On d'Alembertian and Liouvillian Sequences

Marko Petkovšek, Helena Zakrajšek

University of Ljubljana

Limoges, March 2015

Outline

1. P-recursive sequences

Notation:

K ... algebraically closed field of characteristic 0

Definition

A sequence $\langle a(n) \rangle_{n=0}^{\infty} \in K^{\mathbb{N}}$ is *P-recursive* $\iff$
$\exists\, d \in \mathbb{N}$, $\exists\, p_0, p_1, \ldots, p_d \in K[n]$, $p_d \neq 0$:

$$p_d(n)a(n+d) + p_{d-1}(n)a(n+d-1) + \cdots + p_0(n)a(n) = 0$$

for all $n \geq 0$.

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following unary operations $a \to b$*:*

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following unary operations $a \to b$:

$$1.\ scalar\ multiplication: \qquad b(n) = \lambda a(n)$$

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following unary operations $a \to b$:

$$\begin{array}{ll} 1.\ \textit{scalar multiplication}: & b(n) = \lambda a(n) \\ 2.\ \textit{shift}: & b(n) = (Ea)(n) = a(n+1) \end{array}$$

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following unary operations $a \to b$:

$$\begin{array}{ll} 1.\ \textit{scalar multiplication}: & b(n) = \lambda a(n) \\ 2.\ \textit{shift}: & b(n) = (Ea)(n) = a(n+1) \\ 3.\ \textit{indefinite summation}: & b(n) = \sum_{i=0}^{n} a(i) \end{array}$$

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following unary operations $a \to b$:

1. *scalar multiplication* : $b(n) = \lambda a(n)$
2. *shift* : $b(n) = (Ea)(n) = a(n+1)$
3. *indefinite summation* : $b(n) = \sum_{i=0}^{n} a(i)$
4. *multisection* : $b(n) = a(mn+r)$

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following polyadic operations $(a_0, a_1, \ldots, a_{k-1}) \rightarrow b$*:*

Theorem

The set of P-recursive sequences is closed under the following polyadic operations $(a_0, a_1, \ldots, a_{k-1}) \to b$:

$$5.\ addition: \qquad b(n) = \sum_{j=0}^{k-1} a_j(n)$$

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following polyadic operations $(a_0, a_1, \ldots, a_{k-1}) \to b$:

$$\begin{array}{ll} 5.\ \textit{addition}: & b(n) = \sum_{j=0}^{k-1} a_j(n) \\ 6.\ \textit{multiplication}: & b(n) = \prod_{j=0}^{k-1} a_j(n) \end{array}$$

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following polyadic operations $(a_0, a_1, \ldots, a_{k-1}) \to b$:

$$5.\ addition: \qquad b(n) = \sum_{j=0}^{k-1} a_j(n)$$

$$6.\ multiplication: \qquad b(n) = \prod_{j=0}^{k-1} a_j(n)$$

$$7.\ convolution:$$

$$b(n) = \sum_{i_0+i_1+\cdots+i_{k-1}=n} a_0(i_0)a_1(i_1)\cdots a_{k-1}(i_{k-1})$$

1. P-recursive sequences

Theorem

The set of P-recursive sequences is closed under the following polyadic operations $(a_0, a_1, \ldots, a_{k-1}) \to b$*:*

$$\begin{aligned}
&5.\ addition: && b(n) = \textstyle\sum_{j=0}^{k-1} a_j(n)\\
&6.\ multiplication: && b(n) = \textstyle\prod_{j=0}^{k-1} a_j(n)\\
&7.\ convolution:\\
&b(n) = \sum_{i_0+i_1+\cdots+i_{k-1}=n} a_0(i_0)a_1(i_1)\cdots a_{k-1}(i_{k-1})\\
&8.\ interlacing: && b(n) = \left(\Lambda_{j=0}^{k-1} a_j\right)(n)\\
& && \quad\;\; = a_{n \bmod k}(n\ div\ k)
\end{aligned}$$

Definition

A sequence $a \in K^{\mathbb{N}}$ is *hypergeometric* if:

1. P-recursive sequences

Definition

A sequence $a \in K^{\mathbb{N}}$ is *hypergeometric* if:

1. $\exists p, q \in K[n] \setminus \{0\}$:

$$p(n)\, a(n+1) + q(n)\, a(n) \ = \ 0$$

   for all $n \geq 0$,

Definition

A sequence $a \in K^{\mathbb{N}}$ is *hypergeometric* if:

1. $\exists p, q \in K[n] \setminus \{0\}$:

$$p(n)\,a(n+1) + q(n)\,a(n) \;=\; 0$$

   for all $n \geq 0$,
2. $\exists N \in \mathbb{N}:\; a(n) \neq 0$ for all $n \geq N$.

1. P-recursive sequences

Definition

A sequence $a \in K^{\mathbb{N}}$ is *hypergeometric* if:

1. $\exists p, q \in K[n] \setminus \{0\}$:

$$p(n)\,a(n+1) + q(n)\,a(n) \;=\; 0$$

   for all $n \geq 0$,
2. $\exists N \in \mathbb{N}: \; a(n) \neq 0$ for all $n \geq N$.

Notation:

$$\mathcal{H}(K) \;\ldots\; \text{hypergeometric sequences in } K^{\mathbb{N}}$$

2. Some difference algebra

Definition

A *difference ring* is a pair (K, σ) where:

2. Some difference algebra

Definition

A *difference ring* is a pair (K, σ) where:

- K is a commutative ring with multiplicative identity,
- $\sigma : \ K \to K$ is a ring automorphism.

2. Some difference algebra

Definition

A *difference ring* is a pair (K, σ) where:

- K is a commutative ring with multiplicative identity,
- $\sigma : \ K \to K$ is a ring automorphism.

If, in addition, K is a field then (K, σ) is a *difference field*.

2. Some difference algebra

Definition

A *difference ring* is a pair (K, σ) where:

- K is a commutative ring with multiplicative identity,
- $\sigma: \ K \to K$ is a ring automorphism.

If, in addition, K is a field then (K, σ) is a *difference field*.

Example

- $(K[x], \sigma)$ with $\sigma x = x + 1$, $\sigma|_K = \mathrm{id}_K$ is a difference ring.

2. Some difference algebra

Definition

A *difference ring* is a pair (K, σ) where:

- K is a commutative ring with multiplicative identity,
- $\sigma: \; K \to K$ is a ring automorphism.

If, in addition, K is a field then (K, σ) is a *difference field*.

Example

- $(K[x], \sigma)$ with $\sigma x = x + 1$, $\sigma|_K = \mathrm{id}_K$ is a difference ring.
- $(K(x), \sigma)$ with $\sigma x = x + 1, \sigma|_K = \mathrm{id}_K$ is a difference field.

2. Some difference algebra

Definition

A *difference ring* is a pair (K, σ) where:

- K is a commutative ring with multiplicative identity,
- $\sigma : \; K \to K$ is a ring automorphism.

If, in addition, K is a field then (K, σ) is a *difference field*.

Example

- $(K[x], \sigma)$ with $\sigma x = x + 1$, $\sigma|_K = \mathrm{id}_K$ is a difference ring.
- $(K(x), \sigma)$ with $\sigma x = x + 1, \sigma|_K = \mathrm{id}_K$ is a difference field.
- $(K^{\mathbb{N}}, E)$ where

$$E : \; \langle a(0), a(1), a(2), \ldots \rangle \mapsto \langle a(1), a(2), a(3) \ldots \rangle$$

is *not* a difference ring.

2. Some difference algebra

For $a, b \in K^{\mathbb{N}}$ define:

2. Some difference algebra

For $a, b \in K^{\mathbb{N}}$ define:

$$a \sim b \iff a(n) = b(n) \text{ for all large enough } n$$

2. Some difference algebra

For $a, b \in K^{\mathbb{N}}$ define:

$$a \sim b \quad \Longleftrightarrow \quad a(n) = b(n) \text{ for all large enough } n$$

Notation:

$$\mathcal{S}(K) \; = \; K^{\mathbb{N}} / \sim \qquad \ldots \qquad \text{ring of germs of sequences}$$

2. Some difference algebra

For $a, b \in K^{\mathbb{N}}$ define:

$$a \sim b \iff a(n) = b(n) \text{ for all large enough } n$$

Notation:

$\mathcal{S}(K) = K^{\mathbb{N}}/\sim$ ... ring of germs of sequences

$\sigma : \mathcal{S}(K) \to \mathcal{S}(K)$ ... unique automorphism of $\mathcal{S}(K)$ defined by $\sigma \circ \varphi = \varphi \circ E$

2. Some difference algebra

For $a, b \in K^{\mathbb{N}}$ define:

$$a \sim b \quad \Longleftrightarrow \quad a(n) = b(n) \text{ \textit{for all large enough} } n$$

Notation:

$\mathcal{S}(K) = K^{\mathbb{N}} / \sim$ ... ring of germs of sequences

$\sigma : \mathcal{S}(K) \to \mathcal{S}(K)$ ... unique automorphism of $\mathcal{S}(K)$ defined by $\sigma \circ \varphi = \varphi \circ E$

Observation

$(\mathcal{S}(K), \sigma)$ is a difference ring.

Example

Example

- $a = 0$ in $\mathcal{S}(K) \iff a(n) = 0$ for large enough n

2. Some difference algebra

Example

- $a = 0$ in $\mathcal{S}(K) \iff a(n) = 0$ for large enough n
- $a = b$ in $\mathcal{S}(K) \iff a(n) = b(n)$ for large enough n

2. Some difference algebra

Example

- $a = 0$ in $\mathcal{S}(K) \iff a(n) = 0$ for large enough n
- $a = b$ in $\mathcal{S}(K) \iff a(n) = b(n)$ for large enough n
- $K[n]$, $K(n)$, $\mathcal{H}(K)$ all naturally embed into $\mathcal{S}(K)$

2. Some difference algebra

Example

- $a = 0$ in $\mathcal{S}(K) \iff a(n) = 0$ for large enough n
- $a = b$ in $\mathcal{S}(K) \iff a(n) = b(n)$ for large enough n
- $K[n]$, $K(n)$, $\mathcal{H}(K)$ all naturally embed into $\mathcal{S}(K)$

Henceforth we work in $(\mathcal{S}(K), \sigma)$.

2. Some difference algebra

Given $d \in \mathbb{N}$ and $r_0, r_1, \ldots, r_d \in K(n)$, define:

2. Some difference algebra

Given $d \in \mathbb{N}$ and $r_0, r_1, \ldots, r_d \in K(n)$, define:

$$
\begin{aligned}
L: \ \mathcal{S}(K) &\rightarrow \mathcal{S}(K), \\
(La)(n) \ &= \ \sum_{k=0}^{d} r_k(n)\, a(n+k)
\end{aligned}
$$

2. Some difference algebra

Given $d \in \mathbb{N}$ and $r_0, r_1, \ldots, r_d \in K(n)$, define:

$$L: \ \mathcal{S}(K) \to \mathcal{S}(K),$$
$$(La)(n) \ = \ \sum_{k=0}^{d} r_k(n)\, a(n+k)$$

$$L \ = \ \sum_{k=0}^{d} r_k\, \sigma^k$$ ... *linear recurrence operator*

$K(n)\langle\sigma\rangle$ ... *algebra of linear recurrence operators with rational coefficients*

2. Some difference algebra

Commutation rule for composition of operators:

2. Some difference algebra

Commutation rule for composition of operators:

$$\sigma \cdot r(n) \quad = \quad r(n+1)\,\sigma$$

2. Some difference algebra

Commutation rule for composition of operators:

$$
\begin{aligned}
\sigma \cdot r(n) &= r(n+1)\,\sigma \\
\sum_{k=0}^{d} r_k(n)\sigma^k \cdot \sum_{j=0}^{e} s_j(n)\sigma^j &= \sum_{k=0}^{d}\sum_{j=0}^{e} r_k(n)s_j(n+k)\sigma^{j+k}
\end{aligned}
$$

2. Some difference algebra

Commutation rule for composition of operators:

$$\begin{aligned} \sigma \cdot r(n) &= r(n+1)\,\sigma \\ \sum_{k=0}^{d} r_k(n)\sigma^k \cdot \sum_{j=0}^{e} s_j(n)\sigma^j &= \sum_{k=0}^{d}\sum_{j=0}^{e} r_k(n)s_j(n+k)\sigma^{j+k} \end{aligned}$$

Right division of $r(n)\,\sigma^k$ by $s(n)\,\sigma^j$:

$$r(n)\,\sigma^k = \left(\frac{r(n)}{s(n+k-j)}\sigma^{k-j}\right) \cdot s(n)\,\sigma^j$$

2. Some difference algebra

This yields an *algorithm for right division* in $K(n)\langle\sigma\rangle$.

Theorem

2. Some difference algebra

This yields an *algorithm for right division* in $K(n)\langle\sigma\rangle$.

Theorem

For $L_1, L_2 \in K(n)\langle\sigma\rangle$, $L_2 \neq 0$, there are $Q, R \in K(n)\langle\sigma\rangle$ such that

2. Some difference algebra

This yields an *algorithm for right division* in $K(n)\langle\sigma\rangle$.

Theorem

For $L_1, L_2 \in K(n)\langle\sigma\rangle$, $L_2 \neq 0$, *there are* $Q, R \in K(n)\langle\sigma\rangle$ *such that*

- $L_1 \;=\; QL_2 + R,$
- $\operatorname{ord} R \;<\; \operatorname{ord} L_2.$

2. Some difference algebra

This yields an *algorithm for right division* in $K(n)\langle\sigma\rangle$.

Theorem

For $L_1, L_2 \in K(n)\langle\sigma\rangle$, $L_2 \neq 0$, *there are* $Q, R \in K(n)\langle\sigma\rangle$ *such that*

- $L_1 \;=\; QL_2 + R$,
- $\operatorname{ord} R \;<\; \operatorname{ord} L_2$.

Hence we obtain in the usual way the *extended right Euclidean algorithm* which yields $S, T, U, V \in K(n)\langle\sigma\rangle$ such that

2. Some difference algebra

This yields an *algorithm for right division* in $K(n)\langle\sigma\rangle$.

Theorem

For $L_1, L_2 \in K(n)\langle\sigma\rangle$, $L_2 \neq 0$, *there are* $Q, R \in K(n)\langle\sigma\rangle$ *such that*

- $L_1 \; = \; QL_2 + R$,
- $\operatorname{ord} R \; < \; \operatorname{ord} L_2$.

Hence we obtain in the usual way the *extended right Euclidean algorithm* which yields $S, T, U, V \in K(n)\langle\sigma\rangle$ such that

- $SL_1 + TL_2 \; = \; \operatorname{gcrd}(L_1, L_2)$,

2. Some difference algebra

This yields an *algorithm for right division* in $K(n)\langle\sigma\rangle$.

Theorem

For $L_1, L_2 \in K(n)\langle\sigma\rangle$, $L_2 \neq 0$, *there are* $Q, R \in K(n)\langle\sigma\rangle$ *such that*

- $L_1 \;=\; QL_2 + R,$
- $\operatorname{ord} R \;<\; \operatorname{ord} L_2.$

Hence we obtain in the usual way the *extended right Euclidean algorithm* which yields $S, T, U, V \in K(n)\langle\sigma\rangle$ such that

- $SL_1 + TL_2 \;=\; \operatorname{gcrd}(L_1, L_2),$
- $UL_1 \;=\; VL_2 \;=\; \operatorname{lclm}(L_1, L_2).$

2. Some difference algebra

Proposition

Let $L \in K(n)\langle\sigma\rangle$*,* $h \in \mathcal{H}(K)$*,* $\frac{\sigma h}{h} = r \in K(n)$*.*
Then:

$$Lh = 0 \quad \Longleftrightarrow \quad \exists Q: \ L = Q(\sigma - r).$$

Proposition

Let $L \in K(n)\langle\sigma\rangle$, $h \in \mathcal{H}(K)$, $\frac{\sigma h}{h} = r \in K(n)$. *Then:*

$$Lh = 0 \quad \Longleftrightarrow \quad \exists Q: \; L = Q(\sigma - r).$$

hypergeometric solutions of $Ly = 0$

$$\updownarrow$$

1^{st}-order right factors of L

2. Some difference algebra

Example

(AMM problem no. 10375)

$$L \;=\; \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3)$$

2. Some difference algebra

Example

(AMM problem no. 10375)

$$L = \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3)$$

Equation $Ly = 0$ has hypergeometric solution $y_n = (2n)!$. Hence $L = L_2 L_1$ where

Example

(AMM problem no. 10375)

$$L = \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3)$$

Equation $Ly = 0$ has hypergeometric solution $y_n = (2n)!$. Hence $L = L_2 L_1$ where

$$\begin{aligned} L_1 &= \sigma - (2n+1)(2n+2), \\ L_2 &= \sigma - (2n+2)(2n+3). \end{aligned}$$

3. D'Alembertian sequences

D'Alembert substitution:

D'Alembert substitution:

If $La = 0$ then substituting $y \leftarrow az$ in $Ly = 0$ yields

3. D'Alembertian sequences

D'Alembert substitution:

If $La = 0$ then substituting $y \leftarrow az$ in $Ly = 0$ yields

$$L_1 \, \Delta z = 0$$

3. D'Alembertian sequences

D'Alembert substitution:

If $La = 0$ then substituting $y \leftarrow az$ in $Ly = 0$ yields

$$L_1 \, \Delta z = 0$$

where $\operatorname{ord} L_1 = \operatorname{ord} L - 1$ and $\Delta = \sigma - 1$.

3. D'Alembertian sequences

D'Alembert substitution:

If $La = 0$ then substituting $y \leftarrow az$ in $Ly = 0$ yields

$$L_1 \Delta z = 0$$

where $\mathrm{ord}\, L_1 = \mathrm{ord}\, L - 1$ and $\Delta = \sigma - 1$.

Definition

$a \in \mathcal{S}(K)$ is *d'Alembertian* $\iff$ there are first-order operators $L_1, L_2, \ldots, L_d \in K(n)\langle\sigma\rangle$ such that

3. D'Alembertian sequences

D'Alembert substitution:

If $La = 0$ then substituting $y \leftarrow az$ in $Ly = 0$ yields

$$L_1 \, \Delta z = 0$$

where $\operatorname{ord} L_1 = \operatorname{ord} L - 1$ and $\Delta = \sigma - 1$.

Definition

$a \in \mathcal{S}(K)$ is *d'Alembertian* $\iff$ there are first-order operators $L_1, L_2, \ldots, L_d \in K(n)\langle\sigma\rangle$ such that

$$L_d \cdots L_2 L_1 \, a \; = \; 0. \qquad (1)$$

3. D'Alembertian sequences

D'Alembert substitution:

If $La = 0$ then substituting $y \leftarrow az$ in $Ly = 0$ yields

$$L_1 \, \Delta z = 0$$

where $\operatorname{ord} L_1 = \operatorname{ord} L - 1$ and $\Delta = \sigma - 1$.

Definition

$a \in \mathcal{S}(K)$ is *d'Alembertian* $\iff$ there are first-order operators $L_1, L_2, \ldots, L_d \in K(n)\langle\sigma\rangle$ such that

$$L_d \cdots L_2 L_1 \, a = 0. \quad (1)$$

Notation:

$$\mathcal{A}(K) \ \ldots \ \text{d'Alembertian sequences in } \mathcal{S}(K)$$

3. D'Alembertian sequences

D'Alembert substitution:

If $La = 0$ then substituting $y \leftarrow az$ in $Ly = 0$ yields

$$L_1 \Delta z = 0$$

where $\operatorname{ord} L_1 = \operatorname{ord} L - 1$ and $\Delta = \sigma - 1$.

Definition

$a \in \mathcal{S}(K)$ is *d'Alembertian* $\iff$ there are first-order operators $L_1, L_2, \ldots, L_d \in K(n)\langle\sigma\rangle$ such that

$$L_d \cdots L_2 L_1 a \;=\; 0. \qquad (1)$$

Notation:

$\mathcal{A}(K)$ ... d'Alembertian sequences in $\mathcal{S}(K)$

$\mathrm{nd}(a) =$ minimal d s.t. (1) holds (*nesting depth* of a)

3. D'Alembertian sequences

3. D'Alembertian sequences

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$:

3. D'Alembertian sequences

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$: d'Alembertian because

$$\left(\sigma - \frac{n+1}{n+2}\right)(\sigma - 1)\, H_n = 0.$$

3. D'Alembertian sequences

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$: d'Alembertian because

$$\left(\sigma - \frac{n+1}{n+2}\right)(\sigma - 1)\, H_n = 0.$$

- Derangement numbers $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$:

3. D'Alembertian sequences

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$: d'Alembertian because

$$\left(\sigma - \frac{n+1}{n+2}\right)(\sigma - 1)\, H_n = 0.$$

- Derangement numbers $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$: d'Alembertian because

$$(\sigma + 1)(\sigma - (n+1))\, d_n = 0.$$

3. D'Alembertian sequences

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$: d'Alembertian because
$$\left(\sigma - \frac{n+1}{n+2}\right)(\sigma - 1)\, H_n = 0.$$
- Derangement numbers $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$: d'Alembertian because
$$(\sigma + 1)(\sigma - (n+1))\, d_n = 0.$$

Notation: $a \in \mathcal{S}(K),\ A \subseteq \mathcal{S}(K)$

$$\Sigma a \;=\; \{b \in \mathcal{S}(K);\ \Delta b = a\} \;=\; \sum_{k=0}^{n-1} a(k) + K$$

3. D'Alembertian sequences

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$: d'Alembertian because

$$\left(\sigma - \frac{n+1}{n+2}\right)(\sigma - 1)\, H_n = 0.$$

- Derangement numbers $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$: d'Alembertian because

$$(\sigma + 1)(\sigma - (n+1))\, d_n = 0.$$

Notation: $a \in \mathcal{S}(K),\ A \subseteq \mathcal{S}(K)$

$$\Sigma a = \{b \in \mathcal{S}(K);\ \Delta b = a\} = \sum_{k=0}^{n-1} a(k) + K$$

$$\Sigma A = \{b \in \mathcal{S}(K);\ \Delta b \in A\}$$

3. D'Alembertian sequences

3. D'Alembertian sequences

Remark

1. $\Delta = \sigma - 1$

3. D'Alembertian sequences

Remark

1. $\Delta = \sigma - 1$
2. $\Delta\Sigma \; = \; 1$

3. D'Alembertian sequences

Remark

1. $\Delta = \sigma - 1$
2. $\Delta\Sigma = 1$
3. $\sigma\Sigma = \Sigma + 1$

3. D'Alembertian sequences

Remark

1. $\Delta = \sigma - 1$
2. $\Delta\Sigma = 1$
3. $\sigma\Sigma = \Sigma + 1$
4. $\Sigma 0 = K$

3. D'Alembertian sequences

Remark

1. $\Delta = \sigma - 1$
2. $\Delta\Sigma \;=\; 1$
3. $\sigma\Sigma \;=\; \Sigma + 1$
4. $\Sigma 0 \;=\; K$

Proposition

Let $r \in K(n)$, $\sigma h = rh$, *and* $f \in \mathcal{S}(K)$. *Then*

$$\{y \in \mathcal{S}(K);\ (\sigma - r)y = f\} \;=\; h\Sigma\frac{f}{\sigma h}.$$

3. D'Alembertian sequences

Hence

$$\begin{aligned}&\mathrm{Ker}\ (\sigma - r_d)\cdots(\sigma - r_2)(\sigma - r_1)\\ =\ \ & h_1 \sum \frac{h_2}{\sigma h_1} \sum \frac{h_3}{\sigma h_2} \cdots \sum \frac{h_d}{\sigma h_{d-1}} \sum 0\end{aligned} \tag{2}$$

where $\sigma h_i = r_i h_i$ for $i = 1, 2, \ldots, d$.

3. D'Alembertian sequences

Hence

$$\begin{aligned}&\mathrm{Ker}\ (\sigma - r_d)\cdots(\sigma - r_2)(\sigma - r_1)\\ &= \quad h_1 \sum \frac{h_2}{\sigma h_1} \sum \frac{h_3}{\sigma h_2} \cdots \sum \frac{h_d}{\sigma h_{d-1}} \sum 0 \end{aligned} \qquad (2)$$

where $\sigma h_i = r_i h_i$ for $i = 1, 2, \ldots, d$.

For *any* $L \in K(n)\langle\sigma\rangle$, the space of all D'Alembertian sequences of $Ly = 0$ is of the form

$$h_1 \sum h_2 \sum h_3 \cdots \sum h_d \sum 0.$$

3. D'Alembertian sequences

Example

(AMM problem no. 10375)

3. D'Alembertian sequences

Example

(AMM problem no. 10375)

$$L = \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3)$$

3. D'Alembertian sequences

Example

(AMM problem no. 10375)

$$
\begin{aligned}
L &= \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3) \\
&= (\sigma - (2n+1)(2n+2))\,(\sigma - (2n+2)(2n+3))
\end{aligned}
$$

Example

(AMM problem no. 10375)

$$
\begin{aligned}
L &= \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3) \\
&= (\sigma - (2n+1)(2n+2))\,(\sigma - (2n+2)(2n+3))
\end{aligned}
$$

It follows from (2) that

3. D'Alembertian sequences

Example

(AMM problem no. 10375)

$$\begin{aligned} L &= \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3) \\ &= (\sigma - (2n+1)(2n+2))\,(\sigma - (2n+2)(2n+3)) \end{aligned}$$

It follows from (2) that

$$\mathrm{Ker}\, L = (2n)!\,\Sigma\frac{(2n+1)!}{(2n+2)!}\Sigma\, 0 = (2n)!\, K\,\Sigma\frac{1}{n+1}$$

Example

(AMM problem no. 10375)

$$\begin{aligned} L &= \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3) \\ &= (\sigma - (2n+1)(2n+2))\,(\sigma - (2n+2)(2n+3)) \end{aligned}$$

It follows from (2) that

$$\begin{aligned} \mathrm{Ker}\,L &= (2n)!\,\Sigma\frac{(2n+1)!}{(2n+2)!}\Sigma\,0 \;=\; (2n)!\,K\,\Sigma\frac{1}{n+1} \\ &= (2n)!\left(K\sum_{k=0}^{n-1}\frac{1}{k+1} + K\right) \;=\; K\,(2n)!\,H_n + K\,(2n)! \end{aligned}$$

3. D'Alembertian sequences

Theorem

$\mathcal{A}(K)$ is closed under the following operations:

3. D'Alembertian sequences

Theorem

$\mathcal{A}(K)$ is closed under the following operations:

1. *addition*
2. *multiplication*
3. *shift*
4. *inverse shift*
5. *indefinite summation*
6. *multisection*

3. D'Alembertian sequences

Theorem

$\mathcal{A}(K)$ is closed under the following operations:

1. *addition*
2. *multiplication*
3. *shift*
4. *inverse shift*
5. *indefinite summation*
6. *multisection*

Corollary

$\mathcal{A}(K)$ is the least subring of $\mathcal{S}(K)$ which:

1. *contains $\mathcal{H}(K)$,*
2. *is closed under σ, σ^{-1}, and Σ.*

3. D'Alembertian sequences

To prove these closure properties, we use

3. D'Alembertian sequences

To prove these closure properties, we use

1. induction on nesting depth,

3. D'Alembertian sequences

To prove these closure properties, we use

1. induction on nesting depth,
2. the following lemma:

3. D'Alembertian sequences

To prove these closure properties, we use

1. induction on nesting depth,
2. the following lemma:

Lemma

Given first-order operators

$$L_1, \ldots, L_k, \; R_1, \ldots, R_m \in K(n)\langle\sigma\rangle,$$

there are operators of order ≤ 1

$$M_1, \ldots, M_m, \; N_1 \ldots, N_k \in K(n)\langle\sigma\rangle \setminus \{0\}$$

such that

$$M_m \cdots M_1 L_k \cdots L_1 \; = \; N_k \cdots N_1 R_m \cdots R_1.$$

4. Liouvillian sequences

Definition

$\mathcal{L}(\mathcal{K})$ is the least subring of $\mathcal{S}(\mathcal{K})$ which:

4. Liouvillian sequences

Definition

$\mathcal{L}(\mathcal{K})$ is the least subring of $\mathcal{S}(\mathcal{K})$ which:

- contains $\mathcal{H}(\mathcal{K})$,

4. Liouvillian sequences

Definition

$\mathcal{L}(\mathcal{K})$ is the least subring of $\mathcal{S}(\mathcal{K})$ which:

- contains $\mathcal{H}(\mathcal{K})$,
- is closed under σ, σ^{-1}, Σ, and interlacing.

4. Liouvillian sequences

Definition

$\mathcal{L}(\mathcal{K})$ is the least subring of $\mathcal{S}(\mathcal{K})$ which:

- contains $\mathcal{H}(\mathcal{K})$,
- is closed under σ, σ^{-1}, Σ, and interlacing.

The elements of $\mathcal{L}(\mathcal{K})$ are *Liouvillian sequences*.

4. Liouvillian sequences

Definition

$\mathcal{L}(\mathcal{K})$ is the least subring of $\mathcal{S}(\mathcal{K})$ which:

- contains $\mathcal{H}(\mathcal{K})$,
- is closed under σ, σ^{-1}, Σ, and interlacing.

The elements of $\mathcal{L}(\mathcal{K})$ are *Liouvillian sequences*.

Example

The sequence

$$n!! = \begin{cases} 2^k k!, & n = 2k, \\ \frac{(2k+1)!}{2^k k!}, & n = 2k+1 \end{cases}$$

is Liouvillian (as an interlacing of two hypergeometric sequences).

4. Liouvillian sequences

Definition

$$AL(K) = \{\Lambda_{j=0}^{k-1} a_j \text{ for some } k \in \mathbb{N} \text{ and } a_j \in \mathcal{A}(K)\}$$

Definition

$AL(K) = \{\Lambda_{j=0}^{k-1} a_j \text{ for some } k \in \mathbb{N} \text{ and } a_j \in \mathcal{A}(K)\}$

Theorem

(Reutenauer 2012)

$$\mathcal{L}(K) \;=\; AL(K)$$

4. Liouvillian sequences

Definition

$AL(K) = \{\Lambda_{j=0}^{k-1} a_j \text{ for some } k \in \mathbb{N} \text{ and } a_j \in \mathcal{A}(K)\}$

Theorem

(Reutenauer 2012)

$$\mathcal{L}(K) \;=\; AL(K)$$

Liouvillian sequences ≡ interlacings of d'Alembertian ones

Sketch of alternative proof:

Clearly, $AL(K) \subseteq \mathcal{L}(K)$.

Sketch of alternative proof:

Clearly, $AL(K) \subseteq \mathcal{L}(K)$.

Lemma

$AL(K)$ is closed under the following operations:

4. Liouvillian sequences

Sketch of alternative proof:

Clearly, $AL(K) \subseteq \mathcal{L}(K)$.

Lemma

$AL(K)$ is closed under the following operations:

1. *addition*
2. *multiplication*
3. *shift*
4. *inverse shift*
5. *indefinite summation*
6. *interlacing*

Sketch of alternative proof:

Clearly, $AL(K) \subseteq \mathcal{L}(K)$.

Lemma

$AL(K)$ is closed under the following operations:

1. *addition*
2. *multiplication*
3. *shift*
4. *inverse shift*
5. *indefinite summation*
6. *interlacing*

Hence $\mathcal{L}(K) \subseteq AL(K)$.