Computer algebra techniques for testing the stability of n-D linear discrete systems

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Problem

- Given an N-D discrete system represented by its transfert function $G(z_1, \ldots, z_n) = N(z_1, \ldots, z_n)/D(z_1, \ldots, z_n) \in \mathbb{R}(z_1, \ldots, z_n)$
- We are interested in the structural stability of this system

Structural stability

An N-D discrete system is structurally stable if and only if $D(z_1, \ldots, z_n)$ is devoid from zero in the closed unit polydisc, i.e.

$$D(z_1, \ldots, z_n) \neq 0 \text{ for } |z_1| \leq 1, \ldots, |z_n| \leq 1.$$

Overview

Previous work : The case $n = 1$

- Numerous algebraic stability criteria : Jury test, Bistritz test, etc
- Discrete time analogues of the Routh-Hurwitz criterion
- Based on Cauchy index computation and sign variation in some polynomial sequences

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Bistritz test

$\mathbf{D(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0}$ is the characteristic pol. and $\mathbf{D^\star(z) = z^n\, D(z^{-1})}$

Compute the sequence of polynomials $T_i(z), i = n, \ldots, 0$ defined as

- $T_n(z) = D(z) + D^\star(z)$, $T_{n-1}(z) = \frac{D(z)+D^\star(z)}{(z-1)}$
- For $i = n-1, \ldots, 1$: $\delta_{i+1} = \frac{T_{i+1}(0)}{T_i(0)}$, $T_{i-1}(z) = \frac{\delta_{i+1}(1+z)T_i(z) - T_{i+1}(z)}{z}$

Criterion : the system is stable if and only if the sequence is normal and the number of sign variation in $\{T_n(1), \ldots, T_0(1)\}$ is zero.

Previous work : The case $n > 1$

The condition $D(z_1, \dots, z_n) \neq 0$ for $|z_1| \leq 1, \dots, |z_n| \leq 1$ is equivalent to

[Strintzis,Huang 1977]

$$
\begin{array}{ll}
D(0, \dots, 0, z_n) \neq 0 & \text{for } |z_n| \leq 1 \\
D(0, \dots, 0, z_{n-1}, z_n) \neq 0 & \text{for } |z_{n-1}| \leq 1, \ |z_n| = 1 \\
\vdots & \\
D(0, z_2, \dots, z_{n-1}, z_n) \neq 0 & \text{for } |z_2| \leq 1, \ |z_3| = \dots = |z_n| = 1 \\
D(z_1, z_2, \dots, z_{n-1}, z_n) \neq 0 & \text{for } |z_1| \leq 1, \ |z_2| = \dots = |z_n| = 1
\end{array}
$$

[DeCarlo et al, 1977]

$$
\begin{array}{ll}
D(z_1, 1, \dots, 1) \neq 0 & \text{for } |z_1| \leq 1 \\
D(1, z_2, 1, \dots, 1) \neq 0 & \text{for } |z_2| \leq 1 \\
\vdots & \\
D(1, \dots, 1, z_n) \neq 0 & \text{for } |z_n| \leq 1 \\
D(z_1, \dots, z_n) \neq 0 & \text{for } |z_1| = \dots = |z_n| = 1
\end{array}
$$

Implementations

- Numerous algorithms in 2D, Bistritz (94,99,02,03,04), Xu et al. 04, Fu et al. 06, etc

- Most of them are based on the Strintzis's conditions

$$\begin{cases} D(z_1, 0) \neq 0, \ |z_1| \leq 1 \\ D(z_1, z_2) \neq 0, \ |z_1| = 1, \ |z_2| \leq 1 \end{cases}$$

- The condition $D(z_1, z_2) \neq 0, \ |z_1| = 1, \ |z_2| \leq 1$ is tested by generalizing the univariate tests

- Few in ND with $N > 2$, Serban and Najim (07), Dumitrescu (08)

Overview

Introduction

We start from the DeCarlo's conditions

$$
\begin{array}{ll}
D(z_1, 1, \ldots, 1) \neq 0 & \text{for } |z_1| \leq 1 \\
D(1, z_2, \ldots, 1) \neq 0 & \text{for } |z_2| \leq 1 \\
\vdots & \\
D(1, \ldots, 1, z_n) \neq 0 & \text{for } |z_n| \leq 1 \\
D(z_1, \ldots.., z_n) \neq 0 & \text{for } |z_1| = \ldots = |z_n| = 1
\end{array}
$$

- All the conditions except the last one can be tested using classical univariate stability tests.
- Focus on the condition $\mathcal{C}$: $D(z_1, \ldots, z_n) \neq 0, |z_1| = \ldots = |z_n| = 1$
  1. Transform the condition into an algebraic condition
  2. Check it by means of solving algebraic systems algorithms

One first approach

If $z_i = x_i + iy_i$ for $i = 1, \ldots, n$ with $x_i, y_i \in \mathbb{R}$, the problem is equivalent to the study of the following algebraic system

$$S = \begin{cases} \mathcal{R}(D(x_1 + iy_1, \ldots, x_n + iy_n)) = D_r(x_1, y_1, \ldots, x_n, y_n) = 0 \\ \mathcal{C}(D(x_1 + iy_1, \ldots, x_n + iy_n)) = D_c(x_1, y_1, \ldots, x_n, y_n) = 0 \\ x_i^2 + y_i^2 - 1 = 0 \ \text{ for } \ i = 1, \ldots, n \end{cases}$$

The condition $\mathcal{C}$ is satisfied if and only if S has no real solutions : $V_{\mathbb{R}}(S) = \emptyset$

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- Case $n = 2$: S is generically zero-dimensional $\rightsquigarrow$ Parametrization of the solution, Triangular Representation, Grobner Basis
- Case $n > 2$: S is of positive dimension $\rightsquigarrow$ Cylindrical Algebraic Decomposition, Critical Points Methods

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Drawback : Such a transformation doubles the number of variables

Alternative transformations

- The unit poly-circle defines a n-D subspace in the $2n$-D complex space.
- The condition can be reduced modulo some transformations, to that of the existence of real zeros
  - Inside the unit hyper-cube $[-1, 1]^n$
  - In the whole real space $\mathbb{R}^n$
- Without increasing the number of variables !

For simplicity we consider the case $n = 2$ i.e. $D(z_1, z_2) \neq 0$ for $|z_1| = |z_2| = 1$

From the unit bi-circle to the unit box

Theorem (N.K. Bose)

Let $H(z) \in \mathbb{R}[z^{\pm}]$ be a Laurent polynomial such that $H(z) = H(z^{-1})$. Then, $H(z)$ can be converted into a polynomial $f(x)$ via $x = \frac{1}{2}(z + z^{-1})$ such that

$$H(z) \neq 0 \text{ for } |z| = 1 \text{ iff } f(x) \neq 0 \text{ for } x \in [-1, 1]$$

From the unit circle to the interval $[-1, 1]$

Given $D(z) \in \mathbb{R}[z]$. Define the polynomial $H(z) = D(z)\, D(z^{-1})$.

- $H(z) = H(z^{-1}) = \sum_{i=0}^{d} c_i (z^i + z^{-i})$
- $x = \frac{1}{2}(z + z^{-1}) \Rightarrow z^i + z^{-i} = 2\, T_i(x)$ where T_i denotes the i-th Tchebychev polynomial

From the unit bi-circle to the unit box

The Case n=2

$D(z_1, z_2) \in \mathbb{R}[z_1, z_2]$. $H(z_1, z_2) = D(z_1, z_2)\, D(z_1^{-1}, z_2)\, D(z_1, z_2^{-1})\, D(z_1^{-1}, z_2^{-1})$

- $H(z_1, z_2)$ can be converted into a polynomial $f(x, y)$ using the transformations $x = \frac{1}{2}(z_1 + z_1^{-1})$ and $y = \frac{1}{2}(z_2 + z_2^{-1})$
- $D(z_1, z_2) \neq 0$ for $|z_1| = |z_2| = 1$ iff $f(x, y) \neq 0$ for $(x, y) \in [-1, 1]^2$

Transformation

- $H(z_1, z_2) = \sum_{k=-d}^{d} \sum_{i=0}^{2d} c_i (z_1^i + z_1^{-i}) \times z_2^k : x = \frac{1}{2}(z_1 + z_1^{-1}) \Rightarrow \sum_{k=-d}^{d} h_k(x) z_2^k$
- $H(x, z_2) = \sum_{k=-d}^{d} \sum_{i=0}^{2d} c_i (z_2^i + z_2^{-i}) \times x^k : y = \frac{1}{2}(z_2 + z_2^{-1}) \Rightarrow f(x, y)$

From the unit circle to $\mathbb{R}^2$

Since we are looking for complex zeros of $D(z_1, z_2)$ on the unit bi-circle

- We can use the parametrization of the complex unit circle.
  - $z_1 = (1 - x^2)/(1 + x^2) + i \times 2x/(1 + x^2)$
  - $z_2 = (1 - y^2)/(1 + y^2) + i \times 2y/(1 + y^2)$
- Define the polynomial $f(x, y) = f_r(x, y) + if_c(x, y)$ as the numerator of $D(\frac{1-x^2}{1+x^2} + i\frac{2x}{1+x^2}, \frac{1-y^2}{1+y^2} + i\frac{2y}{1+y^2})$

Theorem

$D(z_1, z_2) \neq 0$ for $|z_1| = |z_2| = 1$ if and only if $V_{\mathbb{R}}(\{f_r(x, y) = f_c(x, y) = 0\}) = \emptyset$

Summary

The condition $D(z_1, z_2) \neq 0$ for $|z_1| = |z_2| = 1$ can be reduced to

1. $f(x, y) \neq 0$ for $(x, y) \in [-1, 1]^2$

   Or

2. $V_{\mathbb{R}}(\{f_r(x, y) = f_c(x, y) = 0\}) = \emptyset$

$f(x, y)$, $f_r(x, y)$ and $f_c(x, y)$ have total degree twice that of D.

Checking for real zeros in $\mathbb{R}^2$

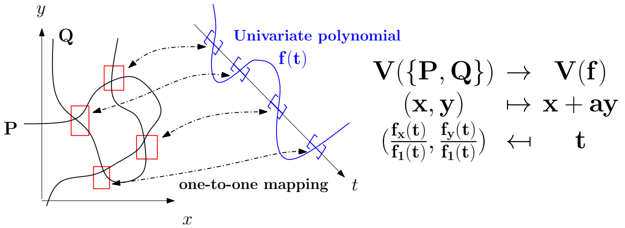
- Generically, the system $\{f_r(x, y), f_c(x, y)\}$ is zero dimensional
- Goal : Compute the number of its real solutions
- Approach : Compute a symbolic representation of the initial system that eases the count and the isolation of its solutions.

A convenient representation is the Rational Univariate Representation

Rational Univariate Representation

Let $\langle P, Q \rangle$ be a zero-dim ideal and V its variety. A RUR of $\langle P, Q \rangle$ is given by :

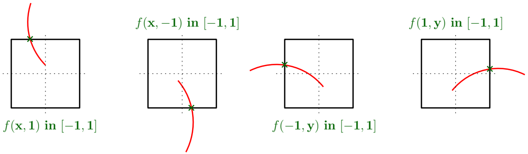
- A linear form $x + ay$ that separates the points of V
- A one-to-one mapping between the roots of an univariate polynomial f and the solutions of V



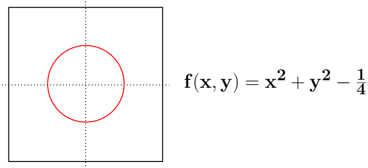
$V(\{P, Q\}) \cap \mathbb{R}^2 = \emptyset$ if and only if $V(f) \cap \mathbb{R} = \emptyset$

Checking for real zeros in $[-1, 1]^2$

- Check if the curve $\mathcal{C}$ defined by the implicite equation $f(x, y) = 0$ intersecte the boundaries of the unit box



- If not ? it may have one or several connected components inside the box



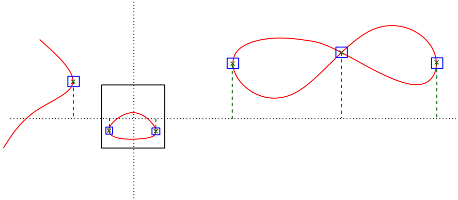
- Question : How to check the existence of real component inside the box ?

Critical points method

- $\pi : (x, y) \mapsto x$ is the projection onto the x-axis.
- The critical points of π restricted to $\mathcal{C}$ are the solutions of the system $\{f(x, y), \frac{\partial f(x,y)}{\partial y}\}$.

Theorem

The set of critical points of π meets the curve $\mathcal{C}$ on each of its real connected components.



- Check if $V(\{f(x, y), \frac{\partial f(x,y)}{\partial y}\}) \cap]-1, 1[^2 = \emptyset$ (RUR+Numerical isolation)

The case $n > 2$

The condition $D(z_1, \ldots, z_n) \neq 0$, $|z_1| = \ldots = |z_n| = 1$ becomes

1. $f(x_1, \ldots, x_n) \neq 0$ for $-1 \leq x_1 \leq 1 \ldots -1 \leq x_n \leq 1$
   - by the transformation $x_i = \frac{1}{2}(z_i + z_i^{-1})$ for $i = 1, \ldots, n$ on the polynomial $H(z_1, \ldots, z_n) = \prod_{z_i \in \{z_i, z_i^{-1}\}} D(z_1, \ldots, z_n)$
2. $\{f_r(x_1, \ldots, x_n) = f_c(x_1, \ldots, x_n) = 0\} \cap \mathbb{R}^n = \emptyset$
   - by the map $(z_1, \ldots, z_n) \mapsto \left(\frac{1-x_1^2}{1+x_1^2} + i\frac{2x_1}{1+x_1^2}, \ldots, \frac{1-x_n^2}{1+x_n^2} + i\frac{2x_n}{1+x_n^2}\right)$

The total degree of $f(x_1, \ldots, x_n)$ is 2^{n-1} times the degree of D.

The total degree of $f_r(x_1, \ldots, x_n)$ and $f_c(x_1, \ldots, x_n)$ is only twice that of D.

Checking for real zeros in $\mathbb{R}^n$

- The systems are no longer zero-dimensional
- Use the critical points method to compute real solutions in each connected component
- More involved when $n > 2$ but still works under mild conditions

Overview

Conclusion

- An implementation on Maple based on the external library RagLib is available.
- Preliminary tests show the relevance of our approach.

Ongoing work :

- Alternative transformations that keep the degree of the polynomials unchanged, e.g. Mobius transformations : $z_j \mapsto \frac{z_j - i}{z_j + i}$
- Certified numerical methods for testing the existence of real solutions.

Thank you !