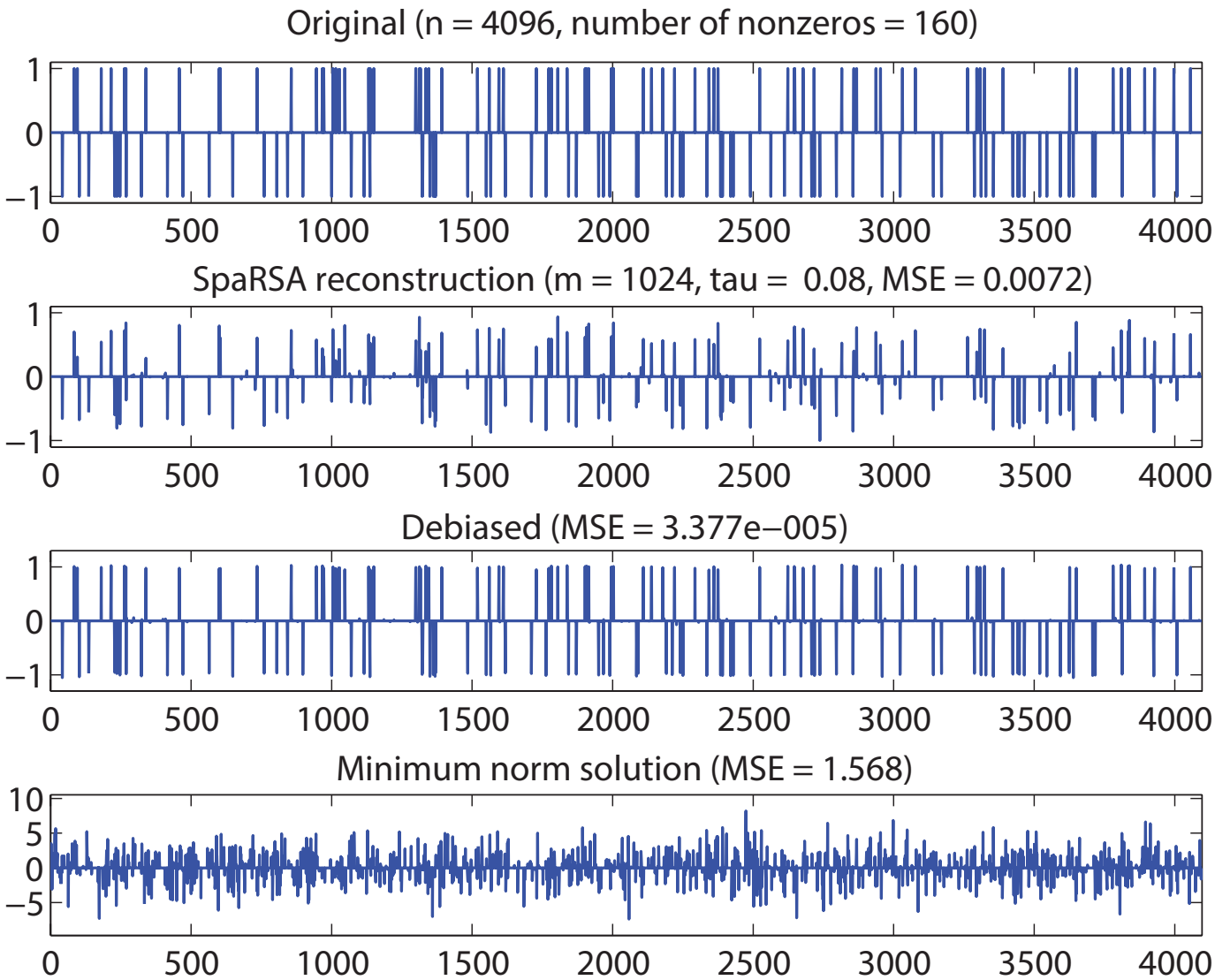
Debiasing:

- ✓ find the zero set (complement of the support of $\widehat{x}$): $\mathcal{Z}(\widehat{x}) = \{1, ..., n\} \setminus \text{supp}(\widehat{x})$.
- ✓ solve $\min_x \|B\,x - b\|_2^2$ s.t. $x_{\mathcal{Z}(\widehat{x})} = 0$. (Fix the zeros and solve an unconstrained problem over the support.)
- ✓ Often, this has to be solved using an algorithm that only uses products by B and B^T, since this matrix cannot be partitioned.

Effect of Debiasing



Final Example: Matrix Recovery (Toh and Yun, 2010)

$$\widehat{M} \in \arg \min_{M \in \mathbb{R}^{n \times n}} \frac{1}{2} \|\Phi(M) - U\|_F^2 + \mu \|M\|_*$$

linear operator

...its adjoint

The proximal algorithm (IST) is as before:

$$X_{k+1} = \text{svt}_{\mu\, \beta_k} \left(X_k - \beta_k \, \Phi^*(\Phi(X_k) - U) \right)$$

Matrix completion: $\Phi(X) = X_\Omega$ (subset of entries) $|\Omega| = p$

Unknown M				IST			APG (FISTA)		
n/r	p	p/d_r	μ	iter	#sv	error	iter	#sv	error
100/10	5666	3	8.21e-03	7723	61	1.88e-01	655	13	1.06e-03
200/10	15665	4	1.05e-02	12180	96	2.45e-01	812	12	1.02e-03
500/10	49471	5	1.21e-02	10900	203	5.91e-01	1132	16	7.63e-04

Unknown M				continuation			APG + continuation		
n/r	p	p/d_r	μ	iter	#sv	error	iter	#sv	error
100/10	5666	3	8.21e-03	429	32	1.06e-03	74	10	1.46e-04
200/10	15665	4	1.05e-02	278	49	4.38e-04	73	10	1.02e-04
500/10	49471	5	1.21e-02	484	125	5.50e-04	72	10	8.06e-05

...the importance of acceleration!

References I

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Part 3: ADMM in Image Processing

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CIMI-ANITI School on Optimisation, 2021

Outline

Alternating Direction Methof of Multipliers (ADMM)

- Canonical problem:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n,\, \mathbf{z}\in\mathbb{R}^m} \quad & f(\mathbf{x}) + g(\mathbf{z}) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} = \mathbf{b} \end{aligned}$$

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- Often used to re-write problems of the form

$$\min_{\mathbf{x}} \; f(\mathbf{x}) + g(\mathbf{H}\mathbf{x})$$

as

$$\min_{\mathbf{x},\mathbf{z}} \; f(\mathbf{x}) + g(\mathbf{z}) \quad \text{subject to} \quad \mathbf{H}\mathbf{x} = \mathbf{z}$$

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$$\begin{aligned} \mathbf{x}_{k+1} &= \arg\min_{\mathbf{x}} f(\mathbf{x}) + \frac{\rho}{2}\left\|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z}_k - \mathbf{b} + \mathbf{u}_k\right\|_2^2 \\ \mathbf{z}_{k+1} &= \arg\min_{\mathbf{z}} g(\mathbf{z}) + \frac{\rho}{2}\left\|\mathbf{A}\mathbf{x}_{k+1} + \mathbf{B}\mathbf{z} - \mathbf{b} + \mathbf{u}_k\right\|_2^2 \\ \mathbf{u}_{k+1} &= \mathbf{u}_{k+1} + \mathbf{A}\mathbf{x}_{k+1} + \mathbf{B}\mathbf{z}_{k+1} - \mathbf{b} \end{aligned}$$

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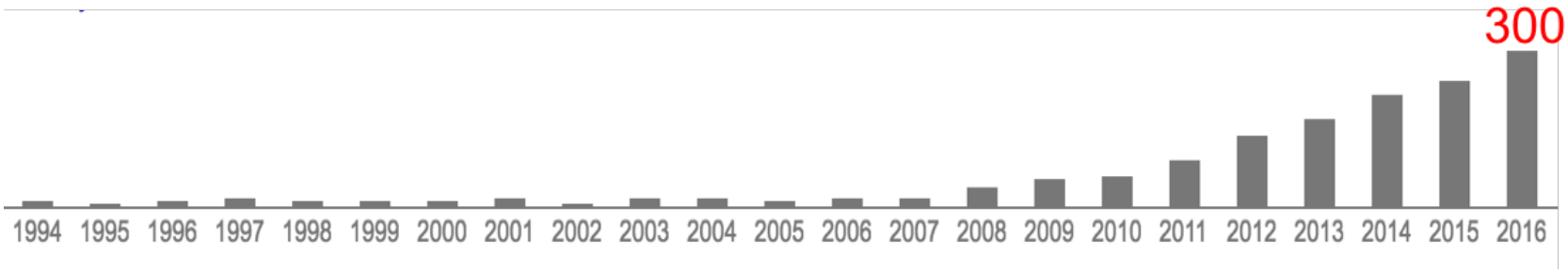
- Introduced by French mathematicians in the 1970s
  [Gabay and Mercier, 1976], [Glowinski and Marrocco, 1975]
- Cornerstone work in the 1990s by Eckstein and Bertsekas [1992]

Explosion of Interest in ADMM

- Citations to paper by Eckstein and Bertsekas [1992]:



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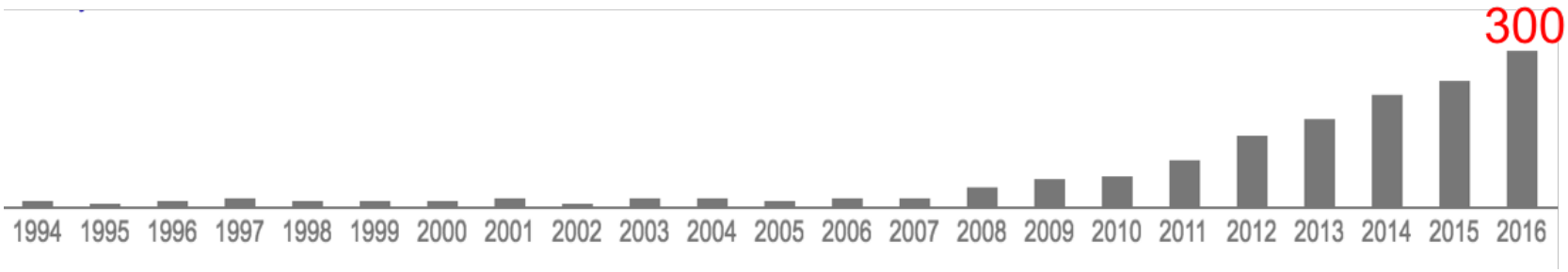

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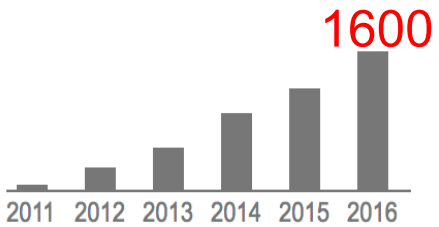

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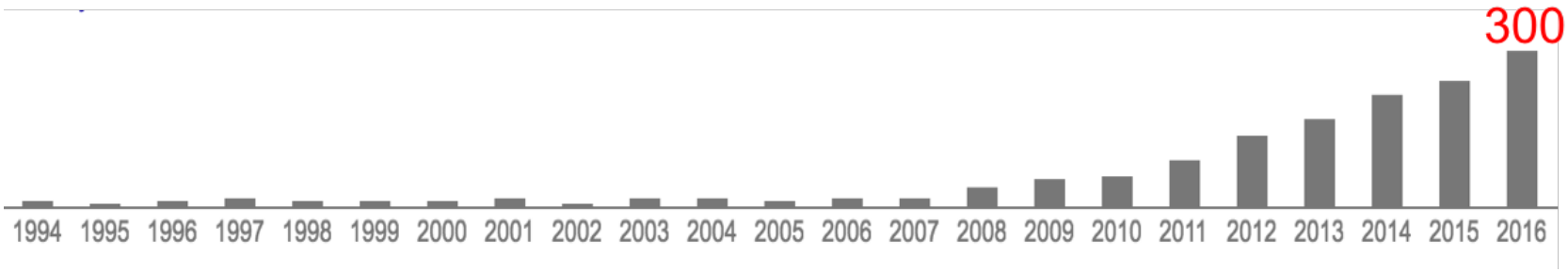

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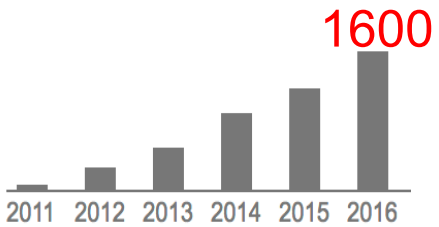

- Convergence properties: active research topic
  [Aspelmeier et al., 2016], [Bauschke et al., 2015], [Davis and Yin, 2014], [Deng and Yin, 2012], [Goldstein et al., 2014], [Nishihara et al., 2015], [Liang et al., 2015], [Patrinos et al., 2014], ...

Classical Convergence Result

- Problem: $\min_{\mathbf{x}} \; f(\mathbf{x}) + g(\mathbf{H}\mathbf{x})$
- ADMM:

$$\begin{aligned}
\mathbf{x}^{(k+1)} &= \arg\min_{\mathbf{x}} f(\mathbf{x}) + \frac{\rho}{2}\left\|\mathbf{H}\mathbf{x} - \mathbf{v}^{(k)} - \mathbf{u}^{(k)}\right\|_2^2 \\
\mathbf{v}^{(k+1)} &= \arg\min_{\mathbf{v}} g(\mathbf{v}) + \frac{\rho}{2}\left\|\mathbf{H}\mathbf{x}^{(k+1)} - \mathbf{v} - \mathbf{u}^{(k)}\right\|_2^2, \\
\mathbf{u}^{(k+1)} &= \mathbf{u}^{(k)} - \mathbf{H}\mathbf{x}^{(k+1)} + \mathbf{v}^{(k+1)},
\end{aligned}$$

Theorem (Eckstein and Bertsekas [1992])

Let $\mathbf{H}$ have full column rank, and $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $g : \mathbb{R}^m \to \bar{\mathbb{R}}$ be closed, proper, convex functions; let $\mathbf{v}_0, \mathbf{u}_0 \in \mathbb{R}^m$, and $\rho > 0$ be given. Then $(\mathbf{x}^{(k)})_{k=1,2,\ldots}$ converges to a solution, if one exists. If no solution exists, then at least one of the sequences $(\mathbf{v}^{(k)})_{k=1,2,\ldots}$ or $(\mathbf{u}^{(k)})_{k=1,2,\ldots}$ diverges.

Two or More Functions

- Problem template:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{j=1}^{J} g_j\left(\mathbf{H}_j\,\mathbf{x}\right)$$

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$$\min_{\mathbf{x}} \; f(\mathbf{x}) + g(\mathbf{H}\mathbf{x}),$$

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ADMM for Two or More Functions

- General problem template: $$\min_{\mathbf{x}\in\mathbb{R}^n} \sum_{j=1}^{J} g_j\left(\mathbf{H}_j\,\mathbf{x}\right)$$

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$$\mathbf{x}_{k+1} = \arg\min_{\mathbf{x}} \sum_{j=1}^{J} \|\mathbf{H}_j\,\mathbf{x} - \mathbf{z}_k^{(j)} + \mathbf{u}_k^{(j)}\|_2^2$$

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$$\mathbf{z}_{k+1}^{(1)} = \arg\min_{\mathbf{v}\in\mathbb{R}^{m_1}} g_1(\mathbf{v}) + \frac{\rho}{2}\left\|\mathbf{H}_1\,\mathbf{x}_{k+1} - \mathbf{v} + \mathbf{u}_k^{(1)}\right\|_2^2$$

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- SALSA, PIDAL, PIDSplit, SDMM

[F and Bioucas-Dias, 2010], [Setzer et al., 2010], [Combettes and Pesquet, 2011]

A Closer Look

- A closer look at the algorithm

$$\mathbf{x}_{k+1} = \left(\sum_{j=1}^{J} \mathbf{H}_j^T \mathbf{H}_j \right)^{-1} \sum_{j=1}^{J} \mathbf{H}_j \left(\mathbf{z}_k^{(j)} - \mathbf{u}_k^{(j)} \right)$$

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$$\mathbf{z}_{k+1}^{(1)} = \text{prox}_{g_1/\rho_k} \left(\mathbf{H}_1 \, \mathbf{x}_{k+1} + \mathbf{u}_k^{(1)} \right)$$

$$\vdots \qquad \vdots$$

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- Decoupled: a linear algebraic problem and a set of proximity operators

A Closer Look

- A closer look at the algorithm

$$\mathbf{x}_{k+1} = \left(\sum_{j=1}^{J} \mathbf{H}_j^T \mathbf{H}_j \right)^{-1} \sum_{j=1}^{J} \mathbf{H}_j \left(\mathbf{z}_k^{(j)} - \mathbf{u}_k^{(j)} \right)$$

$$\mathbf{z}_{k+1}^{(1)} = \text{prox}_{g_1/\rho_k} \left(\mathbf{H}_1 \, \mathbf{x}_{k+1} + \mathbf{u}_k^{(1)} \right)$$

$$\vdots \qquad \vdots$$

$$\mathbf{z}_{k+1}^{(J)} = \text{prox}_{g_J/\rho_k} \left(\mathbf{H}_J \, \mathbf{x}_{k+1} + \mathbf{u}_k^{(J)} \right)$$

$$\mathbf{u}_{k+1} = \mathbf{u}_{k+1} + \mathbf{A}\mathbf{x}_{k+1} + \mathbf{B}\mathbf{z}_{k+1}$$

- Decoupled: a linear algebraic problem and a set of proximity operators
- Hinges on the efficient matrix inversion and simple proximity operators

A Closer Look

- A closer look at the algorithm

$$\mathbf{x}_{k+1} = \left(\sum_{j=1}^{J} \mathbf{H}_j^T \mathbf{H}_j \right)^{-1} \sum_{j=1}^{J} \mathbf{H}_j \left(\mathbf{z}_k^{(j)} - \mathbf{u}_k^{(j)} \right)$$

$$\mathbf{z}_{k+1}^{(1)} = \text{prox}_{g_1/\rho_k} \left(\mathbf{H}_1 \, \mathbf{x}_{k+1} + \mathbf{u}_k^{(1)} \right)$$

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- Decoupled: a linear algebraic problem and a set of proximity operators
- Hinges on the efficient matrix inversion and simple proximity operators
- Matrix inverse independent of ρ_k (good, if not kept constant)

Image Restoration/Reconstruction

- General formulation: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) + \Phi(\mathbf{P}\mathbf{x}) + \iota_C(\mathbf{x})$$

  where $\mathbf{y}$ are observations and $\iota_C(\mathbf{x}) = \begin{cases} 0 & \Leftarrow \quad \mathbf{x} \in C \\ +\infty & \Leftarrow \quad \mathbf{x} \notin C \end{cases}$

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- $\Phi \circ \mathbf{P}$ is a regularizer; *e.g.*, total variation (TV), or Φ is a norm
- $\mathbf{A}$: linear observation operator (blur, projections, ...)

Image Restoration: Observation Models Ψ

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- All the above observation models have simple, component-wise proximity operators

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  ✓ Gaussian observations: $\text{prox}_{\tau\Psi}(u) = \dfrac{\sigma^2 u + \tau y}{\sigma^2 + \tau}$

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- The proximity operator of ι_C is simply an Euclidean projection:

$$\text{prox}_{\iota_C}(\mathbf{u}) = \text{proj}_C(\mathbf{u});$$

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- The proximity operator of ι_C is simply an Euclidean projection:

$$\text{prox}_{\iota_C}(\mathbf{u}) = \text{proj}_C(\mathbf{u});$$

e.g., if $C = \mathbb{R}^n_+$, then $\big(\text{proj}_C(\mathbf{u})\big)_i = \max\{0, u_i\}$

Image Restoration: Regularizers Φ

- General formulation: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) + \Phi(\mathbf{P}\mathbf{x}) + \iota_C(\mathbf{x})$$

Image Restoration: Regularizers Φ

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- Classical regularizers with simple proximity operators

Image Restoration: Regularizers Φ

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- Classical regularizers with simple proximity operators

  ✓ ℓ_1 norm: $\left(\text{prox}_{\tau\|\cdot\|_1}(\mathbf{u})\right)_i = \text{sign}(u_i)\max\{0, u_i - \tau\}$

Image Restoration: Regularizers Φ

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  ✓ Squared ℓ_2 norm: $\text{prox}_{\tau\|\cdot\|_2^2}(\mathbf{u}) = \dfrac{\mathbf{u}}{1+\tau}$

Image Restoration: Regularizers Φ

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  - ✓ Squared ℓ_2 norm: $\text{prox}_{\tau\|\cdot\|_2^2}(\mathbf{u}) = \dfrac{\mathbf{u}}{1+\tau}$ (linear shrinkage)

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Image Restoration: Regularizers Φ

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- Total variation can be written as $\Phi \circ \mathbf{P}$, where

$$\mathbf{P} : \mathbb{R}^n \to (\mathbb{R}^2)^n, \text{ with } (\mathbf{P}\mathbf{x})_i = \begin{bmatrix} x_i - x_{h(i)} \\ x_i - x_{v(i)} \end{bmatrix}, \text{ and } \Phi(\mathbf{v}) = \sum_i \|\mathbf{v}_i\|_2$$

with $h(i)$ and $v(i)$ the horizontal and vertical neighbours of pixel i

Image Restoration: Synthesis Formulation

- General formulation: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) + \Phi(\mathbf{P}\mathbf{x})$$

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- Synthesis formulation: $\mathbf{A} = \mathbf{B}\mathbf{W}$ and $\mathbf{P} = \mathbf{I}$

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  - ✓ $\mathbf{x}$ contains the representation coefficients, not the image itself

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- Using the Sherman-Morrison-Woodbury matrix inversion formula

$$\left(\mathbf{W}^T\mathbf{B}^T\mathbf{B}\mathbf{W} + \mathbf{I}\right)^{-1} = \mathbf{I} - \mathbf{W}^T\mathbf{B}^T\left(\mathbf{B}^T\mathbf{B} + \mathbf{I}\right)^{-1}\mathbf{B}\mathbf{W}$$

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- Can $\mathbf{B}^T\mathbf{B} + \mathbf{I}$ be inverted efficiently?

Image Restoration: Analysis Formulation

- General formulation: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) + \Phi(\mathbf{P}\mathbf{x})$$

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  - ✓ $\mathbf{P}$ the analysis operator of a Parseval frame: $\mathbf{P}^T\mathbf{P} = \mathbf{I}$

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  - ✓ $\mathbf{x}$ contains the image itself

Image Restoration: Analysis Formulation

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- Matrix inversion:

$$\left(\mathbf{B}^T\mathbf{B} + \mathbf{P}^T\mathbf{P}\right)^{-1} = \left(\mathbf{B}^T\mathbf{B} + \mathbf{I}\right)^{-1}$$

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- Can $\mathbf{B}^T\mathbf{B} + \mathbf{I}$ be inverted efficiently?

Image Restoration: Constrained (Morozov) Formulations

- General formulation: $$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) + \Phi(\mathbf{P}\mathbf{x})$$

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- Constrained (or Morozov) formulation:

$$\hat{\mathbf{x}} \in \min_{\mathbf{x}\in\mathbb{R}^n} \Phi(\mathbf{P}\mathbf{x}) \text{ subject to } \Lambda(\mathbf{A}\mathbf{x}, \mathbf{y}) \leq 1$$

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- Can be written in the general formulation, with

$$\Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) = \iota_{D(\mathbf{y})}(\mathbf{A}\mathbf{x}), \text{ with } D(\mathbf{y}) = \{\mathbf{x} : \Lambda(\mathbf{x}, \mathbf{y}) \leq 1\}$$

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$$\Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) = \iota_{D(\mathbf{y})}(\mathbf{A}\mathbf{x}), \;\text{ with }\; D(\mathbf{y}) = \{\mathbf{x} : \Lambda(\mathbf{x}, \mathbf{y}) \leq 1\}$$

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  Thus, $D(\mathbf{y})$ is a unit Euclidean ball around $\mathbf{y}$; projection is trivial.

Image Restoration: Constrained (Morozov) Formulations

- General formulation: $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \Psi(\mathbf{A}\mathbf{x}, \mathbf{y}) + \Phi(\mathbf{P}\mathbf{x})$
- Constrained (or Morozov) formulation:

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- Applies both to synthesis and analysis formulations

Image Restoration: Matrix Inversions

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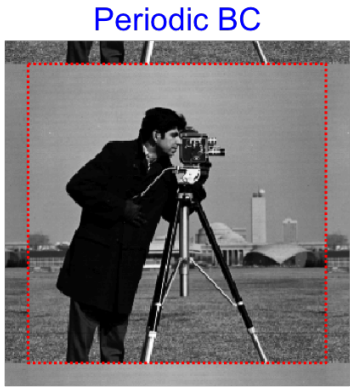
- Cost is at most $O(n\log n)$

Non-periodic Deconvolution



- Periodic boundary conditions are usually unnatural

Non-periodic Deconvolution

Periodic BC

Neumann BC

Dirichlet BC

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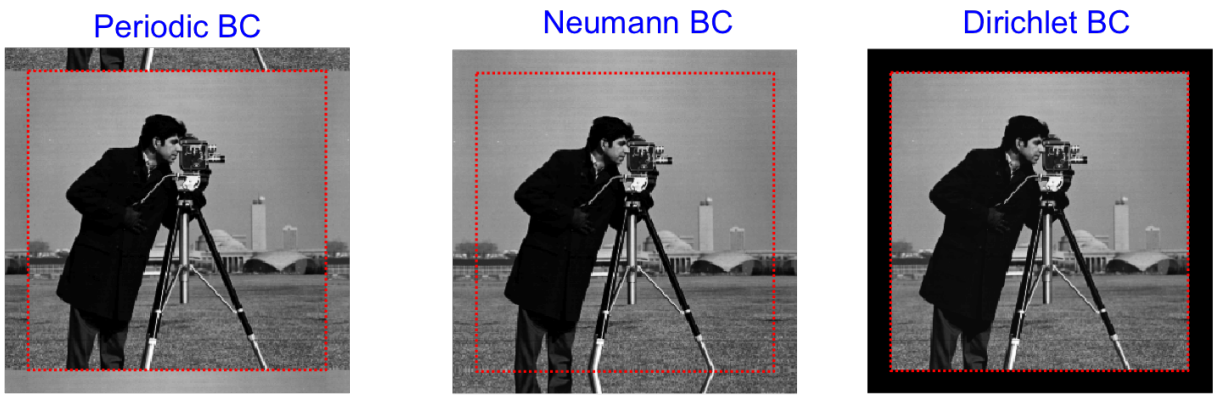
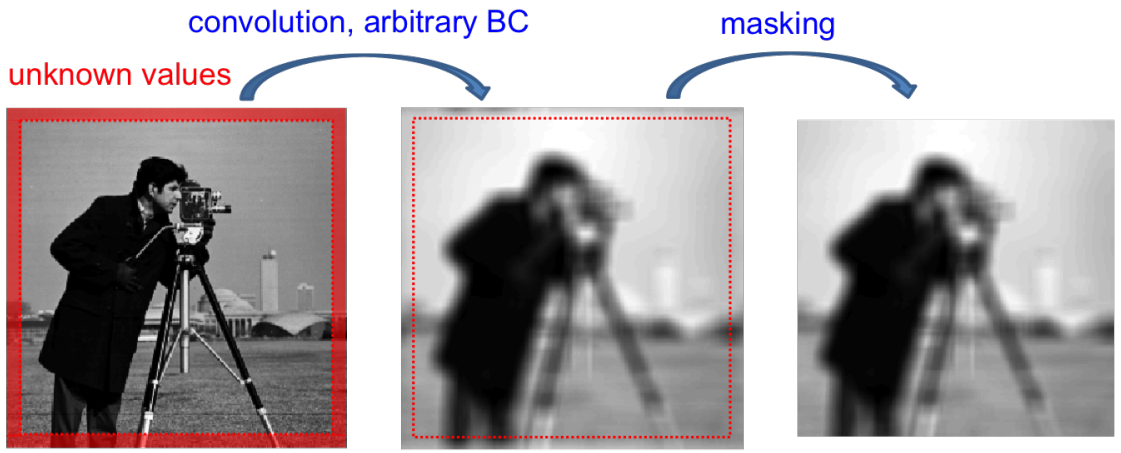
Non-periodic Deconvolution



- Periodic boundary conditions are usually unnatural
- ...as are other standard BC: Neumann, Dirichlet.
- A more natural choice: unknown boundaries [Reeves, 2005], [Chan et al., 2005], [Almeida and F, 2013a], [Ramani and Fessler, 2013]



Non-periodic Deconvolution (2)

- Gaussian noise model: $\Psi(\mathbf{Bx}, \mathbf{y}) = \frac{1}{2\sigma^2} \| \overbrace{\mathbf{M}}^{\text{mask}} \underbrace{\mathbf{U}^H \mathbf{F} \mathbf{U}}_{\text{period. conv.}} \mathbf{x} - \mathbf{y} \|_2^2$

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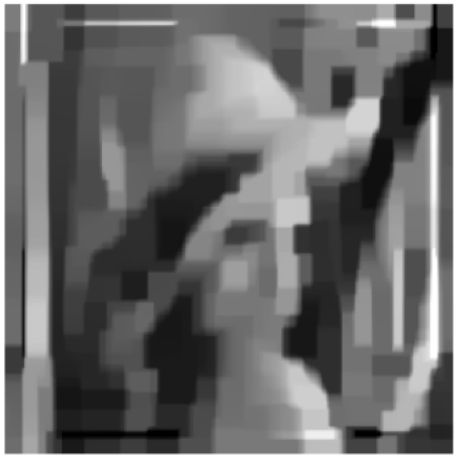
- Similar formulations:
  - ✓ deconvolution + inpainting ($\mathbf{M}$ masks the boundary and missing pixels)
  - ✓ super-resolution (filtering + downsampling mask)

Deconvolution with Unknown Boundaries: Example

original (256×256)

observed (238×238)

Assuming periodic BC



FA-BC (ISNR = -2.52dB)

Edge tapering

FA-ET (ISNR = 3.06dB)

Unknown BC by ADMM

FA-MD (ISRN = 10.63dB)

Deconvolution + Inpainting with Unknown BC: Example

original (256×256)

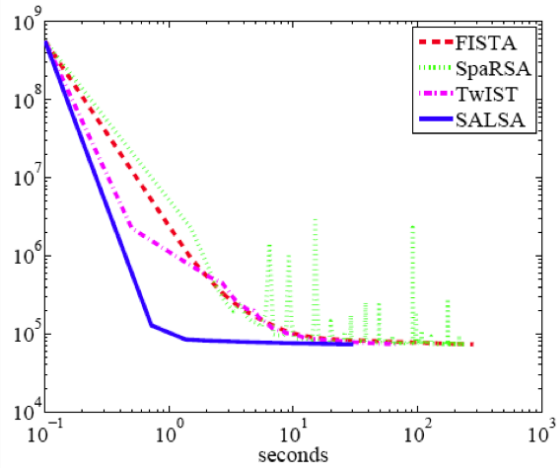
observed (238×238)

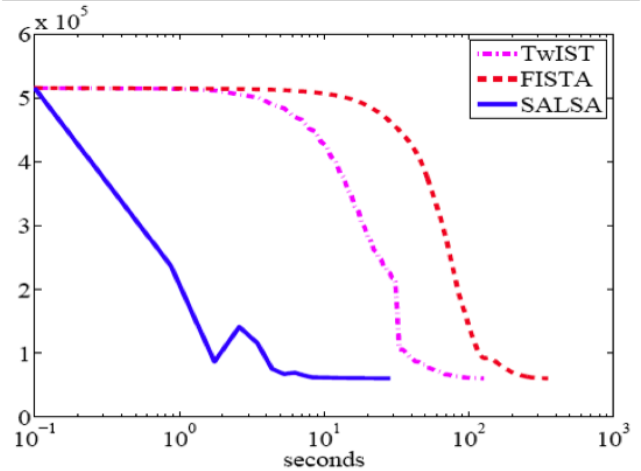
FA-CG (SNR = 20.58dB)

FA-MD (SNR = 20.57dB)

Speed

- Benchmark deblurring problem (9×9 blur, 40dB SNR, Haar frame, ℓ_1) and inpainting problem (50% missing data) [Afonso et al., 2011]





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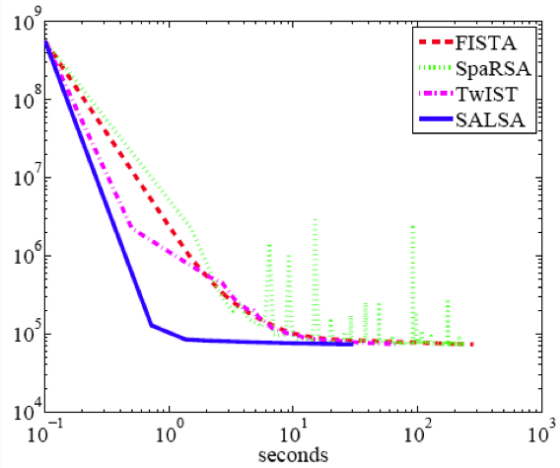
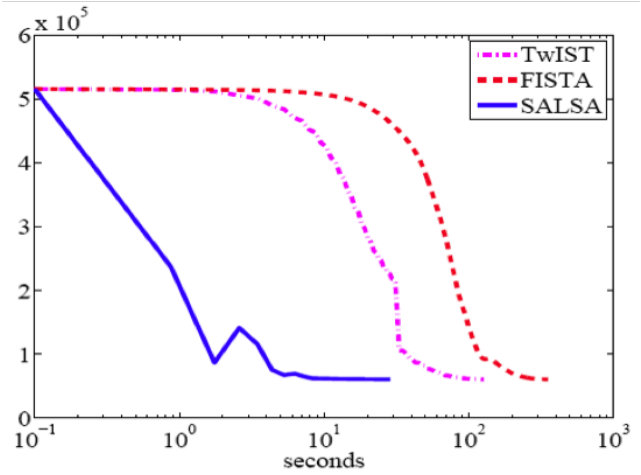
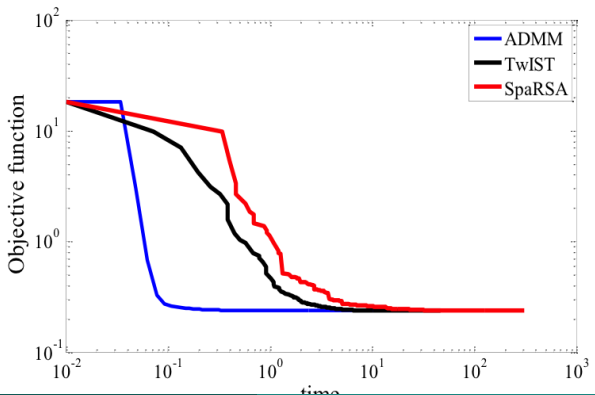
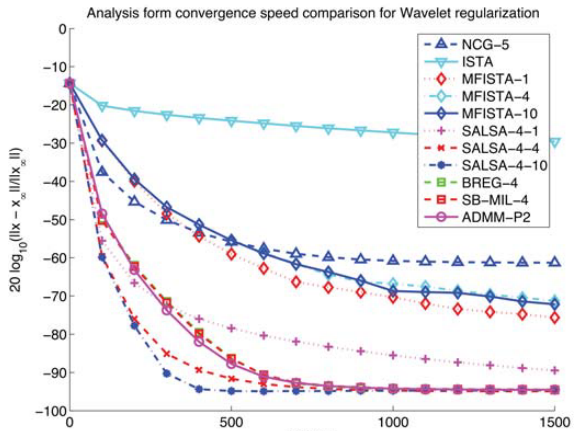




- Deconvolution with unknown BC [Almeida and F, 2013a], [Ramani and Fessler, 2013]





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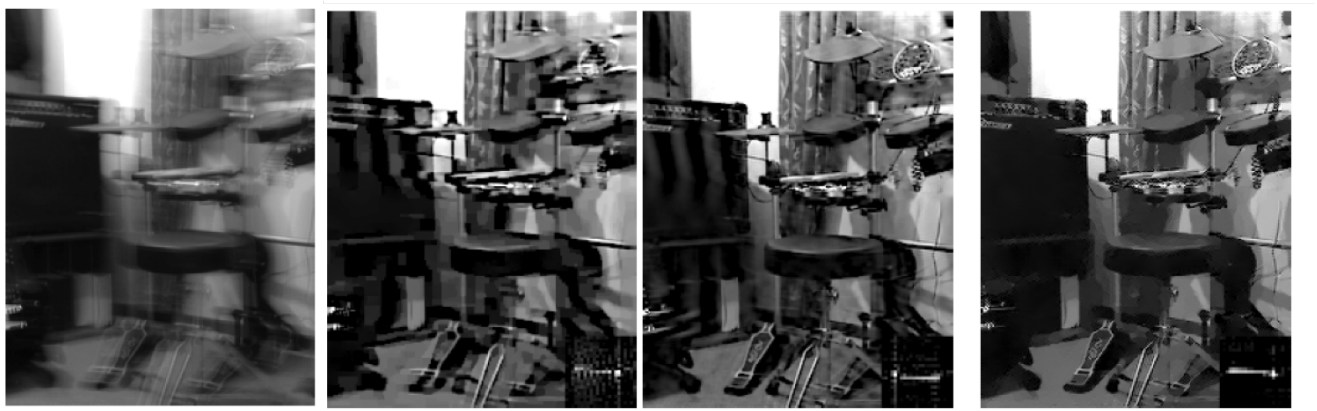
- Current research: choice of parameter ρ_k [Xu et al., 2016]

Blind Deconvolution: Real Examples

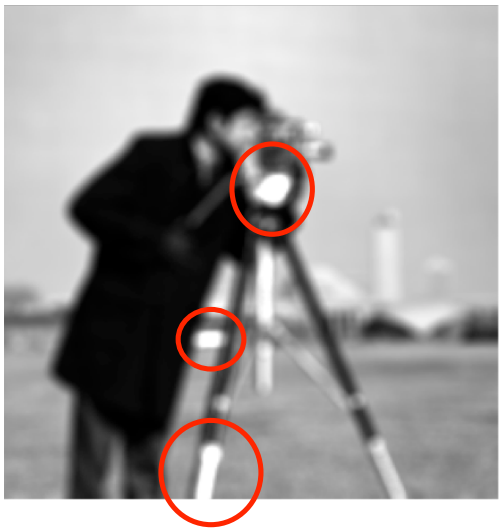


Blurred photo | [14], 70 seconds | [16], 100 seconds | Proposed method, 55 seconds

Observed photo. | [Almeida et al, 2010] | proposed

Results from [Almeida and F, 2013b]

Blind Deconvolution: The Importance of Inpainting



blur and saturations

blind deblurring
ignoring saturations

blind deblurring
accounting for saturations

Denoising Step in ADMM

- Restoration (w/ Gauss noise): $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \Phi(\mathbf{x})$

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- Restoration (w/ Gauss noise): $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \Phi(\mathbf{x})$
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- Can we use one of these denoisers instead of some proximity operator?

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- Plug a black-box denoiser into ADMM [Venkatakrishnan et al., 2013]

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- Empirical results: competitive!

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where $\beta_j(\mathbf{y}) \propto \alpha_j \mathcal{N}(\mathbf{y}|\boldsymbol{\mu}_j, \mathbf{C}_j + \sigma^2\mathbf{I})$, with $\sum_{j=1}^{K} \beta_j(\mathbf{y}) = 1$

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Plug-and-Play ADMM: Deblurring of Generic Images

- Generic GMM prior

Image:	Cameraman						House					
Experiment:	1	2	3	4	5	6	1	2	3	4	5	6
IDD-BM3D [Danielyan et al., 2012]	**8.85**	**7.12**	**10.45**	**3.98**	**4.31**	**4.89**	**9.95**	**8.55**	**12.89**	**5.79**	**5.74**	**7.13**
ADMM-GMM [Teodoro et al., 2016b]	8.39	6.36	9.80	3.47	4.16	4.88	9.66	8.22	12.43	5.50	5.42	6.82

(a) Original

(b) Blurred

(c) IDD-BM3D

(d) ADMM-GMM

- For generic natural images: competitive, but does not beat state-of-the-art

Class-Adapted GMM-Based Restoration

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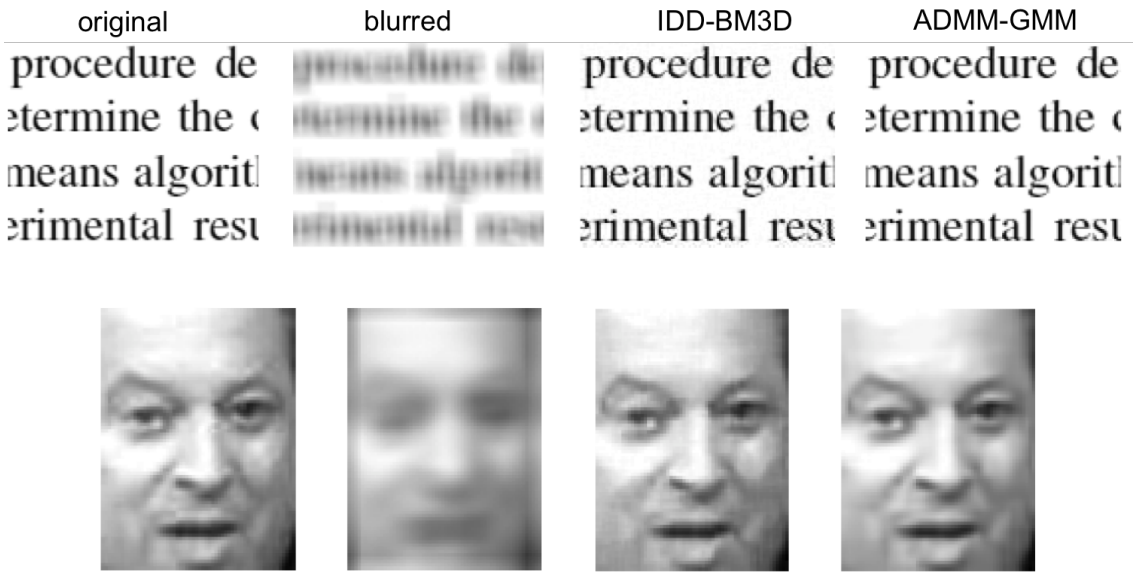

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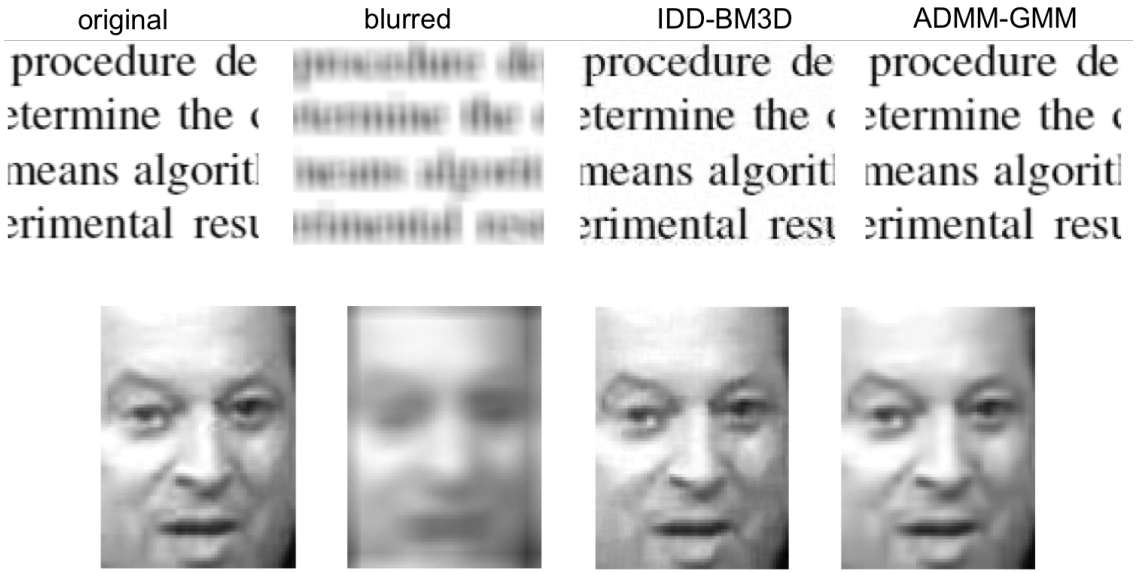

Image class:	Text						Face					
Experiment:	1	2	3	4	5	6	1	2	3	4	5	6
BSNR	26.07	20.05	40.00	15.95	24.78	18.11	28.28	22.26	40.00	15.89	26.22	15.37
Input PSNR	14.14	14.13	12.13	16.83	14.48	28.73	25.61	22.54	20.71	26.49	24.79	30.03
IDD-BM3D	11.97	8.91	16.29	5.88	6.81	4.87	13.66	11.16	14.96	7.31	10.33	6.18
ADMM-GMM	**16.24**	**11.55**	**23.11**	**8.88**	**10.77**	**8.34**	**15.05**	**12.59**	**17.28**	**8.84**	**11.69**	**7.32**

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- Blind image deblurring/deconvolution

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- Even if Φ and Ψ are convex, this is a non-convex problem

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- Filter regularizer: positivity and support, or sparsity

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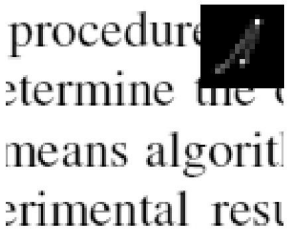
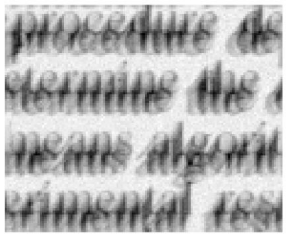
  - ✓ Constraint: positivity and maximum support
  - ✓ Sparsity (adequate for motion blur)

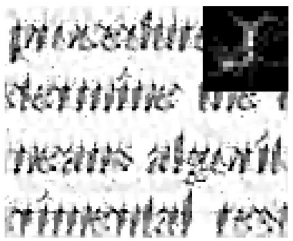
Results: GMM-based prior for text images



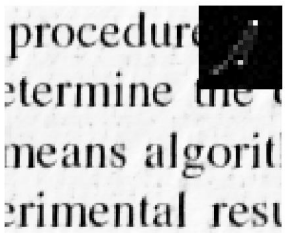
original



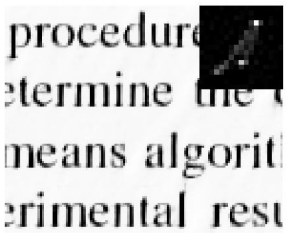
blurred



[Pan et al., 2014]



BM3D: 9.97 dB



GMM: 11.16 dB

Experiments

original

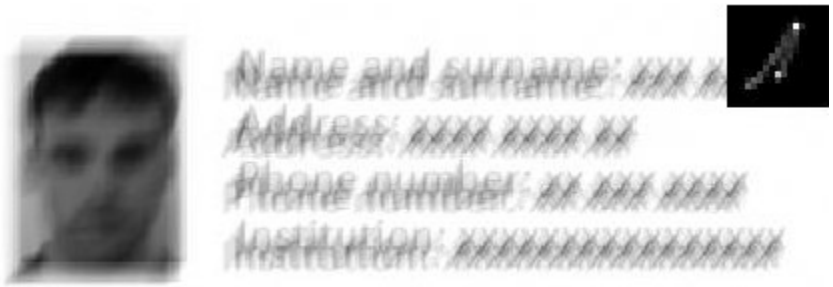
blurred

[Almeida and F, 2013b]
0.36 dB

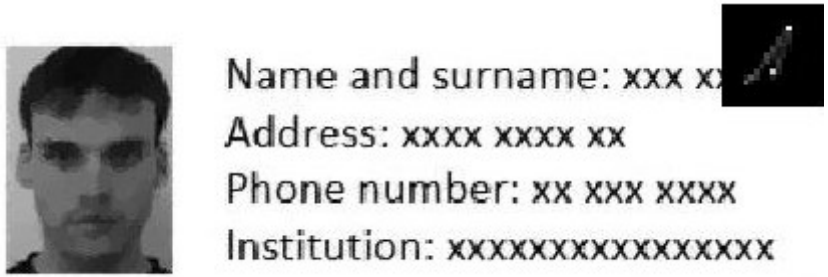
BM3D: 0.66 dB

GMM: 1.19 dB

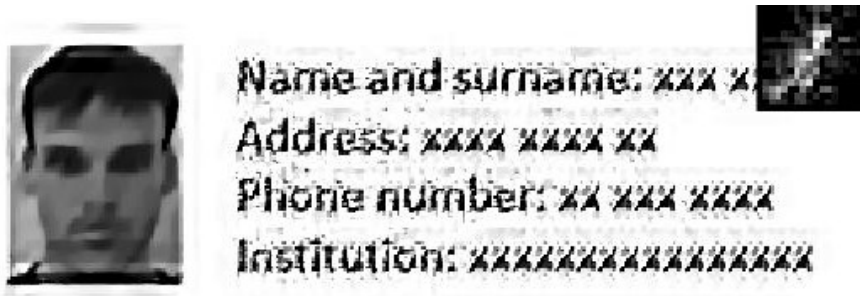
Experiments



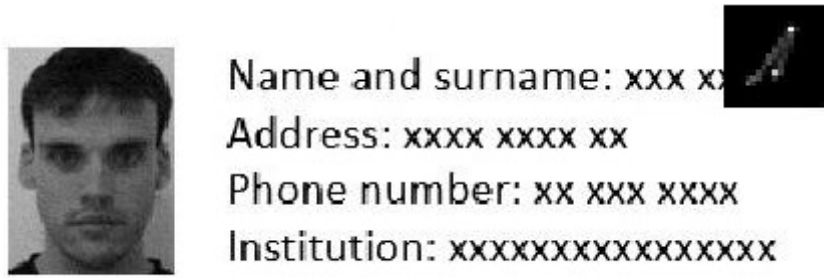
(a) Blurred image



(b) [Almeida and F, 2013b]

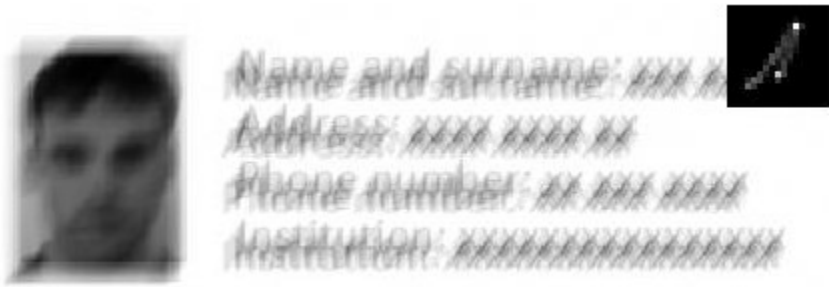


(c) [Pan et al., 2014]

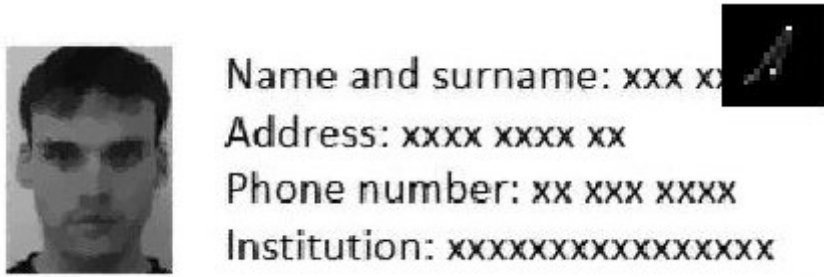


(d) Proposed

Experiments



(a) Blurred image



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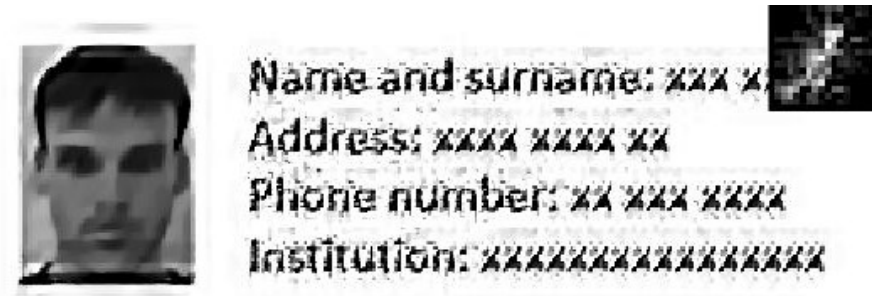

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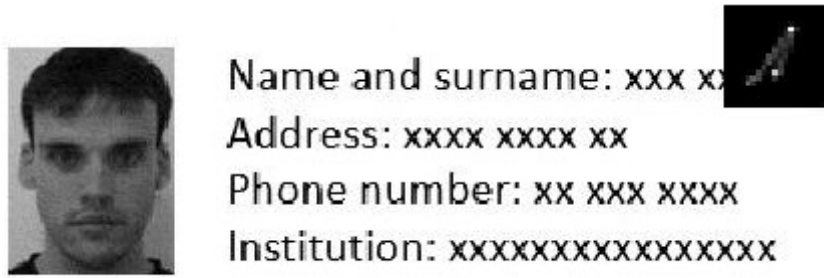


(d) Proposed

- Uses a concatenation of two dictionaries: face and text

Experiments

original

blurred

[Krishnan et al, 2011]

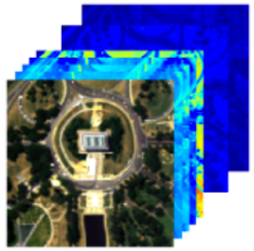
[Xu and Jia, 2011]

[Pan et al, 2014]

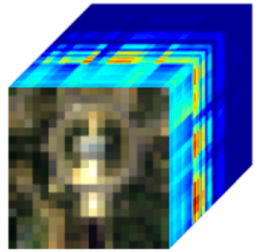
proposed

An Extreme Case of Adaptation: Hyperspectral Fusion

- Spectral-spatial resolution trade-off:



Multi-spectral:
high spatial resolution
low spectral resolution



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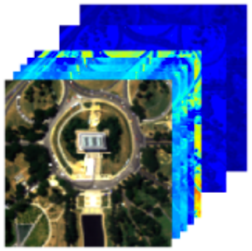

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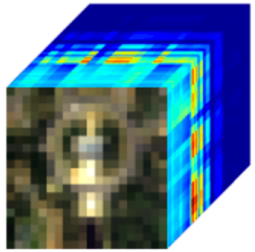
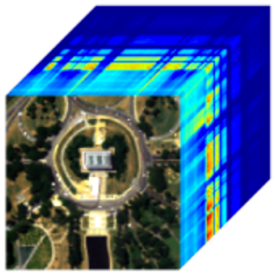


Hyper-spectral:
low spatial resolution
high spectral resolution

- Fuse MS and HS data:

  high spatial & spectral resolutions



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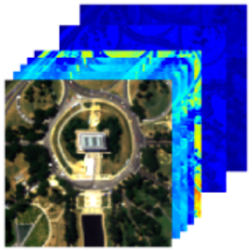

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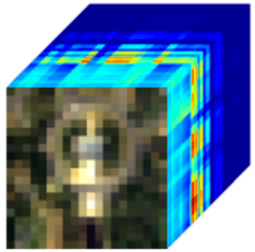

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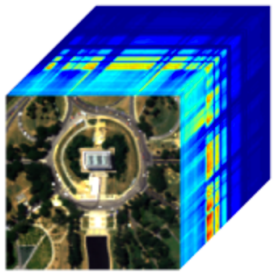

- Extreme case: pansharpening (panchromatic rather than MS image).

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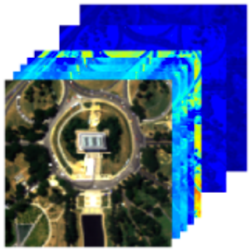

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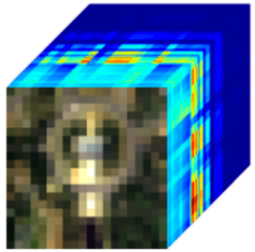

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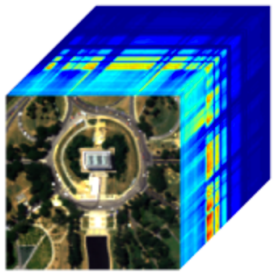

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Hyperspectral Fusion: Formulation

- Observation model [Simões et al., 2015]

$$\begin{aligned} \mathbf{Y}_h &= \overbrace{\mathbf{EX}}^{\mathbf{Z}}\mathbf{BM} + \mathbf{N}_h \\ \mathbf{Y}_m &= \mathbf{R}\underbrace{\mathbf{EX}}_{\mathbf{Z}} + \mathbf{N}_m \end{aligned}$$

hyperspectral data $\in \mathbb{R}^{L_h \times n_h}$

multispectral data $\in \mathbb{R}^{L_m \times n_m}$

$L_h > L_m$ and $n_h < n_m$

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✓ $\mathbf{E} \in \mathbb{R}^{L_h \times p}$: the p-dimensional subspace containing the fused image $\mathbf{Z}$

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- ✓ $\mathbf{N}_h$ and $\mathbf{N}_m$ model noise

Hyperspectral Fusion via PnP-ADMM

- Assuming Gaussian noise:

$$\widehat{\mathbf{X}} \in \arg \min_{\mathbf{X} \in \mathbb{R}^{p \times n_h}} \frac{1}{2}\|\mathbf{EXBM} - \mathbf{Y}_h\|_F^2 + \frac{\lambda_m}{2}\|\mathbf{REX} - \mathbf{Y}_m\|_F^2 + \text{“}\phi(\mathbf{X})\text{”}$$

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- The GMM is learned from patches of $\mathbf{Y}_m$ (high spatial resolution) [Teodoro et al., 2016a]

Convergence

- PnP-ADMM with a patch-based GMM-MMSE denoiser

$$\mathbf{x}_{k+1} = \left(\mathbf{A}^T\mathbf{A} + \rho\mathbf{I}\right)^{-1}\left(\mathbf{A}^T\mathbf{y} + \rho(\mathbf{z}_k + \mathbf{u}_k)\right)$$
$$\mathbf{z}_{k+1} = \text{denoiser}\left(\mathbf{x}_{k+1} - \mathbf{u}_k, 1/\rho\right)$$
$$\mathbf{u}_{k+1} = \mathbf{u}_{k+1} - \mathbf{x}_{k+1} + \mathbf{z}_{k+1}$$

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  $\forall\, \mathbf{x}, \mathbf{x}',\ \|p(\mathbf{x}) - p(\mathbf{x}')\| \leq \|\mathbf{x} - \mathbf{x}'\|$

  b) and p is subgradient of a convex function, *i.e.*,
  $\exists\, \phi : \mathbb{R}^n \to \mathbb{R} :\ p(\mathbf{x}) \in \partial\phi(\mathbf{x}),\ \forall\, \mathbf{x}$

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  b) and p is subgradient of a convex function, *i.e.*,
  $\exists\, \phi : \mathbb{R}^n \to \mathbb{R} :\ p(\mathbf{x}) \in \partial\phi(\mathbf{x}),\ \forall\, \mathbf{x}$

- Does the patch-based GMM-MMSE denoiser satisfy these conditions?

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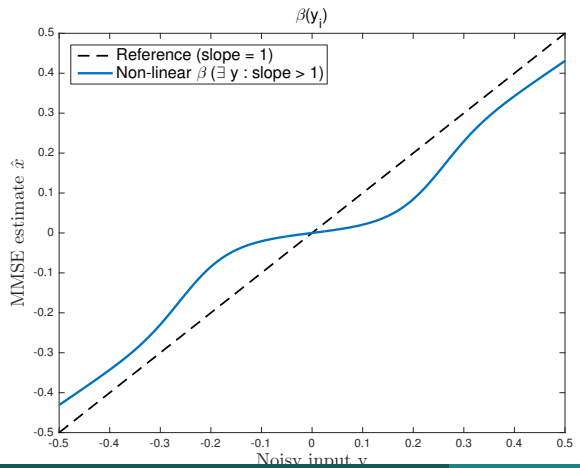
  - ✓ MMSE estimate under Gaussian noise of unit variance:

$$\hat{x} = \mathbb{E}[X|y] = \frac{\frac{\tau_1\, y}{\tau_1+1}\, \beta_1(y) + \frac{\tau_2\, y}{\tau_2+1}\, \beta_2(y)}{\beta_1(y) + \beta_2(y)}, \qquad \text{where } \beta_i(y) = \mathcal{N}(y; 0, \tau_i+1)$$



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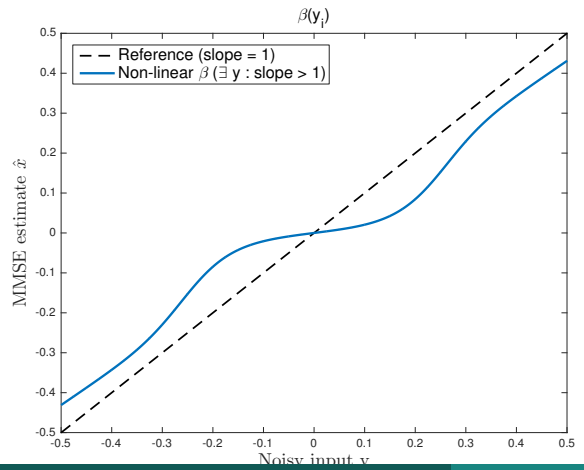
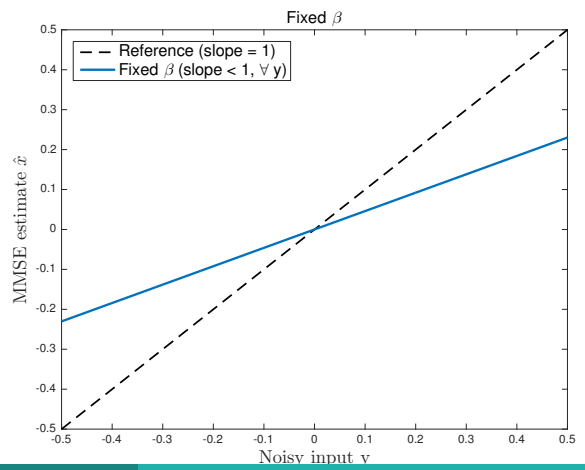
- With β_i fixed: $\hat{x} = y\big(\beta_1 \frac{\tau_1}{\tau_1+1} + \beta_2 \frac{\tau_2}{\tau_2+1}\big)/(\beta_1 + \beta_2)$





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- Key properties of $\mathbf{W}$ [Teodoro et al., 2017b]: for any $\sigma^2 > 0$,

$$\mathbf{W}(\sigma^2) = \mathbf{W}(\sigma^2)^T, \qquad \mathbf{W}(\sigma^2) \succeq 0, \qquad \lambda_{\max}\big(\mathbf{W}(\sigma^2)\big) < 1$$

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- Can we identify the function of which this denoiser is the prox?

$$\phi(\mathbf{x}) = \iota_{S(\mathbf{W})}(\mathbf{x}) + \frac{1}{2}\mathbf{x}^T\bar{\mathbf{Q}}(\bar{\Lambda}^{-1} - \mathbf{I})\bar{\mathbf{Q}}^T\mathbf{x}$$

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- Conclusion: the problem has a solution and PnP-ADMM converges

Hyperspectral Fusion: Synthetic Example

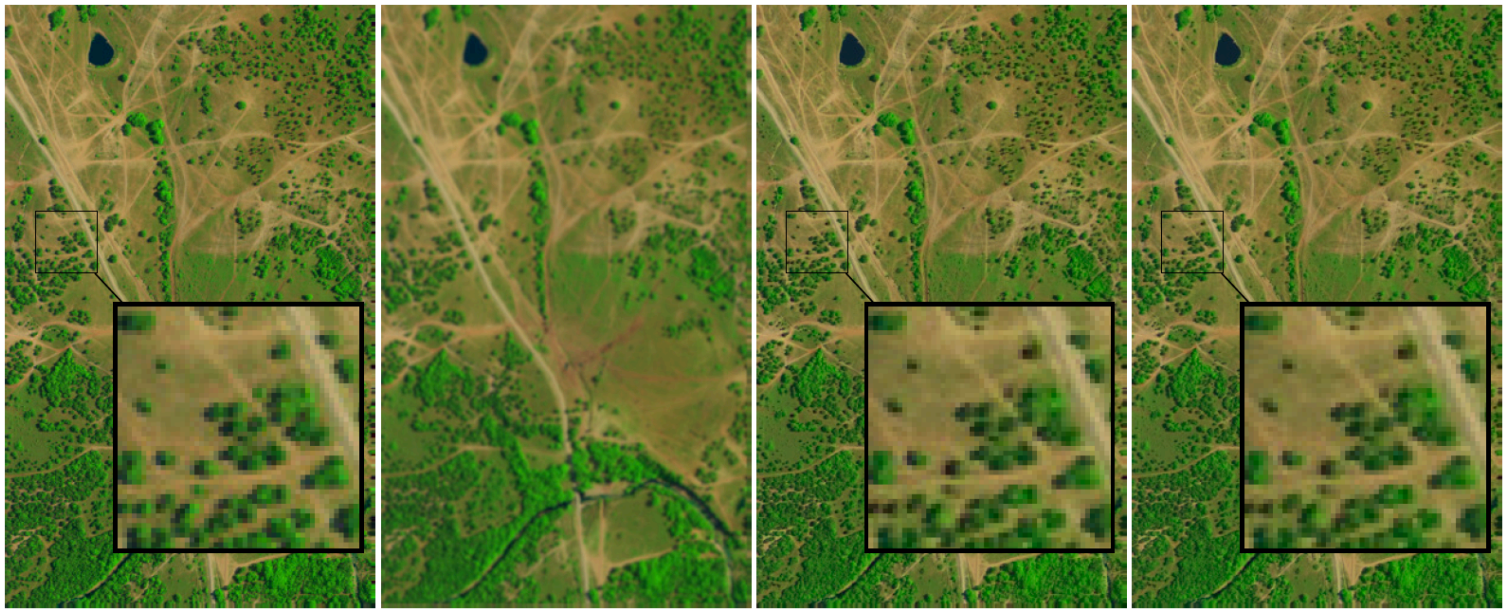
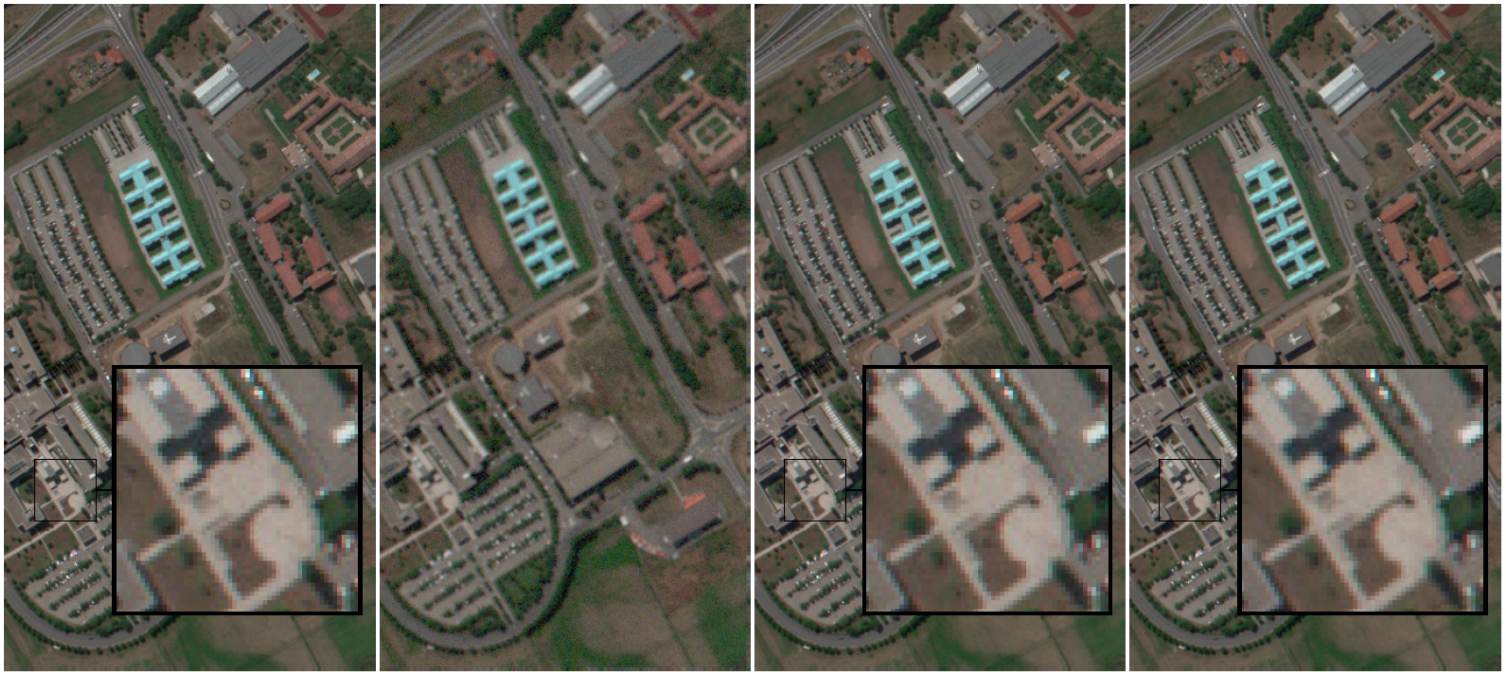


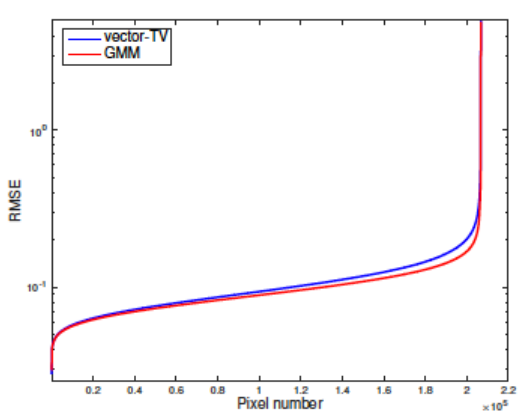
Table 3: HS and MS fusion on RTerrain dataset.

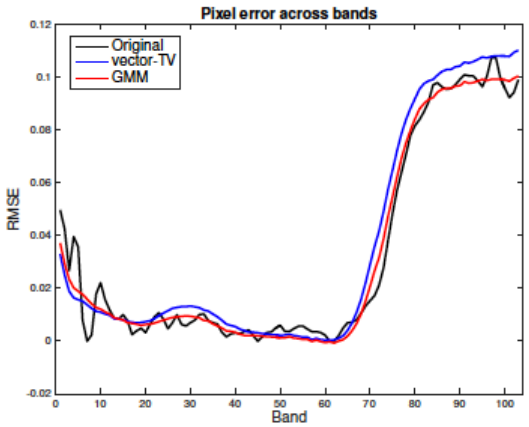
	Exp. 1 (PAN)			Exp. 2 (PAN)			Exp. 3 (R,G,B,N-IR)			Exp. 4 (R,G,B,N-IR)		
SNR ($\mathbf{Y}_m$)	50dB			30dB			50dB			30dB		
SNR ($\mathbf{Y}_h$)	50dB			20dB			50dB			20dB		
Metric	ERGAS	SAM	SRE	ERGAS	SAM	SRE	ERGAS	SAM	SRE	ERGAS	SAM	SRE
HySure	2.62	5.34	21.46	2.77	5.35	20.86	1.08	2.68	28.71	1.53	3.42	26.07
Proposed	2.58	**5.15**	**21.69**	**2.75**	**5.33**	**21.12**	**0.91**	**2.20**	**30.86**	**1.29**	**2.85**	**27.85**
ADMM-BM3D	**2.57**	5.17	21.65	2.76	5.36	21.08	0.93	2.22	30.80	1.31	2.91	27.72

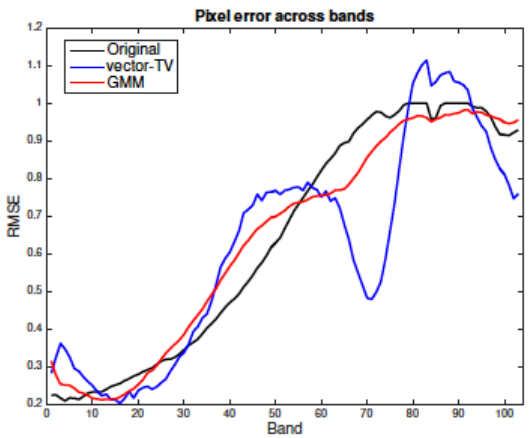
[Teodoro et al., 2016a]

Hyperspectral Fusion: Synthetic Example









Final Remarks

- ADMM/SALSA: a flexible toolbox for a variety inverse problems

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- Its speed hinges on the inversion of $(\mathbf{B}^T\mathbf{B} + \mathbf{I})$ (à la quasi-Newton)
- Plug-and-play (PnP) denoisers "can" be used with ADMM
- Convergence properties of PnP-ADMM with fixed linear denoiser

Thank you.

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