A multilevel fast-marching method

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Abstract: We introduce a numerical method for approximating the solutions for the following class of static Hamilton-Jacobi-Bellman equations arising from minimum time state constrained optimal control problems, after the usual change of variable $v_{\mathrm{s}\to}(x) = 1 - e^{-T_{\mathrm{s}\to}(x)}$:

$$\begin{cases} v_{\mathrm{s}\to}(x) - (\min\limits_{\alpha \in S_1}\{-(\nabla v_{\mathrm{s}\to}(x) \cdot \alpha) f(x,\alpha)\} + 1) = 0,\ x \in \Omega, \\ v_{\mathrm{s}\to}(x) - (\min\limits_{\alpha \in S_1}\{(-\nabla v_{\mathrm{s}\to}(x) \cdot \alpha) f(x,\alpha)\} + 1) \geq 0,\ x \in \partial\Omega, \\ v_{\mathrm{s}\to}(x_{\mathrm{src}}) = 0\ . \end{cases} \tag{1}$$

We are interested in computing $v_{\mathrm{s}\to}(x_{\mathrm{dst}})$ and an optimal trajectory from x_{src} to some given point $x_{\mathrm{dst}} \in \Omega$ only. We show that this is equivalent to solve the above state constrained equation on the subdomain $\mathcal{O}_\varepsilon \subseteq \Omega$ instead of Ω, where $\mathcal{O}_\varepsilon$ is given by:

$$\mathcal{O}_\varepsilon = \{x \mid (v_{\mathrm{s}\to}(x) + v_{\to\mathrm{t}}(x) - v_{\mathrm{s}\to}(x)v_{\to\mathrm{t}}(x)) < \inf_{y\in\Omega}\{(v_{\mathrm{s}\to}(y) + v_{\to\mathrm{t}}(y) - v_{\mathrm{s}\to}(y)v_{\to\mathrm{t}}(y)) + \varepsilon\}\ ,$$

and contains the true optimal trajectories from x_{src} to x_{dst}. Here, $v_{\to\mathrm{t}}$ corresponds to the minimal time from any point to x_{dst} by the usual change of variables.

Our algorithm takes advantage of this good property. Instead of finding the optimal trajectories directly, by solving a discretization of (1) in a fine grid, we first approximately find the subdomain $\mathcal{O}_\varepsilon$ by solving (1) in a coarse-grid. Then, we further discretize $\mathcal{O}_\varepsilon$ with a fine-grid, and solve on that fine-grid, Equation (1) in which Ω is replaced by $\mathcal{O}_\varepsilon$. The computation of approximated value functions on each grid is based on the fast-marching method [2].

We show that using our algorithm, the error estimation for the computation of $v_{\mathrm{s}\to}(x_{\mathrm{dst}})$ is as good as the one obtained using the fine grid to discretize the whole domain [1]. Moreover, we show that the number of computation operations as well as the memory allocation needed for an error of ε is in the order of $(\frac{1}{\varepsilon})^d$, whereas classical methods need at least $(\frac{1}{\varepsilon})^{2d}$. Under regularity conditions on f and the value function, this complexity bound reduces to $C^d \log(1/\varepsilon)$.

References:

[1] M. Bardi and M. Falcone, An approximation scheme for the minimum time function, *Siam Journal on Control and Optimization*, 28:950-965, 1990.

[2] J. A. Sethian and A. Vladimirsky, Ordered upwind methods for static Hamilton–Jacobi equations: Theory and algorithms, *SIAM Journal on Numerical Analysis*, 41(1):325–363,2003.