# Infinite-Dimensional Sums-of-Squares for Optimal Control


- **Eloïse Berthier** (Inria - Ecole Normale Supérieure - PSL Research University)
- Justin Carpentier (Inria - Ecole Normale Supérieure - PSL Research University)
- Alessandro Rudi (Inria - Ecole Normale Supérieure - PSL Research University)
- Francis Bach (Inria - Ecole Normale Supérieure - PSL Research University)



**Keywords:** optimal control, kernel methods, numerical methods.



**Abstract:** We introduce an approximation method to solve an optimal control problem *via* the Lagrange dual of its weak formulation. Our method is based on a sum-of-squares representation of smooth non-negative functions, recently proposed as a non-parametric convex model, and concurrently applied to non-convex optimization [1], as well as optimal transport, sampling or probability modelling.

Concerning polynomials, sum-of-squares representations of non-negative functions date back to important existence results in the 1990s like Krivine-Stengle's and Putinar's Positivstellensatz. While it is clear that a sum-of-squares of polynomials is a non-negative polynomial, these theorems state that conversely, under suitable conditions on its domain, a non-negative polynomial can be written as a sum-of-squares of polynomials of a certain degree. Because these sums-of-squares can be efficiently handled numerically, this has given birth to the whole field of semi-algebraic optimization, namely with the celebrated Moment-SoS or Lasserre's hierarchy [2] in the 2000s. One of the many applications of this framework is the numerical resolution of polynomial optimal control problems [3].

Yet, Lasserre's hierarchy is limited to fully polynomial problems, and the size of the resulting hierarchy of semi-definite programs scales exponentially with the ambient space dimension. In the field of control, this limits its practical applicability to low-dimensional systems, hence excluding applications to robotics. In the field of machine learning, kernel methods have proved their practical efficiency when dealing with high-dimensional problems, way beyond the typical dimensions encountered in robotics. However, while some classical algorithms can be adapted to kernel methods, there is no unique and straightforward application of kernel methods to optimal control. We propose a generic way of doing so [4].

The representation of sum-of-squares in reproducing kernel Hilbert spaces (RKHS), as proposed in [1], is a versatile framework that encompasses Sobolev spaces, polynomials, as well as many other functional spaces. In particular, considering a Sobolev space as our reproducing kernel Hilbert space, we prove that if the Hamiltonian of the control problem is smooth, then it can be written as a sum-of-squares of smooth functions. This plays the same role as a Positivstellensatz, and, importantly, this does not require to build a hierarchy of increasing size. Indeed, the sum-of-squares representation of a non-negative function is directly infinite-dimensional. An important feature of an RKHS is that all practical computations can be performed using only accesses to the kernel function, without requiring to compute any infinite-dimensional feature representation. This is generically called the "kernel trick".

We focus on control problems for which the dynamics and cost function are unknown, but only accessed through observations. This is close to the model-free reinforcement learning paradigm. From the observations, our method computes an approximation of the value function. Since the number of observations is finite, this leads to a solvable optimization problem which is a semi-definite program. This provides approximations of the true solution of the optimal control problem. An important limitation is that, unlike the method of [3], it does not provide certified lower-bounds.


**References:**


[1] A. Rudi, U. Marteau-Ferey, and F. Bach. Finding global minima via kernel approximations. *Technical Report*, arXiv:2012.11978, 2020.

[2] J-B. Lasserre. *An introduction to polynomial and semi-algebraic optimization*, vol.52, Cambridge University Press, 2015.

[3] J-B. Lasserre, D. Henrion, C. Prieur, and E. Trélat. Nonlinear optimal control via occupation measures and LMI-relaxations. *SIAM Journal on Control and Optimization*, no. 4, 1643–1666, 2008.

[4] E. Berthier, J. Carpentier, A. Rudi, and F. Bach. Infinite-Dimensional Sums-of-Squares for Optimal Control. *Technical Report*, arXiv:2110.07396, 2021.