Computations of Hypergeometric Generating Series for Small-Step Walks in the Quarter Plane

Frédéric Chyzak

informatiques mathématiques Inria

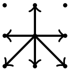
Functional Equation in LIMoges 2015

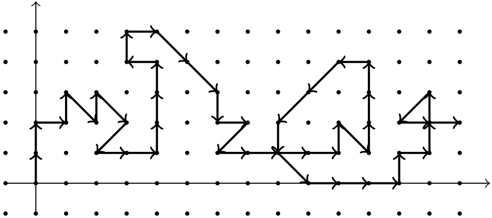
Joint work with A. Bostan, M. Kauers, L. Pech, and M. van Hoeij

▹ Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a prefixed subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▹ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 

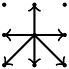


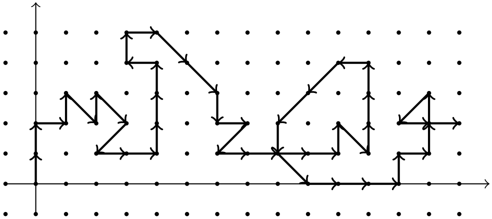
Enumerative Combinatorics of Lattice Walks

▷ Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a prefixed subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 

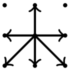


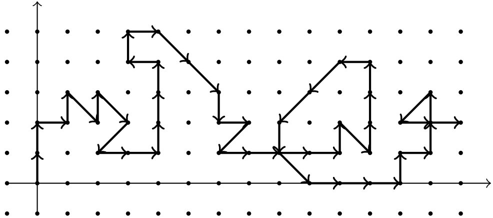
▷ $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a prefixed subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



▷ $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Special, combinatorially meaningful specializations:

- $f_{n;0,0}$ counts walks of length n returning to the origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ counts all walks with prescribed length n.

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

Generating Series and Combinatorial Problems

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x,y][[t]].$$

▷ Special, combinatorially meaningful specializations:

- $F(t;0,0)$ counts walks returning to the origin (“excursions”);
- $F(t;1,1) = \sum_{n\geq 0} f_n t^n$ counts walks with prescribed length.

Generating Series and Combinatorial Problems

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x,y][[t]].$$

▷ Special, combinatorially meaningful specializations:

- $F(t;0,0)$ counts walks returning to the origin ("excursions");
- $F(t;1,1) = \sum_{n\geq 0} f_n t^n$ counts walks with prescribed length.

Combinatorial questions: Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Algebraic nature of F: algebraic? transcendental?
- Explicit form: of F? of f?
- Asymptotics of f?

Generating Series and Combinatorial Problems

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

▷ Special, combinatorially meaningful specializations:

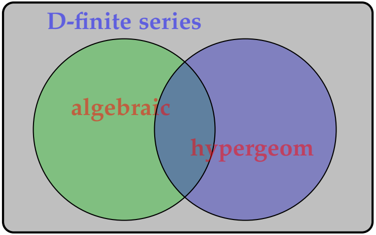
- $F(t;0,0)$ counts walks returning to the origin (“excursions”);
- $F(t;1,1) = \sum_{n \geq 0} f_n t^n$ counts walks with prescribed length.

Combinatorial questions: Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

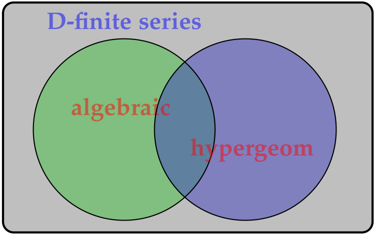
- Algebraic nature of F: algebraic? transcendental?
- Explicit form: of F? of f?
- Asymptotics of f?

Our goal: Use computer algebra to give computational answers.

Important Classes of Univariate Power Series

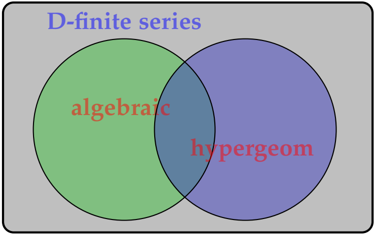


Important Classes of Univariate Power Series



▷ *Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.

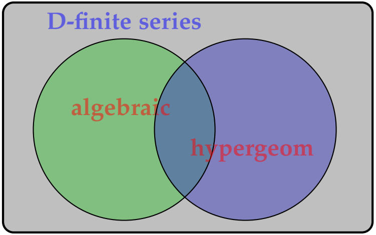
Important Classes of Univariate Power Series



▷ *Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.

▷ *D-finite*: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

Important Classes of Univariate Power Series



▷ *Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.

▷ *D-finite*: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

▷ *Hypergeometric*: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

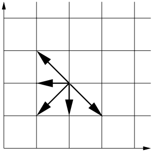
$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1), $$

$$ t(1-t)S''(t) + \big(c - (a+b+1)t\big)S'(t) - abS(t) = 0. $$

Small-Step Models of Interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

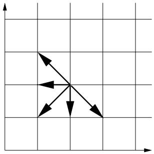
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



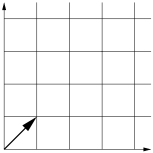
trivial,

Small-Step Models of Interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



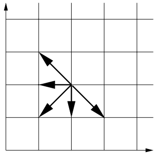
trivial,



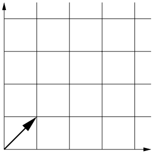
simple,

Small-Step Models of Interest

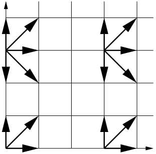
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$, some are:



trivial,



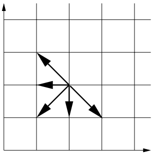
simple,



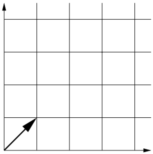
intrinsic to the half plane,

Small-Step Models of Interest

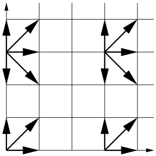
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



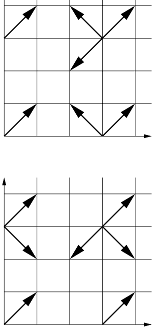
trivial,



simple,



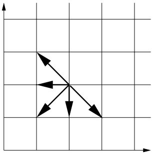
intrinsic to the half plane,



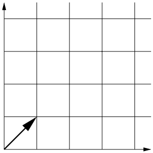
symmetrical.

Small-Step Models of Interest

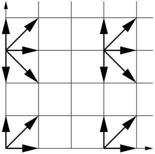
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



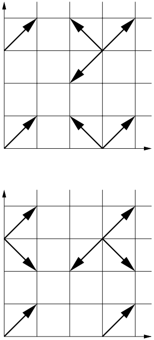
trivial,



simple,



intrinsic to the half plane,



symmetrical.

One is left with 79 interesting distinct models.

Table of All Conjectured D-Finite $F(t;1,1)$ [Bostan & Kauers 2009]

	OEIS	$\mathfrak{S}$	alg	equiv		OEIS	$\mathfrak{S}$	alg	equiv
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

PROVE THIS TABLE!

Previous Work on the 23 Cases

▷ Human proof of D-finiteness/algebraicity for cases 1–22 in [Bousquet-Mélou & Mishna, 2010]:

- based on averaging over a group of rational invariant transformations,
- but implied algorithm too expensive to exhibit ODEs!

▷ Computer proof of algebraicity for case 23 in [Bostan & Kauers, 2010].

Previous Work on the 23 Cases

▷ Human proof of D-finiteness/algebraicity for cases 1–22 in [Bousquet-Mélou & Mishna, 2010]:

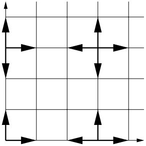
- based on averaging over a group of rational invariant transformations,
- but implied algorithm too expensive to exhibit ODEs!

▷ Computer proof of algebraicity for case 23 in [Bostan & Kauers, 2010].

▷ Human proofs of asymptotics $f_n \sim \kappa n^\alpha \rho^n$:

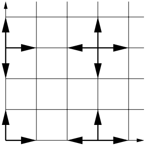
- ρ for all cases in [Fayolle & Raschel, 2012];
- (α, ρ) for cases 1–4,17–23 (zero drift) using [Denisov & Wachtel, 2013];
- (κ, α, ρ) for cases 1–4 (2 axes of sym.) in [Melczer & Mishna, 2014];
- (κ, α, ρ) for cases 17–22 in [Bousquet-Mélou & Mishna, 2010];
- (κ, α, ρ) for case 23 in [Bostan and Kauers, 2010].

The Kernel Equation [≤ Knuth, 1968]: an Example



walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$

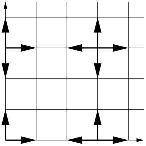
The Kernel Equation [≤ Knuth, 1968]: an Example



walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$,

provided this remains in the quarter plane!

The Kernel Equation [≤ Knuth, 1968]: an Example



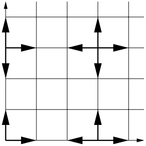
walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$,

provided this remains in the quarter plane!

Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!] f_{n;i,j-1} + [\![0 < i]\!] f_{n;i-1,j} + f_{n;i,j+1}.$$

The Kernel Equation [≤ Knuth, 1968]: an Example



walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$,

provided this remains in the quarter plane!

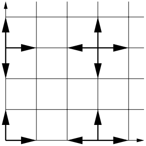
Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!] f_{n;i,j-1} + [\![0 < i]\!] f_{n;i-1,j} + f_{n;i,j+1}.$$

$$f_{n+1;i,j} x^i y^j t^{n+1} = \left(f_{n;i+1,j} x^{i+1} y^j t^n\right) \times \bar{x}t + [\![0 < j]\!] \left(f_{n;i,j-1} x^i y^{j-1} t^n\right) \times yt + [\![0 < i]\!] \left(f_{n;i-1,j} x^{i-1} y^j t^n\right) \times xt + \left(f_{n;i,j+1} x^i y^{j+1} t^n\right) \times \bar{y}t,$$

Notation: $\bar{x} = \dfrac{1}{x}, \quad \bar{y} = \dfrac{1}{y}.$

The Kernel Equation [≤ Knuth, 1968]: an Example



walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$,

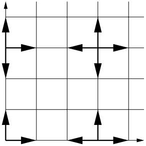
provided this remains in the quarter plane!

Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!] f_{n;i,j-1} + [\![0 < i]\!] f_{n;i-1,j} + f_{n;i,j+1}.$$

$$f_{n+1;i,j} x^i y^j t^{n+1} = \left(f_{n;i+1,j} x^{i+1} y^j t^n\right) \times \bar{x} t + [\![0 < j]\!] \left(f_{n;i,j-1} x^i y^{j-1} t^n\right) \times y t +$$
$$[\![0 < i]\!] \left(f_{n;i-1,j} x^{i-1} y^j t^n\right) \times x t + \left(f_{n;i,j+1} x^i y^{j+1} t^n\right) \times \bar{y} t,$$
$$F(t;x,y) - 1 = \big(F(t;x,y) - F(t;0,y)\big) \times \bar{x} t + F(t;x,y) \times y t +$$
$$F(t;x,y) \times x t + \big(F(t;x,y) - F(t;x,0)\big) \times \bar{y} t,$$

The Kernel Equation [≤ Knuth, 1968]: an Example



walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$,

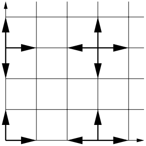
provided this remains in the quarter plane!

Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!] f_{n;i,j-1} + [\![0 < i]\!] f_{n;i-1,j} + f_{n;i,j+1}.$$

Functional Equation in LIMoges:

$$((x + \bar{x} + y + \bar{y})\, t - 1)\, F(t; x, y) = \bar{y} t F(t; x, 0) + \bar{x} t F(t; 0, y) - 1.$$



walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$,

provided this remains in the quarter plane!

Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!] f_{n;i,j-1} + [\![0 < i]\!] f_{n;i-1,j} + f_{n;i,j+1}.$$

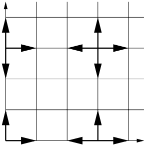
Functional Equation in LIMoges:

$$((x + \bar{x} + y + \bar{y})\, t - 1)\, F(t; x, y) = \bar{y} t F(t; x, 0) + \bar{x} t F(t; 0, y) - 1.$$

Remarks:

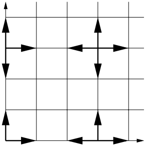
- Erasing the constraint leads to a rational generating series.
- Direct attempt to solve leads to tautologies.

D-Finiteness via the Finite Group: an Example



$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$



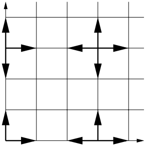
$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

Kernel equation:

$$J(t; x, y)xytF(t; x, y) = txF(t; x, 0) + tyF(t; 0, y) - xy,$$

D-Finiteness via the Finite Group: an Example



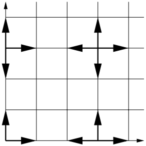
$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

Kernel equation:

$$J(t; x, y)xytF(t; x, y) = txF(t; x, 0) + tyF(t; 0, y) - xy,$$
$$-J(t; x, y)\bar{x}ytF(t; \bar{x}, y) = -t\bar{x}F(t; \bar{x}, 0) - tyF(t; 0, y) + \bar{x}y,$$

D-Finiteness via the Finite Group: an Example



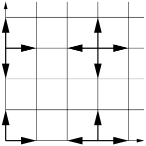
$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

Kernel equation:

$$\begin{aligned} J(t;x,y)xytF(t;x,y) &= txF(t;x,0) + tyF(t;0,y) - xy, \\ -J(t;x,y)\bar{x}ytF(t;\bar{x},y) &= -t\bar{x}F(t;\bar{x},0) - tyF(t;0,y) + \bar{x}y, \\ J(t;x,y)\bar{x}\bar{y}tF(t;\bar{x},\bar{y}) &= t\bar{x}F(t;\bar{x},0) + t\bar{y}F(t;0,\bar{y}) - \bar{x}\bar{y}, \end{aligned}$$

D-Finiteness via the Finite Group: an Example

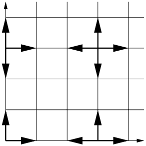


$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

Kernel equation:

$$\begin{aligned}
J(t;x,y)xytF(t;x,y) &= txF(t;x,0) + tyF(t;0,y) - xy,\\
-J(t;x,y)\bar{x}ytF(t;\bar{x},y) &= -t\bar{x}F(t;\bar{x},0) - tyF(t;0,y) + \bar{x}y,\\
J(t;x,y)\bar{x}\bar{y}tF(t;\bar{x},\bar{y}) &= t\bar{x}F(t;\bar{x},0) + t\bar{y}F(t;0,\bar{y}) - \bar{x}\bar{y},\\
-J(t;x,y)x\bar{y}tF(t;x,\bar{y}) &= -txF(t;x,0) - t\bar{y}F(t;0,\bar{y}) + x\bar{y}.
\end{aligned}$$



$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

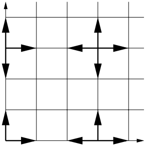
Kernel equation:

$$\begin{aligned} J(t;x,y)xytF(t;x,y) &= txF(t;x,0) + tyF(t;0,y) - xy, \\ -J(t;x,y)\bar{x}ytF(t;\bar{x},y) &= -t\bar{x}F(t;\bar{x},0) - tyF(t;0,y) + \bar{x}y, \\ J(t;x,y)\bar{x}\bar{y}tF(t;\bar{x},\bar{y}) &= t\bar{x}F(t;\bar{x},0) + t\bar{y}F(t;0,\bar{y}) - \bar{x}\bar{y}, \\ -J(t;x,y)x\bar{y}tF(t;x,\bar{y}) &= -txF(t;x,0) - t\bar{y}F(t;0,\bar{y}) + x\bar{y}. \end{aligned}$$

Summing up yields:

$$\sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xyt\,F(t;x,y)\big) = \frac{-xy + \bar{x}y - \bar{x}\bar{y} + x\bar{y}}{J(t;x,y)}.$$

D-Finiteness via the Finite Group: an Example



$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

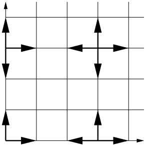
$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

Kernel equation:

$$\begin{aligned} J(t;x,y)xytF(t;x,y) &= txF(t;x,0) + tyF(t;0,y) - xy, \\ -J(t;x,y)\bar{x}ytF(t;\bar{x},y) &= -t\bar{x}F(t;\bar{x},0) - tyF(t;0,y) + \bar{x}y, \\ J(t;x,y)\bar{x}\bar{y}tF(t;\bar{x},\bar{y}) &= t\bar{x}F(t;\bar{x},0) + t\bar{y}F(t;0,\bar{y}) - \bar{x}\bar{y}, \\ -J(t;x,y)x\bar{y}tF(t;x,\bar{y}) &= -txF(t;x,0) - t\bar{y}F(t;0,\bar{y}) + x\bar{y}. \end{aligned}$$

Summing up yields:

$$[x^>][y^>] \sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xyt\,F(t;x,y)\big) = [x^>][y^>] \frac{-xy + \bar{x}y - \bar{x}\bar{y} + x\bar{y}}{J(t;x,y)}.$$



$J = -\frac{1}{t} + \sum_{(i,j)\in\mathfrak{S}} x^i y^j = x + \bar{x} + y + \bar{y} - \frac{1}{t}$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}).$$

Kernel equation:

$$\begin{aligned} J(t;x,y)xytF(t;x,y) &= txF(t;x,0) + tyF(t;0,y) - xy, \\ -J(t;x,y)\bar{x}ytF(t;\bar{x},y) &= -t\bar{x}F(t;\bar{x},0) - tyF(t;0,y) + \bar{x}y, \\ J(t;x,y)\bar{x}\bar{y}tF(t;\bar{x},\bar{y}) &= t\bar{x}F(t;\bar{x},0) + t\bar{y}F(t;0,\bar{y}) - \bar{x}\bar{y}, \\ -J(t;x,y)x\bar{y}tF(t;x,\bar{y}) &= -txF(t;x,0) - t\bar{y}F(t;0,\bar{y}) + x\bar{y}. \end{aligned}$$

Summing up yields:

$$xyt\,F(t;x,y) = [x^>][y^>]\frac{-xy + \bar{x}y - \bar{x}\bar{y} + x\bar{y}}{J(t;x,y)}.$$

Cases 1–19 are D-Finite

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

$$xyt\,F(t;x,y) = [x^>][y^>]\,\frac{\sum_{\theta\in\mathcal{G}}(-1)^\theta\theta(xy)}{J(t;x,y)}.$$

In particular, $F(t;x,y)$ is D-finite.

Cases 1–19 are D-Finite

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

$$xyt\,F(t;x,y) = [x^>][y^>]\,\frac{\sum_{\theta\in\mathcal{G}}(-1)^{\theta}\theta(xy)}{J(t;x,y)}.$$

In particular, $F(t;x,y)$ is D-finite.

Proof: Use [Lipshitz'88] (*"The diagonal of a D-finite power series is D-finite"*) for positive parts of D-finite series.

▷ Constructive proof, but it leads to a highly inefficient algorithm to get an ODE for $F(t;x,y)$.

Cases 1–19 are D-Finite

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

$$xyt\,F(t;x,y) = [x^>][y^>]\,\frac{\sum_{\theta\in\mathcal{G}}(-1)^{\theta}\theta(xy)}{J(t;x,y)}.$$

In particular, $F(t;x,y)$ is D-finite.

Proof: Use [Lipshitz'88] (*"The diagonal of a D-finite power series is D-finite"*) for positive parts of D-finite series.

▷ Constructive proof, but it leads to a highly inefficient algorithm to get an ODE for $F(t;x,y)$; in fact, any such ODE is probably

TOO LARGE TO BE MERELY WRITTEN!

Cases 1–19 are D-Finite

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

$$xyt\,F(t;x,y) = [x^>][y^>]\,\frac{\sum_{\theta\in\mathcal{G}}(-1)^\theta\theta(xy)}{J(t;x,y)}.$$

In particular, $F(t;x,y)$ is D-finite.

Proof: Use [Lipshitz'88] (*"The diagonal of a D-finite power series is D-finite"*) for positive parts of D-finite series.

▷ Constructive proof, but it leads to a highly inefficient algorithm to get an ODE for $F(t;x,y)$; in fact, any such ODE is probably

TOO LARGE TO BE MERELY WRITTEN!

▷ Remark: The formula provides no direct information for $x = y = 1$.

Explicit Expressions for the Cases 1–19

Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Explicit Expressions for the Cases 1–19

Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;1,1)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Example: King walks in the quarter plane (A025595)

$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

Explicit Expressions for the Cases 1–19

Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;1,1)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Example: King walks in the quarter plane (A025595)

$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

Obtained by obtaining and solving:

$$t^2(4t+1)(8t-1)(2t-1)(t+1)y''' + t(576t^4 + 200t^3 - 252t^2 - 33t + 5)y'' + (1152t^4 + 88t^3 - 468t^2 - 48t + 4)y' + (384t^3 - 72t^2 - 144t - 12)y = 0.$$

Explicit Expressions for the Cases 1–19

Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

▷ Proof uses Creative telescoping, ODE factorization, ODE solving:

1. (New) If $R = \sum_\theta \frac{(-1)^\theta\, \theta(xy)}{J(t;x,y)}$, then $[x^>][y^>]R = \mathrm{Res}_u\,(\mathrm{Res}_v\, H)$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.

2. (New) If $L \in \mathbb{Q}(x,y)[t]\langle\partial_t\rangle$ and $U, V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t;x,y)) = 0$.
   Use creative telescoping for finding L (as well as U and V).

3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

Explicit Expressions for the Cases 1–19

Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

▷ Proof uses Creative telescoping, ODE factorization, ODE solving:

1. (New) If $R = \sum_\theta \frac{(-1)^\theta \theta(xy)}{J(t;x,y)}$, then $[x^>][y^>]R = \mathrm{Res}_u \left(\mathrm{Res}_v H\right)$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.
   Taking algebraic residues commutes with specializing x and y!
2. (New) If $L \in \mathbb{Q}(x,y)[t]\langle\partial_t\rangle$ and $U,V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t;x,y)) = 0$.
   Use creative telescoping for finding L (as well as U and V).
   Works in practice with early evaluation $(x,y) = (1,1)$, but not for symbolic (x,y).
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

Explicit Expressions for the Cases 1–19

Theorem [Bostan-Chyzak-van Hoeij-Kauers-Pech, 2014]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

▷ Proof uses Creative telescoping, ODE factorization, ODE solving:

1. (New) If $R = \sum_\theta \frac{(-1)^\theta\,\theta(xy)}{J(t;x,y)}$, then $[x^>][y^>]R = \mathrm{Res}_u\,(\mathrm{Res}_v\,H)$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.
   Taking algebraic residues commutes with specializing x and y!
2. (New) If $L \in \mathbb{Q}(x,y)[t]\langle\partial_t\rangle$ and $U, V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t;x,y)) = 0$.
   Use creative telescoping for finding L (as well as U and V).
   Works in practice with early evaluation $(x,y) = (1,1)$, but not for symbolic (x,y).
   Works also for $(0,0)$, $(x,0)$, and $(0,y)$!
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

For $F(t;x,y)$, run whole process for $F(t;0,0)$, $F(t;x,0)$, and $F(t;0,y)$, then substitute into kernel equation!

Computing Positive Parts as Residues

Problem: Definitions of residues and positive parts of rational functions?

$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} = \frac{1}{1-w} = 1 + w + w^2 + \cdots$$

Computing Positive Parts as Residues

Problem: Definitions of residues and positive parts of rational functions?

$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} = \qquad \frac{1}{1-w} = 1 + w + w^2 + \cdots$$

$$-1 = \operatorname{Res}_w \frac{1}{1-w} = 0$$

Computing Positive Parts as Residues

Problem: Definitions of residues and positive parts of rational functions?

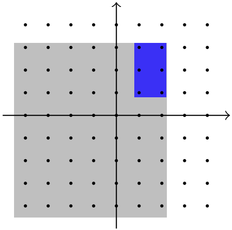
$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} = \qquad \frac{1}{1-w} = 1 + w + w^2 + \cdots$$

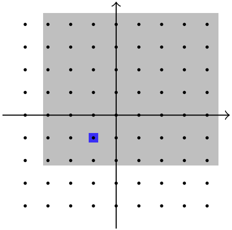
$$0 = [w^>]\frac{1}{1-w} = w + w^2 + \cdots$$

Problem: Definitions of residues and positive parts of rational functions?

$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} = \qquad \frac{1}{1-w} = 1 + w + w^2 + \cdots$$

Our solution:





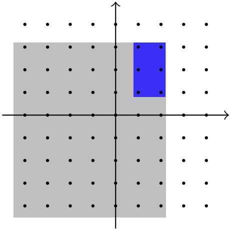
$[\bar{u}^{<}\bar{v}^{<}] : \mathbb{Q}[[\bar{u},\bar{v}]][u,v] \to uv\mathbb{Q}[u,v]$

$\mathrm{Res}_{u,v} : \mathbb{Q}[[u,v]][\bar{u},\bar{v}] \to \mathbb{Q}$

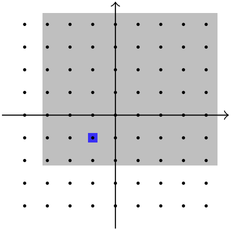
Problem: Definitions of residues and positive parts of rational functions?

$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} = \qquad \frac{1}{1-w} = 1 + w + w^2 + \cdots$$

Our solution:



$[\bar{u}^{<}\bar{v}^{<}] : \mathbb{Q}[[\bar{u},\bar{v}]][u,v] \to uv\mathbb{Q}[u,v]$

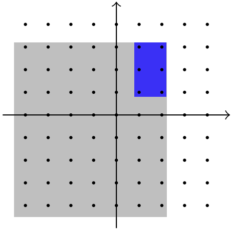


$\mathrm{Res}_{u,v} : \mathbb{Q}[[u,v]][\bar{u},\bar{v}] \to \mathbb{Q}$

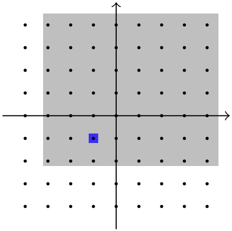
$$Z \in \mathbb{Q}[[u,v]][\bar{u},\bar{v}] \Rightarrow \left(\bar{u}\bar{v}[\bar{u}^{<}\bar{v}^{<}]Z(\bar{u},\bar{v})\right)_{u=a,v=b} = \mathrm{Res}_{u,v} \frac{Z(u,v)}{(1-au)(1-bv)}$$

Computing Positive Parts as Residues

Our solution:



$[\bar{u}^{<}\bar{v}^{<}] : \mathbb{Q}[[\bar{u},\bar{v}]][u,v] \to uv\mathbb{Q}[u,v]$

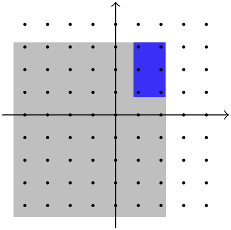


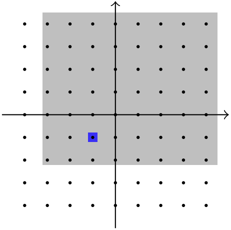
$\mathrm{Res}_{u,v} : \mathbb{Q}[[u,v]][\bar{u},\bar{v}] \to \mathbb{Q}$

Finally, observe $R \in \mathbb{Q}(x)[y,\bar{y}][[t]]$, so that:

$$F(t;x,y) = \frac{1}{xy}[x^{>}][y^{>}]R = \bar{x}\bar{y}[\bar{x}^{<}\bar{y}^{<}]R$$

Our solution:





$[\bar{u}^< \bar{v}^<] : \mathbb{Q}[[\bar{u},\bar{v}]][u,v] \to uv\mathbb{Q}[u,v]$

$\mathrm{Res}_{u,v} : \mathbb{Q}[[u,v]][\bar{u},\bar{v}] \to \mathbb{Q}$

Finally, observe $R \in \mathbb{Q}(x)[y,\bar{y}][[t]]$, so that:

$$F(t;x,y) = \frac{1}{xy}[x^>][y^>]R = \bar{x}\bar{y}[\bar{x}^< \bar{y}^<]R = \mathrm{Res}_{u,v} \frac{R(1/u,1/v)}{(1-xu)(1-yv)}.$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x, y)$. Given $H \in \mathbb{Q}'(t, u, v)$:

- Stage 1: for $r = 0, 1, \dots,$ search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j (H)(t,u,v) = D_v\big(\Phi^k(t,u,v) H(t,u,v)\big)$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x, y)$. Given $H \in \mathbb{Q}'(t, u, v)$:

- Stage 1: for $r = 0, 1, \dots$, search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j (H)(t,u,v) = D_v\big(\Phi^k(t,u,v) H(t,u,v)\big)$$

  by a variant of Abramov's algorithm

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x,y)$. Given $H \in \mathbb{Q}'(t,u,v)$:

- Stage 1: for $r = 0, 1, \dots$, search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(H)(t,u,v) = D_v\big(\Phi^k(t,u,v) H(t,u,v)\big)$$

$$\to \sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\, dv.$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x,y)$. Given $H \in \mathbb{Q}'(t,u,v)$:

- Stage 1: for $r = 0,1,\dots$, search for rational $\eta^k_{i,j}$ and Φ^k s. t.

$$\sum_{0\le i,j,\ i+j\le r} \eta^k_{i,j}(t,u) D_t^i D_u^j(H)(t,u,v) = D_v\big(\Phi^k(t,u,v)H(t,u,v)\big)$$

$$\rightarrow \sum_{0\le i,j,\ i+j\le r} \eta^k_{i,j}(t,u) D_t^i D_u^j(\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\,dv.$$

- Stage 2: for $\hat{r} = 0,1,\dots$, search for rational $\hat{\eta}_i$ and coefficients of $\hat{\Phi}$ s. t.

$$\sum_{0\le i\le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u)\hat{H}(t,u)\big)$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x,y)$. Given $H \in \mathbb{Q}'(t,u,v)$:

- Stage 1: for $r = 0, 1, \dots$, search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(H)(t,u,v) = D_v\big(\Phi^k(t,u,v)H(t,u,v)\big)$$

$$\to \sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\,dv.$$

- Stage 2: for $\hat{r} = 0, 1, \dots$, search for rational $\hat{\eta}_i$ and coefficients of $\hat{\Phi}$ s. t.

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u)\hat{H}(t,u)\big)$$

  by uncoupling + a variant of Abramov's algorithm, or by a variant of Barkatou's algorithm

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x,y)$. Given $H \in \mathbb{Q}'(t,u,v)$:

- Stage 1: for $r = 0, 1, \dots$, search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(H)(t,u,v) = D_v\big(\Phi^k(t,u,v) H(t,u,v)\big)$$

$$\to \sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\,dv.$$

- Stage 2: for $\hat{r} = 0, 1, \dots$, search for rational $\hat{\eta}_i$ and coefficients of $\hat{\Phi}$ s. t.

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u)\hat{H}(t,u)\big)$$

$$\to \sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{\hat{H}})(t) = 0 \quad \text{for} \quad \hat{\hat{H}} = \oint \oint H(t,u,v)\,du\,dv.$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x, y)$. Given $H \in \mathbb{Q}'(t, u, v)$:

- Stage 1: for $r = 0, 1, \dots$, search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(H)(t,u,v) = D_v\big(\Phi^k(t,u,v)H(t,u,v)\big)$$

$$\to \sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\,dv.$$

- Stage 2: for $\hat{r} = 0, 1, \dots$, search for rational $\hat{\eta}_i$ and coefficients of $\hat{\Phi}$ s. t.

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u)\hat{H}(t,u)\big)$$

$$\to \sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{\hat{H}})(t) = 0 \quad \text{for} \quad \hat{\hat{H}} = \oint\oint H(t,u,v)\,du\,dv.$$

- Recombine: from the action on $\hat{H}$, there are G^k s. t.

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i = D_u \hat{\Phi}(t,u,D_t,D_u) + \sum_k G^k(t,u,D_t,D_u) \sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j.$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x, y)$. Given $H \in \mathbb{Q}'(t, u, v)$:

- Stage 1: for $r = 0, 1, \dots$, search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j (H)(t,u,v) = D_v\big(\Phi^k(t,u,v) H(t,u,v)\big)$$

$$\to \sum_{0 \le i,j,\ i+j \le r} \eta_{i,j}^k(t,u) D_t^i D_u^j (\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\, dv.$$

- Stage 2: for $\hat{r} = 0, 1, \dots$, search for rational $\hat{\eta}_i$ and coefficients of $\hat{\Phi}$ s. t.

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u) \hat{H}(t,u)\big)$$

$$\to \sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (\hat{\hat{H}})(t) = 0 \quad \text{for} \quad \hat{\hat{H}} = \oint \oint H(t,u,v)\, du\, dv.$$

- Recombine: there are G^k s. t., upon application to H,

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (H) = D_u\big(\hat{\Phi}(t,u,D_t,D_u) H\big) + \sum_k G^k(t,u,D_t,D_u) D_v(\Phi^k H).$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x, y)$. Given $H \in \mathbb{Q}'(t, u, v)$:

- Stage 1: for $r = 0, 1, \dots,$ search for rational $\eta^k_{i,j}$ and Φ^k s. t.

$$\sum_{0 \le i,j,\ i+j \le r} \eta^k_{i,j}(t,u) D_t^i D_u^j (H)(t,u,v) = D_v\big(\Phi^k(t,u,v) H(t,u,v)\big)$$

$$\to \sum_{0 \le i,j,\ i+j \le r} \eta^k_{i,j}(t,u) D_t^i D_u^j (\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\, dv.$$

- Stage 2: for $\hat{r} = 0, 1, \dots,$ search for rational $\hat{\eta}_i$ and coefficients of $\hat{\Phi}$ s. t.

$$\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u) \hat{H}(t,u)\big)$$

$$\to \sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i (\hat{\hat{H}})(t) = 0 \quad \text{for} \quad \hat{\hat{H}} = \oint \oint H(t,u,v)\, du\, dv.$$

- Recombine: there are G^k s. t., upon application to H,

$$\left(\sum_{0 \le i \le \hat{r}} \hat{\eta}_i(t) D_t^i\right)(H) = D_u\big(\hat{\Phi}(t,u,D_t,D_u) H\big) + D_v\left(\sum_k G^k(t,u,D_t,D_u)(\Phi^k H)\right).$$

Creative Telescoping for Double Integration over a Closed Contour

Write $\mathbb{Q}' = \mathbb{Q}(x,y)$. Given $H \in \mathbb{Q}'(t,u,v)$:

- Stage 1: for $r = 0,1,\dots,$ search for rational $\eta_{i,j}^k$ and Φ^k s. t.

$$\sum_{0\le i,j,\ i+j\le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(H)(t,u,v) = D_v\big(\Phi^k(t,u,v)H(t,u,v)\big)$$

$$\to \sum_{0\le i,j,\ i+j\le r} \eta_{i,j}^k(t,u) D_t^i D_u^j(\hat{H})(t,u) = 0 \quad \text{for} \quad \hat{H} = \oint H(t,u,v)\,dv.$$

- Stage 2: for $\hat{r} = 0,1,\dots,$ search for rational $\hat{\eta}_i$ and coefficients of $\hat{\Phi}$ s. t.

$$\sum_{0\le i\le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{H})(t,u) = D_u\big(\hat{\Phi}(t,u,D_t,D_u)\hat{H}(t,u)\big)$$

$$\to \sum_{0\le i\le \hat{r}} \hat{\eta}_i(t) D_t^i(\hat{\hat{H}})(t) = 0 \quad \text{for} \quad \hat{\hat{H}} = \oint \oint H(t,u,v)\,du\,dv.$$

- Recombine: there are $L \in \mathbb{Q}'[t]\langle D_t\rangle$ and $U, V \in \mathbb{Q}'(t,u,v)$ s. t.

$$L(H) = D_u(U) + D_v(V).$$

	hyp$_1$	hyp$_2$	w		hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2} \\ 2\end{matrix}\middle\vert\, w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4} \\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ 1\end{matrix}\middle\vert\, w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{5}{2} \\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ 2\end{matrix}\middle\vert\, w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4} \\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ 2\end{matrix}\middle\vert\, w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4} \\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4} \\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4} \\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2} \\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4} \\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 2\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4} \\ 3\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2} \\ 1\end{matrix}\middle\vert\, w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ 2\end{matrix}\middle\vert\, w\right)$	$16t^2$

Hypergeometric Series Occurring in Explicit Expressions for $F(t;1,1)$

	hyp$_1$	hyp$_2$	w		hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\vert w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{5}{2}\\ 3\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\vert w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\vert w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ 2\end{matrix}\middle\vert w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\vert w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ 2\end{matrix}\middle\vert w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ 1\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ 3\end{matrix}\middle\vert w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ 2\end{matrix}\middle\vert w\right)$	$16t^2$

Observations: Pairs of related hyps $+{}_2F_1\left(\begin{matrix}a\ b\\ c\end{matrix}\middle\vert w\right)$ with $m = c - (a+b) \in \mathbb{Z}$.

Proofs of Algebraicity/Transcendence of $F(t;x,y)$ and $F(t;1,1)$

Theorem

- In cases 1–19, $F(t;x,y)$ is transcendental since $F(t;0,0)$ is.
- In cases 1–16 and 19, $F(t;1,1)$ is transcendental.
- Specific simplifications prove algebraicity of $F(t;1,1)$ in cases 17–18.

Proof: Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- F is algebraic $\Longrightarrow$ G is algebraic.
- Computing a few coefficients of G shows that this is not 0 on all cases of interest.
- Applying Kovacic's algorithm to L_2 decides whether L_2 has nonzero algebraic solutions.

Coefficient Asymptotics $\kappa n^\alpha \rho^n$ *(in progress)*

Local theory of D-finite functions $\longrightarrow$
Systematic method for coefficient asymptotics
(Flajolet and Odlyzko's singularity analysis)

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \qquad \longrightarrow \qquad f_n \sim \cdots$$

Singularity analysis [Flajolet & Odlyzko (1990)]

- Determine dominant singularities of the *complex-analytic function* f.
- Find asymptotic expansion

$$f(z) =_{z \to s} \sum_{\alpha,\gamma} c_{\alpha,\gamma} (s-z)^\alpha \left(\ln \frac{1}{s-z} \right)^\gamma .$$

- Syntactic transfer into an asymptotic expansion for f_n.

Three Formulas from (DLMF 15.8) on ${}_2F_1$s

▷ To ensure that $c - a - b \in \mathbb{N}$: for $m \in \mathbb{N}$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b-m \end{matrix} \middle| z\right) = (1-z)^{-m} {}_2F_1\left(\begin{matrix} a-m \quad b-m \\ (a-m)+(b-m)+m \end{matrix} \middle| z\right). $$

▷ To bring $\pm\infty$ at 1^-: for $z < 1/2$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| z\right) = (1-2z)^{-a} {}_2F_1\left(\begin{matrix} \frac{1}{2}a \quad \frac{1}{2}a+\frac{1}{2} \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| \frac{4z(z-1)}{(1-2z)^2}\right). $$

▷ Local logarithmic behaviour at 1: for $m \in \mathbb{N}$, $z \in D(1,1) \setminus [0,1]$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+m \end{matrix} \middle| z\right) = $$

polynomial of degree $m-1$ in $1-z$

$+$ term in $(1-z)^m \ln(1-z)$ $+$ higher order terms.

Three Formulas from (DLMF 15.8) on ${}_2F_1$s

▷ To ensure that $c-a-b \in \mathbb{N}$: for $m \in \mathbb{N}$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b-m \end{matrix} \middle| z\right) = (1-z)^{-m} {}_2F_1\left(\begin{matrix} a-m \quad b-m \\ (a-m)+(b-m)+m \end{matrix} \middle| z\right). $$

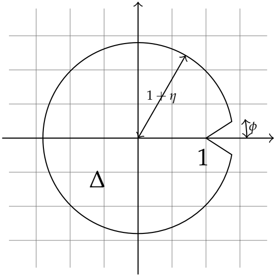
▷ To bring $\pm\infty$ at 1^-: for $z < 1/2$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| z\right) = (1-2z)^{-a} {}_2F_1\left(\begin{matrix} \frac{1}{2}a \quad \frac{1}{2}a+\frac{1}{2} \\ \frac{1}{2}(a+b+1) \end{matrix} \middle| \frac{4z(z-1)}{(1-2z)^2}\right). $$

▷ Local logarithmic behaviour at 1: for $m \in \mathbb{N},\ z \in D(1,1) \setminus [0,1]$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+m \end{matrix} \middle| z\right) = $$
$$ \frac{1}{\Gamma(a+m)\Gamma(b+m)} \sum_{k=0}^{m-1} (-1)^k \frac{(a)_k (b)_k (m-k-1)!}{k!} (1-z)^k $$
$$ - (-1)^m \frac{(1-z)^m}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a+m)_k (b+m)_k}{k!(k+m)!} (1-z)^k \Big(\ln(1-z) $$
$$ - \psi(k+1) - \psi(k+m+1) + \psi(a+k+m) + \psi(b+k+m)\Big). $$

Transfer Theorems [Flajolet & Odlyzko 1990]



For $f(z) = \sum_{n=0}^{\infty} f_n z^n$ analytic in $\Delta \setminus \{1\}$:

$f(z)$	f_n	assumptions
$O\big((1-z)^\alpha\big)$	$O\big(n^{-(\alpha+1)}\big)$	$\alpha \in \mathbb{R}$
$o\big((1-z)^\alpha\big)$	$o\big(n^{-(\alpha+1)}\big)$	$\alpha \in \mathbb{R}$
$\sim C(1-z)^\alpha$	$\sim \frac{Cn^{-(\alpha+1)}}{\Gamma(-\alpha)}$	$\alpha \in \mathbb{R} \setminus \mathbb{N}$
$\sum_{j=0}^{m-1} c_j (1-z)^{\alpha_j} + O\big((1-z)^A\big)$	$\sum_{j=0}^{m-1} \frac{c_j n^{-(\alpha_j+1)}}{\Gamma(-\alpha_j)} + O\big(n^{-(A+1)}\big)$	$\alpha_1 \leq \dots \leq \alpha_{m-1} < A$
$O\big((1-z)^\alpha (\ln(1-z)^{-1})^\gamma\big)$	$O\big(n^{-(\alpha+1)} (\ln n)^\gamma\big)$	$\alpha, \gamma \in \mathbb{R}$
$o\big((1-z)^\alpha (\ln(1-z)^{-1})^\gamma\big)$	$o\big(n^{-(\alpha+1)} (\ln n)^\gamma\big)$	$\alpha, \gamma \in \mathbb{R}$
$\sim C(1-z)^\alpha (\ln(1-z)^{-1})^\gamma$	$\sim \frac{Cn^{-(\alpha+1)} (\ln n)^\gamma}{\Gamma(-\alpha)}$	$\alpha, \gamma \in \mathbb{R} \setminus \mathbb{N}$
$\vdots$	$\vdots$	

Example of Asymptotic Behaviour Driven by the ${}_2F_1$:  at $(1,1)$

$$F(t;1,1) = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)} = 1 - \frac{(1-6t)(1-2t)}{(1+2t)(1+6t)}.$$

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t\to\frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \qquad \longrightarrow \qquad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t\to-\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \qquad \longrightarrow \qquad \frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

Example of Asymptotic Behaviour Driven by the ${}_2F_1$:  at $(1,1)$

$$F(t;1,1) = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)} = 1 - \frac{(1-6t)(1-2t)}{(1+2t)(1+6t)}.$$

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w=1, w=\infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t\to\frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \quad\longrightarrow\quad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t\to-\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \quad\longrightarrow\quad \frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

$$f \quad\longrightarrow\quad f_n \sim \frac{\sqrt{6}}{\pi}6^n$$

Example of Asymptotic Behaviour Driven by the ${}_2F_1$:  at $(1,1)$

$$F(t;1,1) = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}\,{}_2F_1\left(\begin{matrix}\frac{3}{2} \quad \frac{3}{2}\\ 2\end{matrix}\,\middle|\, w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)} = 1 - \frac{(1-6t)(1-2t)}{(1+2t)(1+6t)}.$$

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t\to\frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \quad \longrightarrow \quad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t\to-\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \quad \longrightarrow \quad \frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

$$\int f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi}\frac{6^{n-1}}{n}$$

$$F(t;1,1) = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)} = 1 - \frac{(1-6t)(1-2t)}{(1+2t)(1+6t)}.$$

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w=1, w=\infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t\to\frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \quad \longrightarrow \quad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t\to-\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \quad \longrightarrow \quad \frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

$$\frac{1}{t}\int f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi}\frac{6^n}{n+1}$$

Example of Asymptotic Behaviour Driven by the ${}_2F_1$:  at $(1,1)$

$$F(t;1,1) = \frac{1}{t}\int f \quad \text{for} \quad f = (1-2t)(1+2t)^{-3/2}(1+6t)^{-3/2}{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix}\,\middle|\, w\right)$$

$$\text{where} \quad w = \frac{16t}{(1+2t)(1+6t)} = 1 - \frac{(1-6t)(1-2t)}{(1+2t)(1+6t)}.$$

Singularities: $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{6}, w = 1, w = \infty \longrightarrow$ Dominant singularities = $\pm\frac{1}{6}$.

$$f(t) \sim_{t \to \frac{1}{6}^-} \frac{\sqrt{6}}{\pi}(1-6t)^{-1} \qquad \longrightarrow \qquad \frac{\sqrt{6}}{\pi}6^n$$

$$f(t) \sim_{t \to -\frac{1}{6}^+} \frac{\sqrt{6}}{4\pi}\ln(1+6t) \qquad \longrightarrow \qquad \frac{\sqrt{6}}{4\pi}\frac{6^n}{n}$$

$$\frac{1}{t}\int f \quad \longrightarrow \quad f_n \sim \frac{\sqrt{6}}{\pi}\frac{6^n}{n}$$

Example of Behaviour Not Driven by the ${}_2F_1$:  at $(1,1)$

$$F(t;1,1) = \frac{1}{t(1-t)} \int \frac{t(4+\int f)}{(1-4t)^{3/2}} \quad \text{where}$$

$$f = \frac{(1+2t)(1-4t)^{1/2}}{2t^2}\left(1 + \frac{1}{2t(1+2t)(1+4t^2)^{1/2}}h\right) = \frac{1}{t^2} + O(1),$$

$$h = (1+t)(1-4t+8t^2)\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, w\right) - (1-t)\,{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, w\right),$$

$$w = \frac{16t^2}{1+4t^2} = 1 - \frac{1-12t^2}{1+4t^2}.$$

Singularities: $\frac{1}{4}$, $-\frac{1}{2}$, $\pm\frac{i}{2}$, 1, $w=1$, $w=\infty \longrightarrow$ Dominant singularity = $\frac{1}{4}$.

Example of Behaviour Not Driven by the ${}_2F_1$: at $(1,1)$

$$F(t;1,1) = \frac{1}{t(1-t)} \int \frac{t(4+\int f)}{(1-4t)^{3/2}} \quad \text{where}$$

$$f = \frac{(1+2t)(1-4t)^{1/2}}{2t^2} \left(1 + \frac{1}{2t(1+2t)(1+4t^2)^{1/2}} h\right) = \frac{1}{t^2} + O(1),$$

$$h = (1+t)(1-4t+8t^2)\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, w\right) - (1-t)\,{}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, w\right),$$

$$w = \frac{16t^2}{1+4t^2} = 1 - \frac{1-12t^2}{1+4t^2}.$$

Singularities: $\frac{1}{4}, -\frac{1}{2}, \pm\frac{i}{2}, 1, w = 1, w = \infty \longrightarrow$ Dominant singularity = $\frac{1}{4}$.

$f_{n;1,1} \sim \frac{4}{3}\sqrt{\frac{1}{\pi}}\frac{4^n}{\sqrt{n}}$ holds under the conjecture $\int_0^{\frac{1}{4}} \left(f(t) - \frac{1}{t^2}\right) dt = 2.$

Conclusions

Summary: three kinds of conjectures now proved:

- differential operators that witness D-finiteness,
- algebraic vs transcendental nature of series,
- asymptotics of coefficients (in progress).

Conclusions

Summary: three kinds of conjectures now proved:

- differential operators that witness D-finiteness,
- algebraic vs transcendental nature of series,
- asymptotics of coefficients (in progress).

This is Functional Equation in LIMoges:

- recurrence relation on $f_{n;i,j}$,
- kernel equation on $F(t;x,y)$,
- ODE on $F(t;1,1)$.