No self-concordant barrier interior point method is strongly polynomial

- Xavier Allamigeon (INRIA & CMAP, CNRS, Ecole polytechnique, IP Paris)
- Stéphane Gaubert (INRIA & CMAP, CNRS, Ecole polytechnique, IP Paris)
- **Nicolas Vandame** (CMAP, CNRS, Ecole polytechnique, IP Paris & INRIA)

Keywords: interior points methods, self-concordant barriers, tropical geometry

Abstract: The theory of self-concordant barriers was introduced by Nesterov and Nemirovskii [1]. Self-concordant barrier interior point methods (IPMs) transform the problem of minimizing the linear function $x \mapsto \langle c, x\rangle$ over a polyhedron in a parametric family of problems consisting in minimizing $f(x) + \eta\langle c, x\rangle$ for a parameter $\eta > 0$, where f is a self-concordant barrier. A self-concordant barrier is a strictly convex function which blows up at the border of the feasible set and whose Hessian locally doesn't change much. For each $\eta > 0$, the parametric problem has a unique minimizer $x(\eta)$, and the curve $\eta \mapsto x(\eta)$ is called the *central path*. As $\eta \to +\infty$, the central path converges to an optimal value of the unparametrized problem. IPMs follow the central path with increasing values of η to find an approximated solution.

One of the major open problems is computational optimization is to find a strongly polynomial algorithm for linear programming. This is known as Smale's ninth problem. Recall that a *strongly polynomial algorithm* is a polynomial time algorithm that uses a number of arithmetic operations bounded only by a polynomial in the number of numerical inputs (rather than the bitsize of these inputs). In this context, making any substantial progress on the understanding of the worst-case number of iterations performed by IPMs is a notorious open question.

Our contribution generalizes the result in [2], which showed that IPMs based on the logarithmic barrier are not strongly polynomial. To do so, the authors study the *tropicalization* of the central path of parametric families of linear programs. Tropical geometry studies the first order geometric features of a parametric problem by looking at its asymptotic growth rate as the parameter goes to $+\infty$. The tropicalization of the central path is a piecewise linear curve called the tropical central path.

We show here that *no* self-concordant barrier IPM is strongly polynomial. To achieve this, we show that any self-concordant barrier included in the positive orthant behaves essentially like the logarithmic barrier. We use this to show that IPMs draw polygonal curves in a multiplicative neighborhood of the central path whose log-limit coincides with the tropical central path, independently of the barrier. We provide an explicit parametric linear program that falls in the same class as the Klee–Minty counterexample, *i.e.*, in which the feasible set is a combinatorial cube and whose tropical central path is composed of $2^n - 1$ segments. When the parameter is large, the trajectory of the IPM approximates the tropical central path. This means that this trajectory must contain $2^n - 1$ "segments", or equivalently, that the IPM must perform that many iterations, thus breaking strong polynomiality. These results are presented in [3].

References:

[1] Yurii Nesterov and Arkadii Nemirovskii. Interior Point Polynomial Algorithms in Convex Programming. *Society for Industrial and Applied Mathematics*, 1994.

[2] Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert and Michael Joswig. Log-Barrier Interior Point Methods Are Not Strongly Polynomial. *SIAM Journal on Applied Algebra and Geometry*, Volume 2, Pages 140-178, 2018.

[3] Xavier Allamigeon, Stéphane Gaubert and Nicolas Vandame. No self-concordant barrier interior point method is strongly polynomial. https://arxiv.org/abs/2201.02186. To appear in the proceedings of STOC'2022, 54th Annual ACM Symposium on Theory of Computing.