Geometric Singularities of Algebraic Differential Equations

Werner M. Seiler
Institut für Mathematik, Universität Kassel
(joint work with **Markus Lange-Hegermann**, RWTH Aachen)

singularities of differential equations

$\neq$

singularities of *solutions* of differential equations

- related, but different topics
- no discussion of shocks, blow-ups, etc.

"Interpolation" between three domains:

- differential algebra
- differential topology
- differential algebraic equations

together with techniques from (differential) algebraic geometry

Current goal: detect all singularities of given system of (ordinary or partial) differential equations

Differential Algebra

- here mainly *differential ideal theory*
- covers automatically systems and all orders
- founded by Ritt in early 20th century
- central goal: understanding *singular integrals*
  - oldest example: *Taylor* (1715)
  - best known example: *Clairaut equation* (1734)

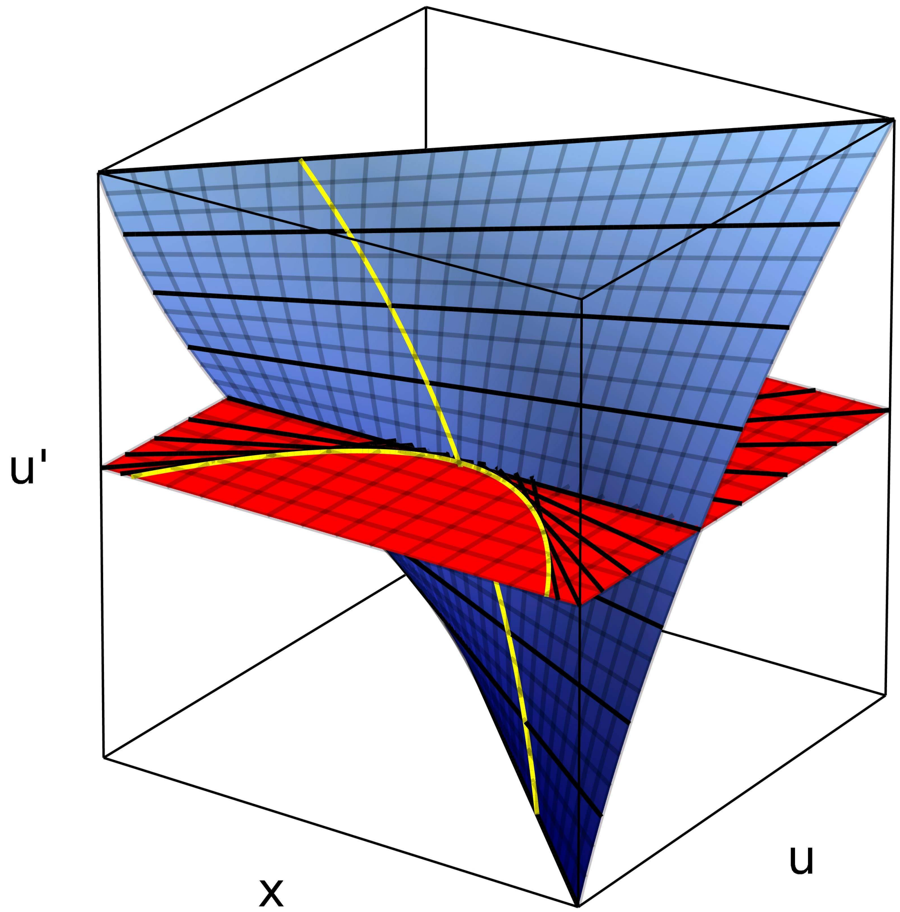
$$u = xu' + f(u') \qquad \text{mit } f''(z) \neq 0 \ \forall z$$

general integral: $u(x) = cx + f(c)$

singular integral:

$$x(\tau) = -f'(\tau), \quad u(\tau) = -\tau f'(\tau) + f(\tau)$$

Differential Algebra

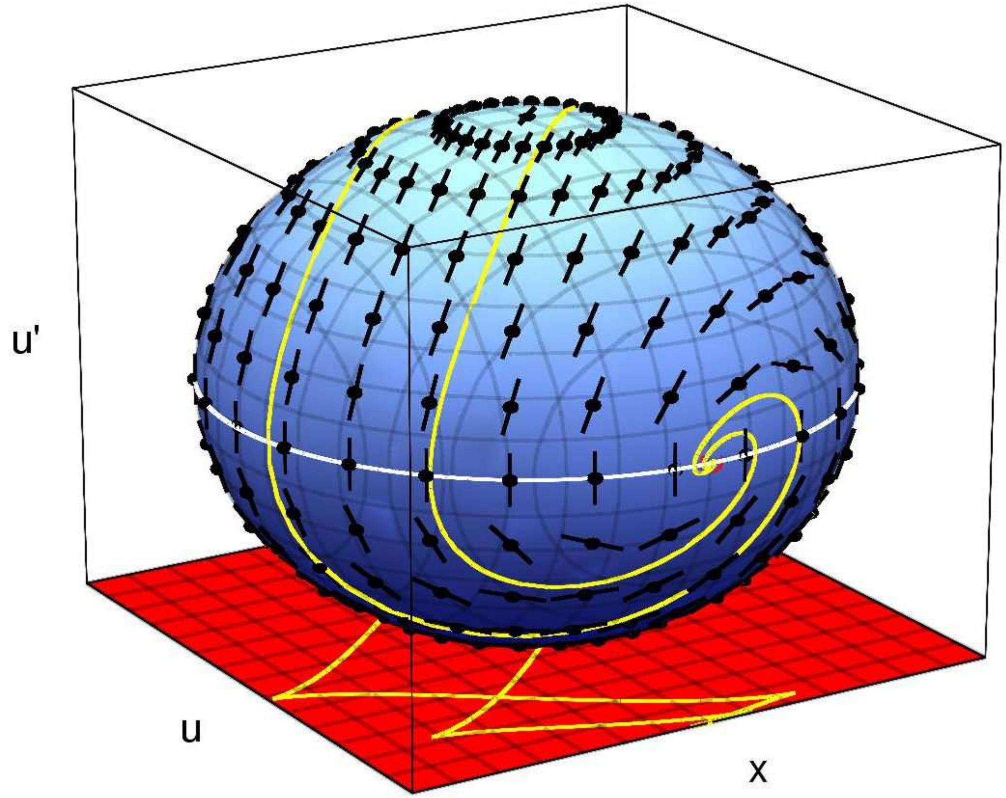


$$f(z) = -\frac{1}{4}z^2$$

Differential Topology

- *singularities* of smooth maps between manifolds
- submanifolds of jet bundles provide geometric model for differential equations
- natural projections between jet bundles of different order $\rightsquigarrow$ critical points $=$ *geometric singularities*
- distinction *regular* and *irregular* singularities
- complete classifications of singularities of *scalar* ordinary differential equations of *first or second order*
- hardly any works on (general) *systems*

Differential Topology



$$(u')^2 + u^2 + x^2 = 1$$

Differential Algebraic Equations

- mainly *analytic* theory of *quasi-linear systems* $A(x, \mathbf{u})\mathbf{u}' = F(x, \mathbf{u})$ (including very large systems!)
- A not necessarily of *maximal* rank and rank may *jump*
- *impasse points* already discussed in 1960s by electrical engineers
  - $\rightsquigarrow$ lead to *jump phenomena* in solutions
- on one side interpreted as sign of bad model. . .
- . . . on the other side experiments often show similar behaviour

- consider *holomorphic* function $\mathbf{f} : \mathcal{U} \subseteq \mathbb{C}^n \to \mathbb{C}^m$, $\mathbf{u} = \mathbf{f}(\mathbf{z})$
  from now on: n, m fixed, $\mathcal{U}$ ignored
- q-*jet* $[\mathbf{f}]^{(q)}_{\mathbf{z}}$ $\rightsquigarrow$ equivalence class of all holomorphic functions $\mathbf{g} : \mathbb{C}^n \to \mathbb{C}^m$ with same Taylor polynomial of degree q around $\mathbf{z} \in \mathcal{U}$ as $\mathbf{f}$
- *jet bundle* $\mathcal{J}_q = \mathcal{J}_q(\mathbb{C}^n, \mathbb{C}^m)$ $\rightsquigarrow$ set of all q-jets $[\mathbf{f}]^{(q)}_{\mathbf{z}}$
  - manifold of dimension $d_q = n + m\binom{n+q}{q}$
    (may be identified with $\mathbb{C}^{d_q}$)
  - local coordinates $(\mathbf{z}, \mathbf{u}^{(q)})$ corresponding to expansion point $\mathbf{z}$ and derivatives up to order q
  - natural projections for $0 \leq r < q$

$$\pi^q_r : \begin{cases} \mathcal{J}_q & \longrightarrow \quad \mathcal{J}_r \\ [\mathbf{f}]^{(q)}_{\mathbf{z}} & \longmapsto \quad [\mathbf{f}]^{(r)}_{\mathbf{z}} \end{cases} \qquad \pi^q : \begin{cases} \mathcal{J}_q & \longrightarrow \quad \mathbb{C}^n \\ [\mathbf{f}]^{(q)}_{\mathbf{z}} & \longmapsto \quad \mathbf{z} \end{cases}$$

Definition:

- *algebraic jet set* of order q $\rightsquigarrow$ locally Zariski closed set $\mathcal{R}_q \subseteq \mathcal{J}_q$ (i.e.: difference of two varieties)
- *algebraic differential equation* of order q $\rightsquigarrow$ algebraic jet set $\mathcal{R}_q \subseteq \mathcal{J}_q$ such that restricted projection $\pi^q|_{\mathcal{R}_q}$ dominant (i.e.: image Zariski dense in $\mathbb{C}^n$)

both generalisation and restriction of classical geometric definition:

- only *polynomial non-linearities* admitted
- $\mathcal{R}_q$ may have *algebraic singularities*
- equations and *inequations* admitted
- dominance replaces *surjectivity*, permits "special points" in $\mathbb{C}^n$
- $\pi^q|_{\mathcal{R}_q}$ not necessarily *submersive*

holomorphic function $\mathbf{f}$ defines *section*

$$\sigma_{\mathbf{f}} : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^m = \mathcal{J}_0, \;\; \mathbf{z} \mapsto \big(\mathbf{z}, \mathbf{f}(\mathbf{z})\big) = [\mathbf{f}]^{(0)}_{\mathbf{z}}$$

(graph of $\mathbf{f}$ is image of $\sigma_{\mathbf{f}}$)

consider *prolonged section*

$$j_q\sigma_{\mathbf{f}} : \mathbb{C}^n \to \mathcal{J}_q, \;\; \mathbf{z} \mapsto [\mathbf{f}]^{(q)}_{\mathbf{z}}$$

Def: $\mathbf{f}$ (resp. $\sigma_{\mathbf{f}}$) *(classical) solution* of differential equation $\mathcal{R}_q \subseteq \mathcal{J}_q$
$\rightsquigarrow$ $\operatorname{im}(j_q\sigma_{\mathbf{f}}) \subseteq \mathcal{R}_q$

What distinguishes $\mathcal{J}_q$ from $\mathbb{C}^{d_q}$? $\rightsquigarrow$ *contact structure* on $\mathcal{J}_q$

Def: *contact distribution* $\mathcal{C}_q \subset T\mathcal{J}_q$ generated by vector fields

$$C_i^{(q)} = \partial_{z^i} + \sum_\alpha \sum_{0\leq|\mu|<q} u^\alpha_{\mu+1_i}\partial_{u^\alpha_\mu} \qquad 1 \leq i \leq n$$

$$C^\mu_\alpha = \partial_{u^\alpha_\mu} \qquad 1 \leq \alpha \leq m,\ |\mu| = q$$

Prop: section $\gamma : \mathbb{C}^n \to \mathcal{J}_q$ of the form $\gamma = j_q\sigma_{\mathbf{f}}$ for function $\mathbf{f}$
$\Longleftrightarrow$ $T\mathrm{im}(\gamma) \subset \mathcal{C}_q$

Proof: *chain rule!*

Vessiot Distribution and Generalised Solutions

consider prolonged solution $j_q\sigma_{\mathbf{f}}$ of equation $\mathcal{R}_q \subseteq \mathcal{J}_q$:

- *integral elements* $\rightsquigarrow$ $T_\rho\big(\operatorname{im}(j_q\sigma_{\mathbf{f}})\big)$ für $\rho \in \operatorname{im}(j_q\sigma_{\mathbf{f}})$
- solution of $\mathcal{R}_q$ $\Longrightarrow$ $T_\rho\big(\operatorname{im}(j_q\sigma_{\mathbf{f}})\big) \subseteq T_\rho\mathcal{R}_q$
- prolonged section $\Longrightarrow$ $T_\rho\big(\operatorname{im}(j_q\sigma_{\mathbf{f}})\big) \subseteq \mathcal{C}_q|_\rho$

consider prolonged solution $j_q\sigma_{\mathbf{f}}$ of equation $\mathcal{R}_q \subseteq \mathcal{J}_q$:

- *integral elements* $\rightsquigarrow$ $T_\rho\big(\mathrm{im}(j_q\sigma_{\mathbf{f}})\big)$ für $\rho \in \mathrm{im}(j_q\sigma_{\mathbf{f}})$
- solution of $\mathcal{R}_q$ $\Longrightarrow$ $T_\rho\big(\mathrm{im}(j_q\sigma_{\mathbf{f}})\big) \subseteq T_\rho\mathcal{R}_q$
- prolonged section $\Longrightarrow$ $T_\rho\big(\mathrm{im}(j_q\sigma_{\mathbf{f}})\big) \subseteq \mathcal{C}_q|_\rho$

Def: *Vessiot space* in point ρ on algebraic jet set $\mathcal{R}_q$

$$\mathcal{V}_\rho[\mathcal{R}_q] = T_\rho\mathcal{R}_q \cap \mathcal{C}_q|_\rho$$

- $\dim \mathcal{V}_\rho[\mathcal{R}_q]$ generally depends on ρ $\rightsquigarrow$ regular distribution only on Zariski open subset of $\mathcal{R}_q$
- computing Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ requires only linear algebra

consider prolonged solution $j_q\sigma_{\mathbf{f}}$ of equation $\mathcal{R}_q \subseteq \mathcal{J}_q$:

- *integral elements* $\rightsquigarrow$ $T_\rho\big(\mathrm{im}(j_q\sigma_{\mathbf{f}})\big)$ für $\rho \in \mathrm{im}(j_q\sigma_{\mathbf{f}})$
- solution of $\mathcal{R}_q$ $\Longrightarrow$ $T_\rho\big(\mathrm{im}(j_q\sigma_{\mathbf{f}})\big) \subseteq T_\rho\mathcal{R}_q$
- prolonged section $\Longrightarrow$ $T_\rho\big(\mathrm{im}(j_q\sigma_{\mathbf{f}})\big) \subseteq \mathcal{C}_q|_\rho$

Def: *Vessiot space* in point ρ on algebraic jet set $\mathcal{R}_q$

$$\mathcal{V}_\rho[\mathcal{R}_q] = T_\rho\mathcal{R}_q \cap \mathcal{C}_q|_\rho$$

- *(geometric) symbol:* $\mathcal{N}_{q,\rho} = T_\rho\mathcal{R}_q \cap V_\rho\pi^q_{q-1} \subseteq \mathcal{V}[\mathcal{R}_q]$
- decompose $\mathcal{V}[\mathcal{R}_q] = \mathcal{N}_q \oplus \mathcal{H}$ with *complement* $\mathcal{H}$ (non-unique)
- *if* $\dim\mathcal{H} = n$ $\rightsquigarrow$ $\mathcal{H}$ horizontal space of *Vessiot connection*
- *if* Vessiot connection *flat* (i. e. differential equation integrable)
  $\rightsquigarrow$ integral manifolds images of prolonged sections

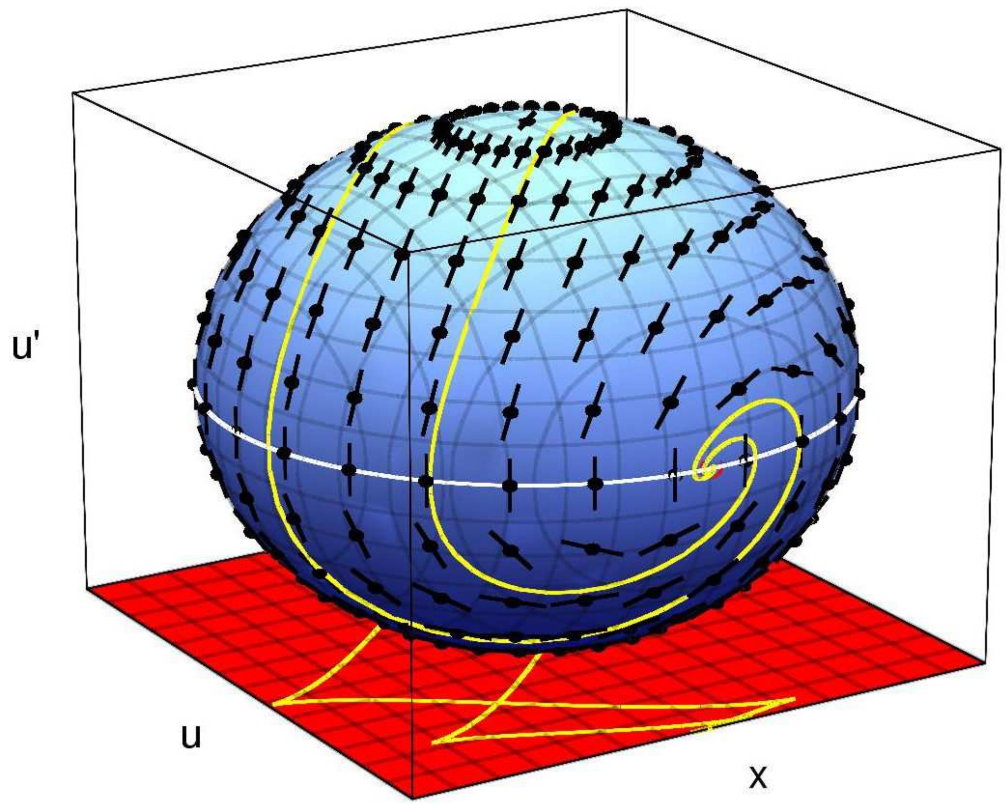
Def: differential equation $\mathcal{R}_q \subseteq \mathcal{J}_q$

- *generalised solution* $\rightsquigarrow$ n-dimensional integral manifold $\mathcal{N} \subseteq \mathcal{R}_q$ of Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$
- *geometric solution* $\rightsquigarrow$ projection $\pi^q_0(\mathcal{N})$ of generalised solution $\mathcal{N} \subseteq \mathcal{R}_q$

comparison with classical solutions:

- geometric solution *not* necessarily graph of function $\mathbf{f}$
- geometric solution $\pi^q_0(\mathcal{N})$ graph of classical solution $\Longleftrightarrow$ $\mathcal{N}$ everywhere transversal to π^q
- geometric solution allow for modelling of *multivalued solutions* (“breaking waves”)

Vessiot distribution and generalised solutions for sphere example



$$(u')^2 + u^2 + x^2 = 1$$

$\mathcal{P} = \mathbb{C}[x_1, \dots, x_n]$ polynomial ring in n variables, point $\rho \in \mathcal{J}_q$

$\Phi : \mathcal{J}_q \to \mathbb{C}$ holomorphic function; $\bar{q}$ maximal order of jet variable u^α_μ actually appearing in Φ $\rightsquigarrow$ *principal part* of Φ in ρ

$$\mathrm{pp}_\rho\,\Phi = \sum_{\alpha=1}^{m} \sum_{|\mu|=\bar{q}} \frac{\partial \Phi}{\partial u^\alpha_\mu}(\rho) x^\mu \mathbf{e}_\alpha \in \mathcal{P}^m$$

$\mathcal{R}_q$ described by equations $\Phi_1 = 0, \dots, \Phi_r = 0$ (inequations irrelevant)
$\rightsquigarrow$ in every point $\rho \in \mathcal{R}_q$ polynomial module
$\mathcal{M}[\rho] = \langle \mathrm{pp}_\rho\,\Phi_1, \dots, \mathrm{pp}_\rho\,\Phi_r \rangle$

Def: *Hilbert function* of $\mathcal{R}_q$ in ρ $\rightsquigarrow$
Hilbert function of factor module $\mathcal{P}^m / \mathcal{M}[\rho]$

Def: algebraic differential equation $\mathcal{R}_q \subseteq \mathcal{J}_q$ *regular* $\rightsquigarrow$

- $\mathcal{R}_q$ smooth (i.e. manifold)
- Hilbert functions independent of point $\rho \in \mathcal{R}_q$

idea: uniform behaviour of all "characteristic values" over $\mathcal{R}_q$, in particular

- $\dim \mathcal{V}_\rho[\mathcal{R}_q]$
- $\dim \mathcal{N}_{q,\rho}$
- size of formal solution space in ρ

consider algebraic differential equation $\mathcal{R}_q \subseteq \mathcal{J}_q$

- *algebraic singularities* $\rightsquigarrow$ singularities in the sense of algebraic geometry
  - ignored in sequel (not much known)
  - determination classical problem in algebraic geometry (Jacobi criterion)
- *geometric singularities* $\rightsquigarrow$ critical points of restricted projection $\hat{\pi}^q_{q-1} : \mathcal{R}_q \longrightarrow \pi^q_{q-1}(\mathcal{R}_q)$ (i.e. $T_\rho \hat{\pi}^q_{q-1}$ not surjective)
  $\rightsquigarrow$ points where dimension of symbol jumps

let $\mathcal{R}_q \subseteq \mathcal{J}_q$ be union of algebraic jet sets; smooth point $\rho \in \mathcal{R}_q$ is

- *regular* $\rightsquigarrow$ ρ has open neighbourhood where $\mathcal{V}[\mathcal{R}_q]$ regular and $\mathcal{V}[\mathcal{R}_q] = \mathcal{N}_q \oplus \mathcal{H}$ with $\dim \mathcal{H} = n$
- *regular singular* $\rightsquigarrow$ ρ has open neighbourhood where $\mathcal{V}[\mathcal{R}_q]$ regular, but $\dim \mathcal{H}_\rho < n$
- *irregular singular* $\rightsquigarrow$ $\mathcal{V}[\mathcal{R}_q]$ not regular on any neighbourhood of ρ
- *purely irregular singular* $\rightsquigarrow$ irregular singularity with $\dim \mathcal{H}_\rho = n$

difference to classical definitions:

- *partial* differential equations require consideration of neighbourhood of point $\rightsquigarrow$ “right” dimension of $\mathcal{V}[\mathcal{R}_q]$ a priori not known
- for *ordinary* differential equations pointwise analysis sufficient

Geometric Singularities

formally integrable ODE: $\mathcal{R}_q \subseteq \mathcal{J}_q$

local description: $\mathbf{\Phi}(z, \mathbf{u}^{(q)}) = 0$

$\mathcal{R}_q$ of finite type $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Prop: point $\rho \in \mathcal{R}_q$

- regular $\Longleftrightarrow$ $\operatorname{rank}\left(\mathbf{C}^{(q)}\mathbf{\Phi}\right)_\rho = m$
- regular singular $\Longleftrightarrow$ ρ not regular and

$$\operatorname{rank}\left(\mathbf{C}^{(q)}\mathbf{\Phi} \mid C^{(q)}_{\text{trans}}\mathbf{\Phi}\right)_\rho = m$$

formally integrable ODE: $\mathcal{R}_q \subseteq \mathcal{J}_q$

local description: $\mathbf{\Phi}(z, \mathbf{u}^{(q)}) = 0$

$\mathcal{R}_q$ of finite type $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Thm: assume $\mathcal{R}_q$ *without* irregular singularities

- $\rho \in \mathcal{R}_q$ regular point $\Longrightarrow$
  - **(i)** unique classical solution $\mathbf{f}$ exists with $\rho \in \operatorname{im} j_q\sigma_{\mathbf{f}}$
  - **(ii)** solution $\mathbf{f}$ may be continued in any direction until $j_q\sigma_{\mathbf{f}}$ reaches either boundary of $\mathcal{R}_q$ or a regular singularity
- $\rho \in \mathcal{R}_q$ regular singularity $\Longrightarrow$ dichotomy
  - **(i)** either *two* classical solutions $\mathbf{f}_1, \mathbf{f}_2$ exist with $\rho \in \operatorname{im} j_q\sigma_{\mathbf{f}_i}$ (either both start or both end in ρ)
  - **(ii)** or *one* classical solution $\mathbf{f}$ exists with $\rho \in \operatorname{im} j_q\sigma_{\mathbf{f}}$ whose derivative of order $q+1$ in $z = \pi^q(\rho)$ is not defined

formally integrable ODE: $\mathcal{R}_q \subseteq \mathcal{J}_q$

local description: $\mathbf{\Phi}(z, \mathbf{u}^{(q)}) = 0$

$\mathcal{R}_q$ of finite type $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Proof: $\mathcal{V}[\mathcal{R}_q]$ locally generated by vector field X

ρ regular singularity $\Longrightarrow$ X vertical wrt π^q

dichotomy $\rightsquigarrow$ does ∂_z-component of X change sign in ρ?

formally integrable ODE: $\mathcal{R}_q \subseteq \mathcal{J}_q$
local description: $\mathbf{\Phi}(z, \mathbf{u}^{(q)}) = 0$
$\mathcal{R}_q$ of finite type $\rightsquigarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

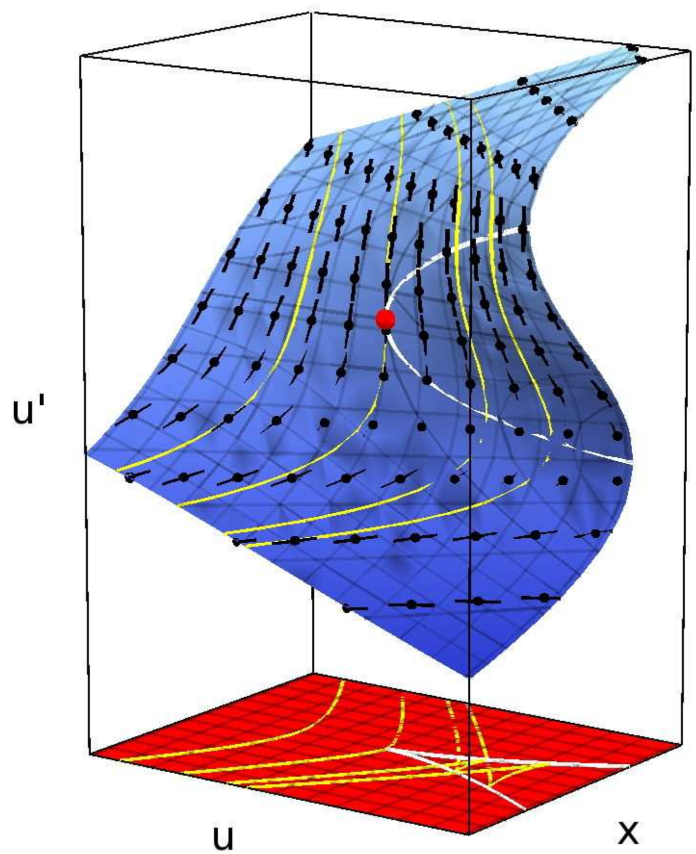
let $\rho \in \mathcal{R}_q$ be an *irregular* singularity

- consider simply connected domain $\mathcal{U} \subset \mathcal{R}_q$ without irregular singularities such that $\rho \in \overline{\mathcal{U}}$
- in $\mathcal{U}$ Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ generated by vector field X

Thm: generically every continuation of X to ρ vanishes

Consequence: solution behaviour in neighbourhood of isolated irregular singularity ρ analysable with dynamical systems theory (mainly determined by eigenstructure of $\operatorname{Jac}_\rho X$)

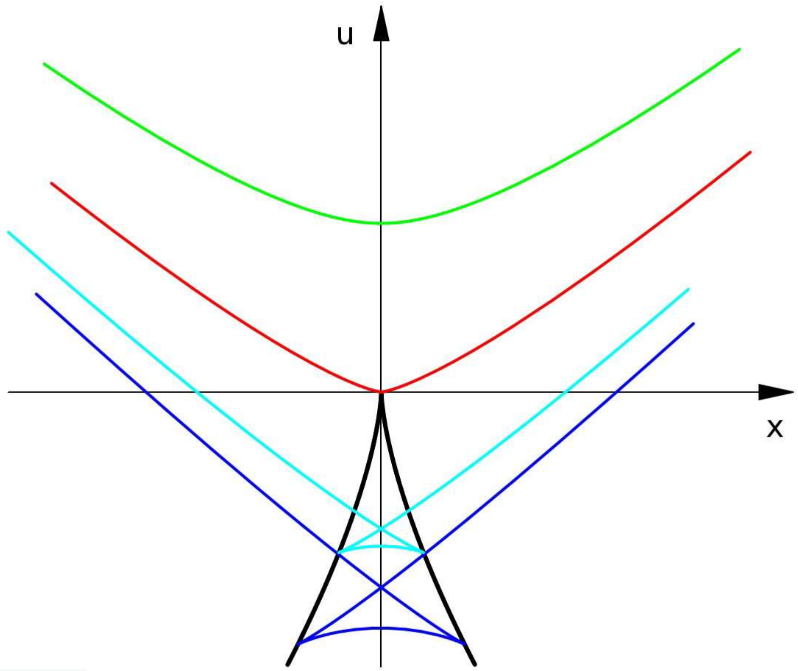
Example: $(u')^3 + uu' - x = 0$ *(hyperbolic gather)*



singularity curve (*criminant*):

$$3(u')^2 + u = 0$$

Example: $(u')^3 + uu' - x = 0$ *(hyperbolic gather)*

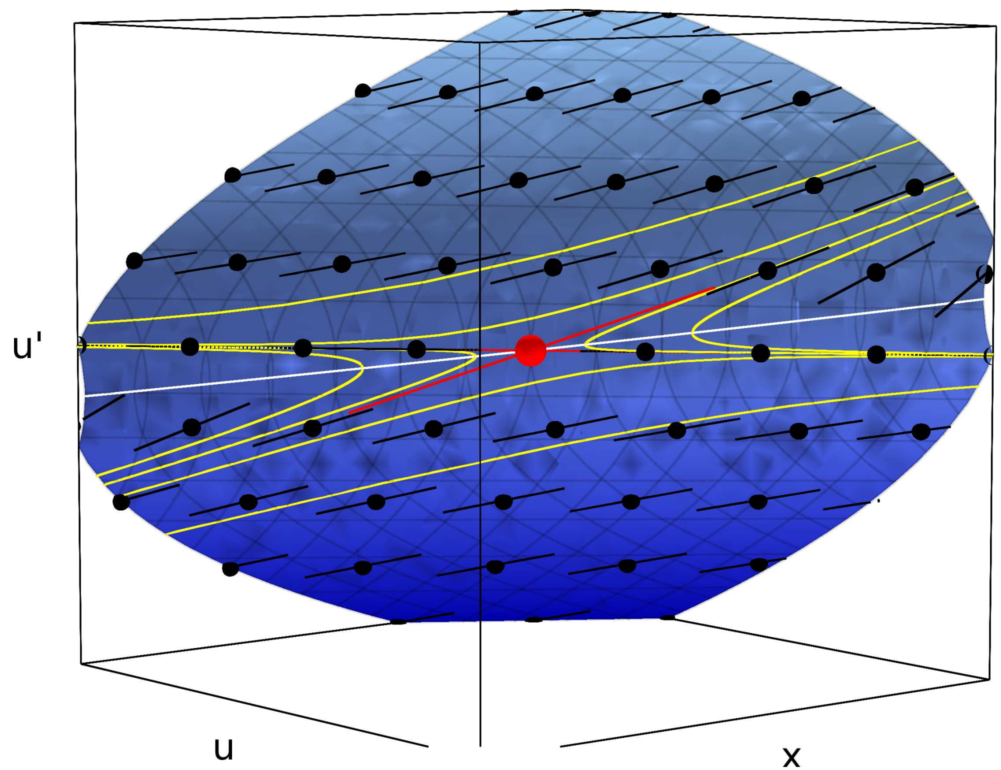


second derivative of solution touching "tip" of discriminant does not exist

solutions "change direction" when crossing discriminant

Example: $(u')^3 + uu' - x = 0$ *(hyperbolic gather)*

neighbourhood of irregular singularity



the two solutions tangential to eigenvectors of $\operatorname{Jac} X$ intersect

Algebraic Case

polynomial ring $\mathbb{C}[z_1, \dots, z_n]$ with total ordering on variables

- *leader* $\operatorname{ld} p$ ⇝ largest variable in polynomial p
- consider p as univariate polynomial in $\operatorname{ld} p$
  - *initial* $\operatorname{init} p$ ⇝ leading coefficient of p
  - *separant* $\operatorname{sep} p$ ⇝ $\partial p / \partial(\operatorname{ld} p)$

algebraic system ⇝ finite set of polynomial equations and inequations

$$\mathcal{S} = \{p_1 = 0, \dots, p_s = 0,\ q_1 \neq 0, \dots, q_t \neq 0\}$$

solution set (locally closed wrt Zariski topology)

$$\operatorname{Sol} \mathcal{S} = \{\mathbf{z} \in \mathbb{C}^n \mid p_i(\mathbf{z}) = 0,\ q_j(\mathbf{z}) \neq 0\}$$

Algebraic Case

Def: *simple* algebraic system

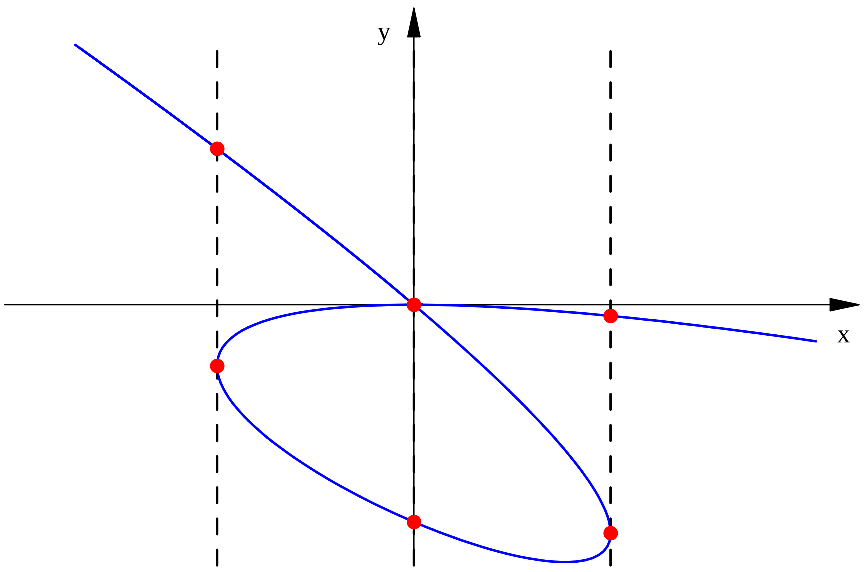
- *triangular:* $\bigl|\{\operatorname{ld} p_i,\ \operatorname{ld} q_j\} \setminus \{1\}\bigr| = s + t$
- *non-vanishing initials:* no equation $\operatorname{init} p_i = 0$ or $\operatorname{init} q_j = 0$ has solution in $\operatorname{Sol} \mathcal{S}$
- *square-free:* dito for separants

Def: *Thomas decomposition* of algebraic system $\mathcal{S}$ $\rightsquigarrow$ finitely many simple systems $\mathcal{S}_1, \dots, \mathcal{S}_k$ such that $\operatorname{Sol} \mathcal{S}$ *disjoint* union of all $\operatorname{Sol} \mathcal{S}_i$

- *exists* always
- depends on *ordering* of variables
- decomposes according to *fibre cardinality* for *coordinate projections*
- can be determined *algorithmically*
  (subresultants, case distinctions $\rightsquigarrow$ expensive!)

Algebraic Case

consider $\mathcal{V}\big(y^3 + (3x+1)y^2 + (3x^2+2x)y + x^3\big)$



Thomas decomposition

- $S_1 = \big\{y^3 + (3x+1)y^2 + (3x^2+2x)y + x^3 = 0,\ 27x^3 - 4x \neq 0\big\}$
- $S_2 = \big\{6y^2 - (27x^2 - 12x - 6)y - 3x^2 + 2x = 0,\ 27x^3 - 4x = 0\big\}$

differential case

ring of differential polynomials

- $\mathbb{K} = \mathbb{C}(z_1, \dots, z_n)$ rational functions
- derivations $\delta_i = \partial/\partial z_i$
- differential unknowns: $U = \{u^1, \dots, u^m\}$
  $\rightsquigarrow$ *jet variables* $u^\alpha_\mu = \delta^\mu u^\alpha$
- $\mathbb{K}\{U\} = \mathbb{K}\left[u^\alpha_\mu \mid 1 \leq \alpha \leq m, \mu \in \mathbb{N}_0^n\right]$
  (polynomial ring in infinitely many variables)
  derivations can be extended: $\delta_i u^\alpha_\mu = u^\alpha_{\mu+1_i}$
- distinguish:
  - *algebraic ideal:* $\langle p_1, \dots, p_s \rangle$
  - *differential ideal:* $\langle p_1, \dots, p_s \rangle_\Delta$
- set $\mathcal{D} = \mathbb{C}\left[z_i, u^\alpha_\mu\right] \subset \mathbb{K}\{U\}$
  $\rightsquigarrow$ $\mathcal{D}_q = \mathbb{C}\left[z_i, u^\alpha_\mu \mid |\mu| \leq q\right]$ *coordinate ring of* $\mathcal{J}_q$

differential case

ranking on $\mathbb{K}\{U\}$

- *total ordering* $\prec$ of jet variables
- $u^\alpha \prec \delta_i u^\alpha$
- $u^\alpha_\mu \prec u^\beta_\nu \quad\Longrightarrow\quad \delta_i u^\alpha_\mu \prec \delta_i u^\beta_\nu$

extend concepts like *leader*, *initial* or *separant*

differential system $\rightsquigarrow$ finite set of differential polynomial equations and inequations

$$\mathcal{S} = \{p_1 = 0, \dots, p_s = 0,\ q_1 \neq 0, \dots, q_t \neq 0\}$$

solution set $\rightsquigarrow$ consider formal solutions
(different function spaces possible)

differential case

Def: *simple* differential system

- simple as *algebraic* system in the finitely many occuring jet variables
- *involutive* for Janet division
- no leader of inequation derivative of leader of equation

Def: *Thomas decomposition* of differential system $\mathcal{S}$ $\rightsquigarrow$ finitely many simple systems $\mathcal{S}_1, \dots, \mathcal{S}_k$ such that $\mathrm{Sol}\,\mathcal{S}$ *disjoint* union of all $\mathrm{Sol}\,\mathcal{S}_i$

- *exists* always
- *algorithmically* computable via combination of algebraic Thomas decomposition and Janet-Riquier theory

starting point: differential system $\mathcal{S}$

goal: all geometric singularities in given order q

differential computation:

- differential Thomas decomposition (other methods also possible) $\rightsquigarrow$ *simple* differential systems $\mathcal{S}_i$
- one $\mathcal{S}_i$ corresponds to *general integral*
  all others yield *singular integrals*
- all other kind of singularities *eliminated*

starting point: differential system $\mathcal{S}$

goal: all geometric singularities in given order q

algebraic analysis of *simple* differential system $\mathcal{S}$

- introduce suitable ideals
  - $I(\mathcal{S}) = \langle p_1, \dots, p_s \rangle_\Delta : h^\infty \subseteq \mathcal{D}$ with $h = \prod_i \operatorname{sep}(p_i) \operatorname{init}(p_i)$
  - $I_q(\mathcal{S}) = I(\mathcal{S}) \cap \mathcal{D}_q, \quad K_q = \langle q_j \mid \operatorname{ord}(q_j) \leq q \rangle_{\mathcal{D}_q}$

$\rightsquigarrow$ algebraic jet set

$$\mathcal{R}_q = \operatorname{Sol}\big(I_q(\mathcal{S})\big) \setminus \operatorname{Sol}\big(K_q(\mathcal{S})\big) \subseteq \mathcal{J}_q$$

starting point: differential system $\mathcal{S}$

goal: all geometric singularities in given order q

algebraic analysis of *simple* differential system $\mathcal{S}$

- ansatz for *Vessiot distribution* of $\mathcal{R}_q$

$$X = \sum_i a^i C_i^{(q)} + \sum_{\alpha,\mu} b_\mu^\alpha C_\alpha^\mu$$

  extended polynomial ring $\mathcal{D}_q^V = \mathcal{D}_q[\mathbf{a}, \mathbf{b}]$ with $\mathbf{b} \succ \mathbf{a} \succ \mathbf{u} \succ \mathbf{z}$

- compute algebraic Thomas decomposition of system over $\mathcal{D}_q^V$ consisting of generators of $I_q(\mathcal{S})$ plus equations for Vessiot distribution (linear in $\mathbf{a}, \mathbf{b}$) $\rightsquigarrow$ solve "parametric linear system"
  $\rightsquigarrow$ simple systems $\mathcal{S}_i^V$ and $\mathcal{S}_i = \mathcal{S}_i^V \cap \mathcal{D}_q$

starting point: differential system $\mathcal{S}$

goal: all geometric singularities in given order q

Def: *regularity decomposition* in order q of simple differential system

$\rightsquigarrow$ write $\operatorname{Sol} I_q(\mathcal{S}) \subseteq \mathcal{J}_q$ as *disjoint* union of finitely many *regular* algebraic jet sets $\mathcal{R}_q^{(i)} \subset \mathcal{J}_q$ (*components* in order q)

- *singular closure* $\overline{\mathcal{R}_q^{(i)}}$ of component $\mathcal{R}_q^{(i)}$ $\rightsquigarrow$ union with all components $\mathcal{R}_q^{(j)}$ lying in Zariski closure of $\mathcal{R}_q^{(i)}$
- *constituent* in order q $\rightsquigarrow$ algebraically simple system $\mathcal{S}'$ such that $\operatorname{Sol}(\mathcal{S}') \subseteq \operatorname{Sol}\big(I_q(\mathcal{S})\big)$ and set of leaders of equations in $\mathcal{S}'$ equal to set of jet variables in $\langle \operatorname{ld} p_1, \dots, \operatorname{ld} p_s \rangle_\Delta \cap \mathcal{D}_q$

starting point: differential system $\mathcal{S}$

goal: all geometric singularities in given order q

final analysis:

- **Thm:** algorithm yields regularity decomposition with all constituents *regular algebraic differential equations*
- **Prop:** union of solution sets of constituents *Zariski dense* in $\mathrm{Sol}\left(I_q(\mathcal{S})\right)$
- consider for each constituent *singular closure* (component may lie in closure of several constituents!)
- taxonomy of singularities via leaders of systems $\mathcal{S}_i^V$ (do variables $\mathbf{a}$ appear as leader?) and comparison with constituent