Classifying Standard Equations

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Functional Equations in Limoges
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Klein's Theorem

A classical result of F. Klein (1877) states that:
If

$$y'' + a(t)y' + b(t)y = 0 \quad (a(t), b(t) \in \mathbb{C}(t))$$

has finite projective monodromy group G, then two solutions to the equation are of the form:

$$y(t) = f(t) \cdot {}_2F_{1(\alpha,\beta;\gamma)}(P(t))$$

where $f(t)$ is a solution to a first order homogeneous linear differential equation i.e. $\frac{f'}{f}(t) \in \mathbb{C}(t)$, ${}_2F_{1(\alpha,\beta;\gamma)}(t)$ is a hypergeometric function and $P(t) \in \mathbb{C}(t)$.

Klein's Theorem

$$y(t) = f(t) \cdot {}_2F_{1(\alpha,\beta,\gamma)}(P(t))$$

Nowadays we say

"$y'' + a(t)y' + b(t)y = 0$ is projectively equivalent to the pullback of a hypergeometric equation by a rational map."

This result was revisited by Dwork and Baldassari (1979) and algorithmically implemented in a joint work of M. Berkenbosch, M. van Hoeij and J.A. Weil (2005). The three latter authors realized that the $f(z)$ and the $P(z)$ can be computed very efficiently.

Klein's Theorem

$$y(t) = f(t) \cdot {}_2F_{1(\alpha,\beta,\gamma)}(P(t))$$

The triple $(\alpha, \beta; \gamma)$ is chosen so that the projective monodromy group of the corresponding hypergeometric equation corresponds to the deck transformations of the Galois covering of the Riemann sphere by the Riemann sphere.

one finite group $G \subseteq PSL_2(\mathbb{C})$, one hypergeometric equation.

Klein's Theorem

$$y(t) = f(t) \cdot {}_2F_{1(\alpha,\beta,\gamma)}(P(t))$$

All this means that, starting from the original equation

$$y'' + a(t)y' + b(t)y = 0,$$

we change the dependent variable to

$$u = \frac{1}{f}y$$

and the independent variable to

$$z = P(t),$$

and it becomes a hypergeometric equation with finite Galois group.The f and $P(t)$ are read-off the coefficient $a(t)$ and the rational linear first integral of minimum degree (Kovacic's algorithm).

Klein's Theorem

Equation $y'' + a(t)y' + b(t)y = 0$ (with algebraic solutions)
Solution $y(t) = f(t) \cdot {}_2F_{1(\alpha,\beta,\gamma)}(P(t))$
Change of variables $u = \frac{1}{f}y$ $z = P(t)$
New equation $z(1-z)u'' + [c - (a+b+1)z]u' - abu = 0$

$$4a = 1 - \lambda^2, \quad 4b = 1 - \mu^2, \quad 4c = 1 - (\lambda^2 + \mu^2 + \nu^2)$$

where (λ, μ, ν) is

$$\begin{aligned} D_{2\cdot n} &\rightsquigarrow (1/2, 1/2, 1/n) \\ A_4 &\rightsquigarrow (1/3, 1/2, 1/3) \\ S_4 &\rightsquigarrow (1/3, 1/2, 1/4) \\ A_5 &\rightsquigarrow (1/3, 1/2, 1/5) \end{aligned}$$

Klein's Theorem - Example

Pépin's equation (1881)

$$y'' + \frac{21}{100}\frac{t^2-t+1}{t^2(t-1)^2}y = 0;$$

has projective monodromy group isomorphic to A_5.

$$\sqrt[3]{t^2-t}\cdot F_{\left(-\frac{1}{60},\frac{11}{60};\frac{2}{3}\right)}\left(\frac{4}{27}\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right)$$

$$\sqrt[3]{\frac{108}{t^2(t-1)^2}}(t^2-t+1)^3\cdot F_{\left(\frac{19}{60},\frac{31}{60};\frac{4}{3}\right)}\left(\frac{4}{27}\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right)$$

Standard 2nd order Equations

$$z(1-z)u'' + [c - (a+b+1)z]u' - abu = 0$$

with

$$4a = 1 - \lambda^2, \quad 4b = 1 - \mu^2, \quad 4c = 1 - (\lambda^2 + \mu^2 + \nu^2)$$

where (λ, μ, ν) is

$$\begin{aligned} D_{2\cdot n} &\rightsquigarrow (1/2, 1/2, 1/n) \\ A_4 &\rightsquigarrow (1/3, 1/2, 1/3) \\ S_4 &\rightsquigarrow (1/3, 1/2, 1/4) \\ A_5 &\rightsquigarrow (1/3, 1/2, 1/5) \end{aligned}$$

What about higher order?

Ulmer's G_{54}-equation (2003):

$$y''' + \frac{3(3t^2-1)}{t(t-1)(t+1)}y'' + \frac{221t^4-206t^2+5}{12t^2(x-1)^2(t+1)^2}y' + \frac{374t^6-673t^4+254t^2+5}{54t^3(t-1)^3(t+1)^3}y = 0$$

$$\frac{1}{\sqrt{t(t-1)(t+1)}} \cdot F_{(-\frac{1}{12},\frac{1}{6},\frac{2}{3};\frac{1}{2},\frac{3}{4})}\left(\frac{1}{16}\frac{(t^2+1)^4}{t^2(t+1)^2(t-1)^2}\right)$$

$$\frac{(t^2+1)^2}{\sqrt{t^3(t+1)^3(t-1)^3}} \cdot F_{(\frac{5}{12},\frac{2}{3},\frac{7}{6};\frac{5}{4},\frac{3}{2})}\left(\frac{1}{16}\frac{(t^2+1)^4}{t^2(t+1)^2(t-1)^2}\right)$$

$$\frac{(t^2+1)}{t(t+1)(t-1)} \cdot F_{(\frac{1}{6},\frac{1}{3},\frac{11}{12};\frac{3}{4},\frac{5}{4})}\left(\frac{1}{16}\frac{(t^2+1)^4}{t^2(t+1)^2(t-1)^2}\right)$$

What about higher order?

Geilselmann-Ulmer F_{36}-equation (1997):

$$y''' + \frac{5(9t^2+14t+9)}{48t^2(t+1)^2}y' - \frac{5(81t^3+185t^2+229t+81)}{432t^3(t+1)^3}y = 0$$

$$t^{\frac{1}{2}}(t-1)^{\frac{3}{4}} \cdot F_{(-\frac{1}{12},\frac{1}{6},\frac{2}{3};\frac{1}{2},\frac{3}{4})}\left(\frac{4(t-1)}{t^2}\right)$$

What about higher order?

These two equations are projectively equivalent to a pullback of

$$u''' + \frac{3}{4}\frac{5z-3}{z(z-1)}u'' + \frac{43z-9}{24z^2(z-1)}u' - \frac{1}{108z^2(z-1)}u = 0,$$

which is the generalized hypergeometric equation defining ${}_3F_{2(-\frac{1}{12},\frac{1}{6},\frac{2}{3};\frac{1}{2},\frac{3}{4})}(z)$.

3rd order Klein-style Theorem

M. Berkenbosch (2006):
If

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0 \quad (a(t), b(t), c(t) \in \mathbb{C}(t))$$

has finite Galois group, then the solutions to the equation are of the form:

$$y(t) = f(t) \cdot F_{Std}(P(t))$$

where $f(t)$ is a solution to a first order homogeneous linear differential equation i.e. $\frac{f'}{f}(t) \in \mathbb{C}(t)$, $F_{Std}(t)$ is a solution to a Standard equation and $P(t) \in \mathbb{C}(t)$.

.... there are infinitely many standard equations for each finite group of $\mathrm{SL}_3(\mathbb{C})$.

Proving Klein's Theorem - Galois Coverings

If X is a proyective curve and G a finite group acting on X.

$$\begin{aligned} g \in G : X &\longrightarrow X \\ p &\longmapsto g \cdot p \end{aligned}$$

Projecting into the orbits space X/G, we obtain a ramified covering:

$$\begin{aligned} \Pi : X &\longrightarrow X/G \\ p &\longmapsto Gp \end{aligned}$$

Proving Klein's Theorem - Galois Coverings

Algebraically, the action of G corresponds to an action over the field of meromorphic functions over X, $K = \mathbb{C}(X)$.

$$\begin{aligned} g \in G : K &\longrightarrow K \\ f &\longmapsto g^* f : p \mapsto f(g^{-1} \cdot p) \end{aligned}$$

Taking G-invariant functions, $K^G = \{f \mid f = g^* f \ \forall g \in G\}$, we obtain an algebraic extension:

$$\begin{aligned} \imath : K^G &\longrightarrow K \\ f &\longmapsto f \end{aligned}$$

Proving Klein's Theorem - Galois Coverings

$$\mathbb{C}(X/G) = K^G$$

$G = D_{2\cdot 4}$, $X = \mathbb{P}^1(\mathbb{C})$:

$$\{t \mapsto t, t \mapsto \frac{1}{t}, t \mapsto -t, t \mapsto \frac{-1}{t}, t \mapsto \frac{-t+1}{t+1}, t \mapsto \frac{t+1}{t-1}, t \mapsto \frac{t+1}{-t+1}, t \mapsto \frac{t-1}{t+1}\}$$

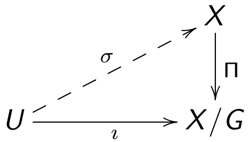
Taking $K = \mathbb{C}(t)$. We get $K^G = \mathbb{C}(z) \to K = \mathbb{C}(t)$

$$z = P(t) = \frac{1}{16}\frac{(t^2+1)^4}{t^2(t+1)^2(t-1)^2}$$

Proving Klein's Theorem - Galois Coverings

Given X and the action of G explicitly, the map $\Pi : X \to X/G$ can be described algebraically. What about the other direction?

Can one describe algebraically the sections of Π?



F. Klein and H. Schwarz knew the answer when $X = \mathbb{P}^1(\mathbb{C})$ and G is finite: *use Gauss' hypergeometric functions*

Proving Klein's Theorem - Galois Coverings

Take two linearly independent solution u_1, u_2 to the equation:

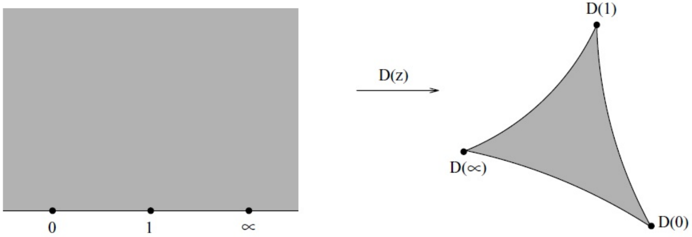
$$z(1-z)u'' + [c - (a+b+1)z]u' - abu = 0.$$

The image of $\mathbb{H} = \{\mathrm{im}(z) > 0\}$ in $\mathbb{P}^1(\mathbb{C})$ by the map

$$D_{(\lambda,\mu,\nu)}(z) = (u_1(z) : u_2(z))$$

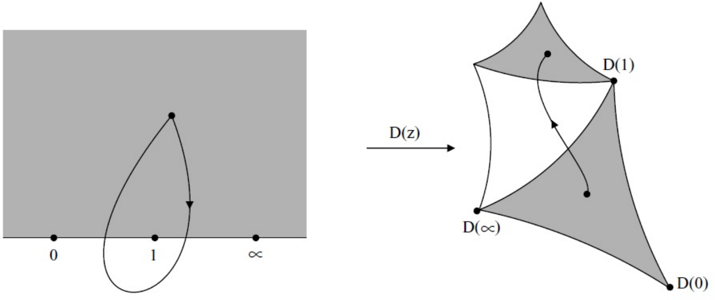
is a curvilinear triangle.

Proving Klein's Theorem - Galois Coverings



F. Beukers, *Gauss' hypergeometric function*

Proving Klein's Theorem - Galois Coverings



F. Beukers, *Gauss' hypergeometric function*

Proving Klein's Theorem - Galois Coverings

Take two linearly independent solution u_1, u_2 to the equation:

$$z(1-z)u'' + [c-(a+b+1)z]u' - abu = 0$$

where (a, b, c) are such that (λ, μ, ν) is $(1/2, 1/2, 1/n)$, $(1/3, 1/2, 1/3)$, $(1/3, 1/2, 1/4)$ or $(1/3, 1/2, 1/5)$ and

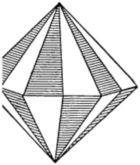
$$4a = 1-\lambda^2, \quad 4b = 1-\mu^2, \quad 4c = 1-(\lambda^2+\mu^2+\nu^2)$$

then the image of $\mathbb{H} = \{\mathrm{im}(z) > 0\}$ in $\mathbb{P}^1(\mathbb{C})$ by the map

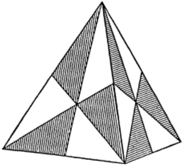
$$D_{(\lambda,\mu,\nu)}(z) = (u_1(z) : u_2(z))$$

is a curvilinear triangle... evenly tiling the sphere.

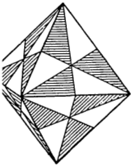
Proving Klein's Theorem - Galois Coverings



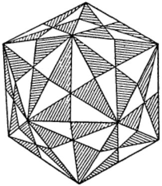
Dihedron



Tetrahedron



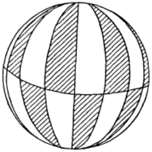
Octahedron



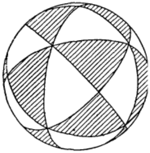
Icosahedron

M. Yoshida, *Hypergeometric functions, My Love*

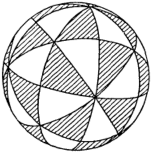
Proving Klein's Theorem - Galois Coverings



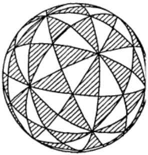
$(2,2,n)$



$(2,3,3)$



$(2,3,4)$



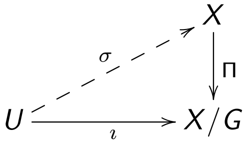
$(2,3,5)$

M. Yoshida, *Hypergeometric functions, My Love*

Proving Klein's Theorem - Galois Coverings

Summary:

It is all about describing the sections of Galois coverings of projective curves



What does projective equivalence mean in linear ODEs?

$$y(t) = f(t) \cdot F_{Std}(P(t))$$

Let $\overline{r} : (a, b) \subseteq \mathbb{R} \to \mathbb{R}^n$ be a smooth curve. Set $\overline{r}'(t) = \overline{y}(t) = (y_1(t), \ldots, y_n(t))$.
There is a linear dependence between $\overline{y}(t), \overline{y}'(t), \ldots, \overline{y}^{(n)}(t)$:

$$0 = a_n(t)\overline{y}^{(n)}(t) + \ldots + a_1(t)\overline{y}'(t) + a_0(t)\overline{y}(t)$$

In particular, $y_1, \ldots, y_n$ are solutions to

$$a_n(t)y^{(n)} + \ldots + a_1(t)y' + a_0(t)y = 0$$

What does projective equivalence mean in linear ODEs?

$\overline{r} : (a, b) \subseteq \mathbb{R} \to \mathbb{R}^n \qquad \overline{r}'(t) = \overline{y}(t) = (y_1(t), \dots, y_n(t))$
Set $z = \int^t f(\tau) d\tau$ so that

$$\frac{d}{dz} t(z) = \frac{1}{\frac{d}{dt} z(t)} = \frac{1}{f(t)}$$

and

$$\frac{d}{dz} \overline{r}(z) = \frac{1}{f(t)} \overline{y}(t) = \overline{u}(z).$$

What does projective equivalence mean in linear ODEs?

It means reparametrization: $z = \int^t f(\tau) d\tau$
And sometimes it is useful to pick a new parameter in order to make geometric invariant quantities explicit (e.g. curvature, torsion, Frenet-Serret formulas... pick $f(t) = \|\overline{y}\|(t)$ so that

$$\langle \frac{d}{dz}\overline{r}, \frac{d}{dz}\overline{r}\rangle(z) = \frac{1}{\|\overline{y}\|^2}\langle \overline{y}, \overline{y}\rangle(t)$$

becomes constant)

What does projective equivalence mean in linear ODEs?

Once a good parameter has been chosen, one can think of a suitable frame to study the curve.
e.g. Frenet-Serret frame: z arc-length parameter and orthonormalization:

$$\overline{u},\ \frac{d}{dz}\overline{u},\dots,\ \frac{d}{dz}^{n-1}\overline{u} \rightsquigarrow \overline{v}_1, \overline{v}_2, \dots, \overline{v}_n$$

becomes

$$\frac{d}{dz}\begin{bmatrix} \overline{v}_1(z) \\ \vdots \\ \overline{v}_n(z) \end{bmatrix} = \begin{bmatrix} 0 & \chi_1(z) & \cdots & 0 \\ -\chi_1(z) & \ddots & \ddots & \\ \vdots & \ddots & 0 & \chi_{n-1}(z) \\ 0 & \cdots & -\chi_{n-1}(z) & 0 \end{bmatrix} \begin{bmatrix} \overline{v}_1(z) \\ \vdots \\ \overline{v}_n(z) \end{bmatrix}$$

How to choose a good parameter?

Regardless of the parameter for our original smooth curve

$$\bar{r} : (a, b) \subseteq \mathbb{R} \to \mathbb{R}^n,$$

the projective curve given by the image of the map

$$\begin{array}{rcl} (a, b) & \longrightarrow & \mathbb{P}^{n-1}(\mathbb{R}) \\ t & \longmapsto & (y_1(t) : \ldots : y_n(t)) \end{array}$$

is always the same. Think of it as a proyective version of a Gauss map.

How to choose a good parameter?

Let us now start from an irreducible linear ODE

$$a_n(t)y^{(n)} + \ldots + a_1(t)y' + a_0(t)y = 0 \quad (a_i \in \mathbb{C}(t))\,,$$

and we assume that its projective monodromy group G is finite. Pick $U \subseteq \mathbb{P}^1(\mathbb{C})$ (simple open, e.g. disk) avoiding all singularities and zeroes of the wronskian. Let $y_1, \ldots, y_n : U \to \mathbb{C}$ be n-linearly independent solutions. Consider the analytic map

$$\begin{array}{rcl} D : U \subseteq \mathbb{P}^1(\mathbb{C}) & \longrightarrow & \mathbb{P}^{n-1}(\mathbb{C}) \\ t & \longmapsto & (y_1(t) : \ldots : y_n(t)) \end{array}$$

We extend analytically all we can the image $D(U)$. We obtain an algebraic curve in $\mathbb{P}^{n-1}(\mathbb{C})$ and its Zariski-closure, X, is a projective curve (Fano Curve).

How to choose a good parameter?

$$U \overset{D}{\dashrightarrow} X \subseteq \mathbb{P}^{n-1}(\mathbb{C})$$

The genus g of the projective curve X can be obtained from the local (rational) exponents at the singularities of the equation.

$$\sum_{p \in S} \left(\frac{1}{e_p} - 1 \right) = -2 - \frac{2(g-1)}{M}$$

where S is the set of singularities, e_p the least common denominator of the differences of all the local exponents at p and M is the size of G.

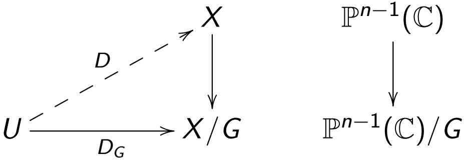
How to choose a good parameter?

$$U \overset{D}{\dashrightarrow} X \subseteq \mathbb{P}^{n-1}(\mathbb{C})$$

letting the projective monodromy group G act by projective transformations on $\mathbb{P}^{n-1}(\mathbb{C})$, we can post-compose D by the quotient

$$\mathbb{P}^{n-1}(\mathbb{C}) \longrightarrow \mathbb{P}^{n-1}(\mathbb{C})/G$$

How to choose a good parameter?



by Galois correspondence the bottom arrow is algebraic; and therefore D_G can be extended to an algebraic map

$$D_G : \mathbb{P}^1(\mathbb{C}) \longrightarrow X/G \subseteq \mathbb{P}^{n-1}(\mathbb{C})/G$$

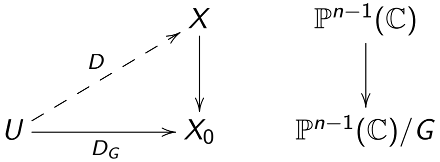
How to choose a good parameter?

$$D_G : \mathbb{P}^1(\mathbb{C}) \longrightarrow X/G \subseteq \mathbb{P}^{n-1}(\mathbb{C})/G$$

By Lüroth Theorem $X/G = D_G\left(\mathbb{P}^1(\mathbb{C})\right) = X_0 \simeq \mathbb{P}^1(\mathbb{C})$. If $\mathbb{C}(X_0) = \mathbb{C}(z)$, we have

$$z = D_G^*(z) = P(t) \in \mathbb{C}(t)$$

Standard Equation



When D_G is birational, $\mathbb{C}(z) = \mathbb{C}(t)$, and we say that the original equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = 0$$

is *Standard*. If it is not then the equation is a pullback by the rational map D_G ($z = P(t)$) of the Standard Equation whose solutions give the sections of the Galois covering $X \to X_0 \simeq \mathbb{P}^1(\mathbb{C})$ (reparametrize it using z).

nth order Klein-style Theorem

If

$$a_n(t)y^{(n)} + \ldots + a_1(t)y' + a_0(t)y = 0 \quad (a_i(t) \in \mathbb{C}(t))$$

has finite Galois group, then the solutions to the equation are of the form:

$$y(t) = f(t) \cdot F_{Std}(P(t))$$

where $f(t)$ is a solution to a first order homogeneous linear differential equation i.e. $\frac{f'}{f}(t) \in \mathbb{C}(t)$, $F_{Std}(t)$ is a solution to a Standard equation and $P(t) \in \mathbb{C}(t)$.

.... Is there a way to classify the involved Standard Equations?.

Classifying Standard Equations

A ruled surface S is the total space of a fiber bundle over a projective curve where each fiber is isomorphic to $\mathbb{P}^1(\mathbb{C})$.

$$\pi : S \longrightarrow C \text{ with } \pi^{-1}(p) \simeq \mathbb{P}^1(\mathbb{C})$$

Classifying Standard Equations

The simplest ruled surface is $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.
In general, a ruled surface over C corresponds to the bundle obtained by the projective spaces defined by the fibers of a rank-2 vector bundle over C:

$$S = \mathbb{P}(\mathscr{O}_C(n) \oplus \mathscr{O}_C(m)).$$

The isomorphism class of the ruled surface S is determined by the quantity $m - n$... or $(m - n)^2$.

Classifying Standard Equations

$$S = \mathbb{P}(\mathscr{O}(n) \oplus \mathscr{O}(m)).$$

Line bundles over $\mathbb{P}^1(\mathbb{C})$ correspond to algebraic maps

$$\mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^N(\mathbb{C})$$

Given a Standard Equation we take the algebraic maps

$$\begin{array}{rcl} D_G : \mathbb{P}^1(\mathbb{C}) & \longrightarrow & X_0 \simeq \mathbb{P}^1(\mathbb{C}) \\ z & \longmapsto & (u_1(z) : \ldots : u_n(z)) \cdot G \end{array}$$

giving us $\mathscr{O}(n)$; and,

$$\begin{array}{rcl} D'_G : \mathbb{P}^1(\mathbb{C}) & \longrightarrow & X'_0 \simeq \mathbb{P}^1(\mathbb{C}) \\ z & \longmapsto & (u'_1(x) : \ldots : u'_n(x)) \cdot G \end{array}$$

giving us $\mathscr{O}(m)$.

Classifying Standard Equations - Examples

$$
\begin{aligned}
A_4 &\rightsquigarrow \mathbb{P}\Big(\mathscr{O}(2) \oplus \mathscr{O}(26)\Big) \\
S_4 &\rightsquigarrow \mathbb{P}\Big(\mathscr{O}(1) \oplus \mathscr{O}(25)\Big) \\
A_5 &\rightsquigarrow \mathbb{P}\Big(\mathscr{O}(1) \oplus \mathscr{O}(61)\Big) \\
D_{2\cdot n} &\rightsquigarrow \mathbb{P}\Big(\mathscr{O}(2) \oplus \mathscr{O}(2[2n+1])\Big) \text{ if } 2 \not| n \\
D_{2\cdot n} &\rightsquigarrow \mathbb{P}\Big(\mathscr{O}(1) \oplus \mathscr{O}(2n+1)\Big) \text{ if } 2 | n
\end{aligned}
$$

Classifying Standard Equations

If the equation is not standard, we can still define a ruled surface through the maps

$$\begin{array}{rcl} D_G : \mathbb{P}^1(\mathbb{C}) & \longrightarrow & X_0 \simeq \mathbb{P}^1(\mathbb{C}) \\ t & \longmapsto & (y_1(t) : \ldots : y_n(t)) \cdot G \end{array}$$

giving us $\mathscr{O}(n)$; and,

$$\begin{array}{rcl} D'_G : \mathbb{P}^1(\mathbb{C}) & \longrightarrow & X_0 \simeq \mathbb{P}^1(\mathbb{C}) \\ t & \longmapsto & (y'_1(t) : \ldots : y'_n(t)) \cdot G \end{array}$$

giving us $\mathscr{O}(m)$.
The value $(n-m)^2$ identifies which standard equation associated to the group G is needed to make-up a solution to our equation.

nth order Klein Theorem

If

$$a_n(t)y^{(n)} + \ldots + a_1(t)y' + a_0(t)y = 0 \quad (a_i(t) \in \mathbb{C}(t))$$

has finite projective monodromy group G, then the solutions to the equation are of the form:

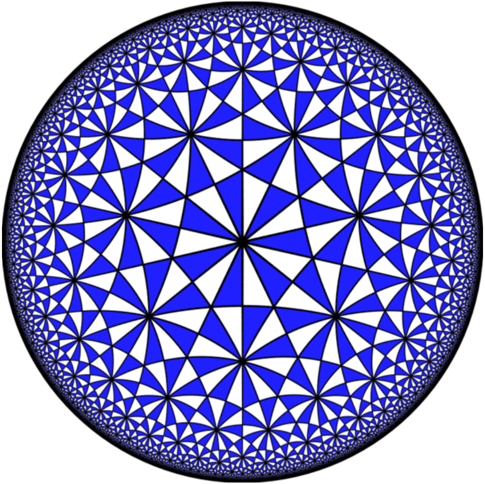
$$y(t) = f(t) \cdot {}_G F_{(n-m)^2}(P(t))$$

where $f(t)$ is a solution to a first order homogeneous linear differential equation i.e. $\frac{f'}{f}(t) \in \mathbb{C}(t)$, ${}_G F_{(n-m)^2}(z)$ is a solution to the Standard Equation $\left(G, (n-m)^2\right)$ and $P(t) \in \mathbb{C}(t)$.

One of the oldest Standard Equations in the literature is Hurwitz':

$$u''' + \frac{1}{4}\frac{13z-7}{z(z-1)}u'' + \frac{1}{112}\frac{137z-14}{z^2(z-1)}u' + \frac{27}{21952}\frac{1}{z^2(z-1)}u = 0$$

The map D associated describes Klein's quartic and taking curvilinear triangles we get:



taken from Wikipedia. Cheers!

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