Bounding the Distance to Unsafe Sets with Convex Optimization

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Abstract: This talk presents a method to lower-bound the distance of closest approach between points on an unsafe set and points along system trajectories [1]. Such a minimal distance is a quantifiable and interpretable certificate of safety of trajectories, as compared to prior art in barrier which offers a binary indication of safety/unsafety [2]. The distance estimation problem is converted into a infinite-dimensional Linear Program (LP) in occupation measures based on existing work in optimal control [3], peak estimation [4] and optimal transport [5]. The moment-Sum of Squares hierarchy is used to obtain a sequence of lower bounds obtained through solving Linear Matrix Inequalities (LMIs) in increasing size [6], and these lower bounds will converge to the true minimal distance as the degree approaches infinity under mild conditions (e.g. Lipschitz dynamics, compact sets). The size of the largest positive semidefinite matrix constraint (moment matrix) grows as $\binom{2n+d}{d}$ in an n-state d-degree problem, and this large matrix size may be reduced to $\binom{n+1+d}{d}$ in the case where the distance function is separable (e.g. squared L_2 distance) through the application of correlative sparsity [7]. Near-optimal trajectories that achieve the minimal distance may be recovered if the solved moment matrices obey rank constraints [8]. The distance estimation problem can be modified to accommodate dynamics with uncertainty [9], and piecewise (norm) distance functions (e.g. L_1 and L_∞ distances) may be treated using the theory of polyhedral liftings [10]. Safety of shapes traveling in an evolving orientation along trajectories may be assured by bounding the set-set distance between points on the shape and points on the unsafe set with a sequence of LMIs.

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