Program of the 8th edition of the conference

Functional Equations in LIMoges

FELIM 2015

March 23-25, 2015

Organised by

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Salle de Conférences XLIM 3

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FELIM 2015, Functional Equations in LIMoges

Monday, March 23

09:00-09:15 Welcome

09:15-10:15 **Werner Seiler**: *Geometric singularities of algebraic differential equations*

10:15-10:45 Coffee break, Discussions

10:45-11:15 **Jorge Mozo Fernández**: *Polynomial asymptotics and summability*

11:15-11:45 **Eckhard Pfluegel**: *Pseudo-linear matrix-transformation techniques and applications to cryptography*

11:45-12:15 **Jean-Claude Yakoubsohn**: *A numerical study for the Newton method in the singular case*

12:15-14:00 Lunch break

14:00-15:00 **Peter Olver** : *Algebras of differential invariants*

15:00-15:30 Coffee break, Discussions

15:30-16:00 **Loïc Teyssier**: *Computing dessins d'enfants*

16:00-16:30 **Juan Viu-Sos**: *On the geometry of line arrangements and polynomial vector fields*

16:30-17:00 **Yacine Bouzidi**: *Algebraic techniques for testing the structural stability of multidimensional systems*

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Tuesday, March 24

09:00-10:00 **Frédéric Chyzak**: *Hypergeometric generating series for small-step walks in the quarter plane*

10:00-10:30 Coffee break, Discussions

10:30-11:00 **Marko Petkovšek**: *On d'Alembertian and Liouvillian sequences*

11:00-11:30 **Charlotte Hulek**: *On the analytic reduction of singularly perturbed differential equations*

11:30-12:00 **Maximilian Jaroschek**: *Formal reduction of a class of Pfaffian systems in several variables*

12:00-14:00 Lunch break

14:00-15:00 **Georg Regensburger**: *Positive steady states and solutions of polynomial systems with real exponents*

15:00-15:30 Coffee break, Discussions

15:30-16:00 **Frédéric Pierret**: *Modeling and scale dynamics*

16:00-16:30 **Alban Quadrat**: *A constructive algebraic analysis approach to Artstein's reductions*

20:00- Dinner

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Wednesday, March 25

09:00-10:00 **Camilo Sanabria**: *Classifying standard equations*

10:00-10:30 Coffee break, Discussions

10:30-11:00 **Thomas Dreyfus**: *Difference equations on elliptic curve*

11:00-11:30 **Thierry Combot**: *Fast computation of Liouvillian solutions of difference equations*

11:30-12:00 **Roman N. Lee**: *Factoring out parameter dependence in linear differential systems*

12:00-14:00 Lunch break

14:00-14:30 **Islam Boussaada**: *Tracking the algebraic multiplicity of crossing imaginary roots for time-delay systems: a functional confluent Vandermonde-based framework*

14:30-15:00 **Sergei Abramov**: *The EG-family of algorithms and procedures for solving linear differential and difference higher-order systems*

15:00-15:30 **Alexander Prokopenya**: *Quantum algorithm for phase estimation: simulation with Wolfram Mathematica*

15:30- Coffee break, Discussions

Abstracts of the talks

Contents

1 Geometric singularities of algebraic differential equations

Werner Seiler
Kassel University, Germany.

Combining ideas and methods from differential algebra, algebraic geometry and differential topology, we develop a rigorous geometric theory of algebraic differential equations and their singularities which is able to handle general, i.e. also under- or overdetermined, systems of ordinary or partial differential equations. This theory covers both singular integrals playing an important role in differential algebra and the notions of regular and irregular singularities appearing in differential topology. It leads to the novel concept of a regularity decomposition of an algebraic differential equation. We present an algorithm for the construction of such a decomposition and thus the automatic detection of all singularities in a given order. The key tools for this algorithm are the algebraic and the differential Thomas decomposition.

2 Polynomial asymptotics and summability

Jorge Mozo Fernández
Universidad de Valladolid, Spain.

We will introduce the notion of asymptotic expansions, and summability, with respect to an analytic germ of holomorphic function, generalizing the monomial asymptotic expansions developed by Canalis-Durand, Mozo and Schäfke. The main tool we will use will be a theorem of monomialization of analytic germs (reduction of singularities), that allows to reduce this new asymptotics to monomial ones. Also, a proof of a generalized Ramis-Sibuya Theorem will be presented. It is joint work with Reinhard Schäfke (Université de Strasbourg).

3 Pseudo-linear matrix-transformation techniques and applications to cryptography

Eckhard Pfluegel
Kingston University, United Kingdom.

Starting point of this talk is our previous work on a probabilistic algorithm that converts secret input data into public and private data for an application in cryptography. The algorithm is based on viewing the input as a suitable square matrix with entries in a finite field. A similarity transformation can then be computed that results in a new matrix which has companion form. We investigate adapting this algorithm to a pseudo-linear setting. We compare and contrast the resulting salient features. In particular, we discuss the complexity and security and point out some advantages of the new scheme.

4 A numerical for the Newton method in the singular case

Jean-Claude Yakoubsohn
Institut de Mathématiques de Toulouse, France.

This is a new theoretical development to recover the quadratic convergence which is lost in the neigbourhood of singularities.

5 Algebras of differential invariants

Peter Olver,
University of Minnesota, United States of America.

A classical theorem of Lie and Tresse, modernized by Kumpera, states that the algebra of differential invariants of a Lie group or (suitable) Lie pseudo-group action is finitely generated. I will present a constructive algorithm, based on the equivariant method of moving frames, that reveals the full structure of such differential invariant algebras, and, in particular, pinpoints generating sets of differential invariants as well as their differential syzygies. A variety of applications and outstanding issues will be discussed, including equivalence and symmetry detection in image processing, and some surprising new results in classical surface geometries.

6 Computing dessins d'enfants

Loïc Teyssier
Université de Strasbourg, France.

A Dessin d'Enfant is a graph whose combinatorial structure is given in the following way, introduced by A. Grothendieck. Its vertices are the roots of a given algebraic function f (meromorphic on a suitable compact Riemann surface X) of Belyi class: the only finite and non-zero critical value is 1 with multiplicity 1. Its edges correspond to the components of $f^{-1}(]0,1])$. Any finite graph can be realized this way. We propose to present an algorithm computing the combinatorial class of a Dessin given its Belyi function. It is based on computing well-chosen trajectories of an algebraic vector field derived from f. We take this study further and show how to determine effectively the combinatorics of the separatrix graph associated to any algebraic vector field on X, answering a question by A. Douady. The question of the inverse problem (reconstruct all Belyi functions giving rise to a given graph) will be discussed, especially taking a geometric viewpoint instead of the current Gröbner basis approach.

7 On the geometry of line arrangements and polynomial vector fields

Juan Viu-Sos
Université de Pau et des Pays de l'Adour, France.

The geometry of an embedded object in a certain category (analytic, algebraic, linear,...) can be studied looking at the set of vector fields which fix this object. The purpose of this talk is to present a first result in the case of algebraic vector fields fixing predefined line arrangements in the plane, describing the influence of the (weak and strong) combinatorics on the minimal degree expected for such a kind of vector fields. We determine that this minimal degree does not depend on the strong combinatorics presenting an explicit example of two line arrangements with same combinatorics but different geometries.

8 Algebraic techniques for testing the structural stability of multidimensional systems

Yacine Bouzidi
INRIA Saclay, France.

In this work, we present new computer algebra based methods for testing the structural stability of N-D discrete linear systems (with $N \geq 2$). Recall that an N-D discrete linear system, described by its transfer function $\frac{N(z_1,\ldots,z_n)}{D(z_1,\ldots,z_n)}$, where $N(z_1,\ldots,z_n)$ and $D(z_1,\ldots,z_n)$ are polynomials in complex variables with real coefficients, is said to be structurally stable if and only if:

$$D(z_1,\ldots,z_n) \neq 0 \text{ for } |z_1| \leq 1,\ldots,|z_n| \leq 1 \tag{1}$$

A simplified form of the above condition has been obtained by Decarlo et al. [1] who gave a set of conditions that are equivalent to condition (1)

$$\begin{array}{ll} D(z_1,1,\ldots,1) \neq 0 & \text{for } |z_1| \leq 1 \\ D(1,z_2,1,\ldots,1) \neq 0 & \text{for } |z_2| \leq 1 \\ \vdots & \\ D(1,\ldots,1,z_n) \neq 0 & \text{for } |z_n| \leq 1 \\ D(z_1,\ldots,z_n) \neq 0 & \text{for } |z_1| = \ldots = |z_n| = 1 \end{array}$$

Starting from these conditions, we first show that the last condition i.e. $D(z_1,\ldots,z_n) \neq 0$ for $|z_1| = \ldots = |z_n| = 1$ (the existence of complex zeros on the unit poly-circle) is equivalent, modulo certain transformations, to the existence of real zeros in some region of the real space. We then check the resulting conditions using classical algorithms for searching for real zeros of algebraic systems. Our methods have been implemented on maple and their efficiency is demonstrated through numerous tests.

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9 Hypergeometric generating series for small-step walks in the quarter plane

Frédéric Chyzak
INRIA, France.

This is a joint work with A. Bostan, M. van Hoeij, M. Kauers and L. Pech.

Lattice walks are combinatorial objects that occur frequently in discrete mathematics, statistical physics, probability theory, or operational research. The generating series that enumerate them under certain constraints interest both combinatorialists and the algorithmists of computer algebra. First, their algebraic properties vary greatly according to the family of admissible steps chosen to define them, making their generating series sometimes rational, sometimes algebraic (and therefore described by a polynomial equation), sometimes D-finite (and therefore described by a linear differential equation), or sometimes with no apparent equation. Since a few years, this has motivated an effort of classification that has resulted in characterizations that are not sufficiently understood yet to be fully explicit. In addition, the computational properties of lattice walks make them an interesting challenge for computer algebra: indeed, their description often leads to equations, whether polynomial or differential, whose degrees, orders, and sizes are so large that it becomes difficult to obtain those descriptions explicitly, and to manipulate them with reasonable efficiency.

Given a family of non-zero vectors of the plane with coordinates ± 1, vectors which we shall call "steps" a small-step walk on the plane, square lattice is a finite succession of steps located one after the other. We are particularly interested in walks that are constrained by being confined to the quarter plane (that is, with non-negative integer coordinates), and counted according to their length (number of steps). In this talk, we shall present a work in progress that makes a bridge between previous works of different natures on the topic of small-step walks on the quarter plane. On the one hand, Bousquet-Mélou and Mishna showed [2] that among the 79 essentially different models of walks, only 19 possess a D-finite and transcendental generating series, and thus correspond to linear differential equations, but without making explicit the differential equations whose existence they were proving. Almost simultaneously, Bostan et Kauers [1] performed non-trivial but heuristic computations to obtain linear differential equations most probably satisfied by the 19 walk models, but without formally proving the correctness of these equations. In the work described, we shall give the first proof that these equations are satisfied by the corresponding generating series. Our approach proceeds by representing the generating series of constrained walks as coefficient extractions in rational series, and by a thorough validation of the use of the creative-telescoping process [3] employed for these extractions.

Once proved, the differential equations allow to compute in a guaranteed way many formulas and properties of the walk series. First, a suitable factorization of the underlying linear differential operators combined with the algorithm of [4] allows to represent the walks geenerating series as variations of iterated primitives of Gauss hypergeometric series. It follows that algebraicity and transcendence properties of enumerative series and specializations that are significant to combinatorics are accessible to computation, as well as asymptotic formulas for a number of walk models counted by lengths.

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10 On d'Alembertian and Liouvillian sequences

Marko Petkovšek
University of Ljubljana, Slovenia.

D'Alembertian sequences are those annihilated by products of first-order linear recurrence operators with polynomial coefficients. Alternatively, d'Alembertian sequences are those expressible as nested indefinite sums with hypergeometric summands. On the other hand, Liouvillian sequences are defined as the elements of the least subring of the ring of germs of sequences that contains all the hypergeometric sequences, and is closed under forward and backward shift, indefinite summation, and interlacing of an arbitrary number of sequences. It is clear that d'Alembertian sequences form a proper subclass of Liouvillian sequences. In 2012, Christophe Reutenauer pointed out (using difference Galois theory) that Liouvillian sequences are precisely the interlacings of d'Alembertian sequences. We give an alternative proof of this characterization by proving the salient closure properties of d'Alembertian sequences and their interlacings. This is a joint work with Helena Zakrajšek.

11 On the analytic reduction of singularly perturbed differential equations

Charlotte Hulek
Université de Strasbourg, France.

In this talk we present a powerful method for the analytic reduction of singularly perturbed differential equations with *turning points*. It is based on a theorem of *slow-fast* factorization of new composite expansions called *Gevrey.* These expansions have recently been introduced by A. Fruchard and R. Schäfke [1]. This method allows us to prove an analytic version of a formal theorem due to R.J. Hanson and D.L. Russell [2]. This version is a result of uniform simplification in a full neighborhood of a turning point for a class of singularly perturbed differential equations of the second order. It is a generalization of a well known theorem due to Y. Sibuya [3]. We also consider differential equations of order greater than two.

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12 Formal reduction of a class of Pfaffian systems in several variables

Maximilian Jaroschek
Max Planck Institute for Informatics, Germany.

In this talk we are interested in the formal reduction of completely integrable Pfaffian systems with normal crossings. Our investigations treat the generalization of known methods for the case of one or two variables to the multivariate setting. We follow the approach of the latter by associating to a given Pfaffian system a set of ordinary linear differential systems from which information on formal invariants can be retrieved. Furthermore, we introduce a variant of rank reduction facilitated by standard bases of modules over power series rings. This is joint ongoing work with Moulay A. Barkatou and Suzy S. Maddah (University of Limoges, XLIM).

13 Positive steady states and solutions of polynomial systems with real exponents

Georg Regensburger
Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austria.

Reaction networks with mass action kinetics give rise to polynomial ODE systems. Chemical reaction network theory (CRNT) provides statements about uniqueness, existence, and stability of positive steady states for all rate constants and initial conditions depending on the underlying network structure alone. In terms of polynomial equations, they guarantee existence and uniqueness of positive solutions for all parameters.

We first survey several results from CRNT, emphasizing the consequences for polynomial equations with real and symbolic exponents and addressing computational aspects. Then we describe an extension to generalized mass action systems where reaction rates are allowed to be power-laws in the concentrations. In this setting, uniqueness and existence for all parameters additionally depend on sign vectors of related vector subspaces. We also illustrate our results with an implementation in the computer algebra system Maple. This is joint work with Stefan Müller. Finally, we discuss an interpretation of the results as a first partial multivariate generalization of the classical Descartes' rule for positive roots of univariate polynomials. This is joint work Carsten Conradi, Alicia Dickenstein, Elisenda Feliu, Stefan Müller, and Anne Shiu.

14 Modeling and scale dynamics

Frédéric Pierret
Observatoire de Paris, France.

Modeling phenomena, for example in Physics or Biology, always begin with a choice of hypothesis on the dynamics such as determinism, randomness, derivability etc. Depending on these choices, different behaviors can be observed. The natural question associated to the modeling problem is the following : "With a finite set of data concerning a phenomenon, can we recover its underlying nature ?" From this problem, we introduce in this talk the definition of multi-scale functions, scale calculus and scale dynamics based on the time-scale calculus (see [1]). These definitions will be illustrated on the multi-scale Okamoto's functions. The introduced formalism explains why there exists different continuous models associated to an equation with different scale regimes whereas the equation is scale invariant. A typical example of such an equation, is the Euler-Lagrange equation (see [2], [3], [4]) which will be discussed. This is a joint work with Jacky Cresson (LMA/UPPA).

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15 A constructive algebraic analysis approach to Artstein's reductions

Alban Quadrat
INRIA Saclay, France.

Artstein's famous reduction ([Artstein (1982)]) proves the equivalence between linear differential systems with delayed inputs and linear differential systems without time-delays. The purpose of this talk is to show how to find again the different Artstein's integral transformations in a mechanical way, i.e., without educated guess or clever thoughts. Within the *algebraic analysis approach* to linear functional systems ([Chyzak et al. (2005), Fliess et al. (1998), Quadrat (2010)]), we first reformulate Artstein's reduction in terms of an *isomorphism problem* ([Cluzeau et al. (2008)]) between two *finitely presented left modules* over a *ring of integro-differential time-delay operators*. These modules are explicitly defined by means of the matrices of functional operators defining the linear functional systems. Considering the different *commutation rules* for the differential, integral, time-delay and dilatation operators, we present an algorithmic method to find again and extend Artstein's reductions. These results advocate for a complete *algorithmic study of noncommutative polynomial rings of integro-differential time-delay/dilatation operators* ([Quadrat (2015)]) and for the development of dedicated packages such as `IntDiffOp` ([Korporal et al. (2012)]) for rings of integro-differential operators (see [Korporal et al. (2012), Quadrat et al. (2013), Regensburger et al. (2009)] and the references therein). Finally, this algorithmic approach can also be used to handle different computations over the ring $\mathcal{E}$ introduced in [Loiseau (2000)] for the study of time-delay systems (see [Quadrat (2015)]).

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16 Classifying standard equations

Camilo Sanabria
Universidad de los Andes, Colombia.

Solutions to ordinary linear differential equations with rational coefficients can be used to parametrize projective curves. Among these curves, of special interest are the ones whose parameterizing functions form a basis of solutions to a common equation, for they have many symmetries (they include at least the projective monodromy group).

Using this point of view we generalize F. Klein's classical theorem on second order equations, which states that, if the solutions are algebraic, the equation is the pullback of a hypergeometric equation by a rational map, to higher order. The hypergeometric equations involved in the original theorem are classified using Schwarz triples, the family of equations appearing in our generalization, called standard equations, are classified using ruled surfaces.

17 Difference equations on elliptic curve

Thomas Dreyfus
Université de Toulouse, France.

Let us consider a difference equation of order two with coefficients that are meromorphic over an elliptic curve. To such equations, we may associate a group, the Galois group, that measures the algebraic relations between the solutions. It is an algebraic group of invertible complex matrices. The goal of the talk is to give necessary conditions for the irreducibility and the imprimitivity of the Galois group. This is a joint work with Julien Roques.

18 Fast computation of Liouvillian solutions of difference equations

Thierry Combot
Institut Mathématique de Bourgogne, France.

We prove that the computation of Liouvillian solutions of a linear difference equation L, with coefficients in $\mathbf{k}(n)$ (with $[\mathbf{k} : \mathbb{Q}] < 1$) is equivalent to finding a first order right factor of L in the space of difference operators over a difference ring A, an infinite difference ring extension of $\mathbf{k}(n)$ (independent of L). We then present an algorithm to find such factors and avoiding the combinatorial explosion in the search of singularities. The approach used is similar to the one used by van Hoeij for differential operators. Additional data on the Galois group is also obtained. We will focus in the complexity with respect to coefficients degrees and dispersion. An analysis of necessary field extensions is also made and we make comparisons to the existing algorithm of van Hoeij/Cluzeau for the classical hypergeometric case: in worst case examples, the complexity is exponential with the number of factors, but also the field extensions used are too large (however still being of polynomial size), leading to a useless exponential cost search, as in this example

$$\begin{aligned}
&4(7n^7 + 114n^6 + 700n^5 + 1776n^4 + 214n^3 - 8681n^2 - 18290n - 14557)(n^{18} + 18n^{17}+\\
&\qquad 172n^{16} + 1006n^{15} + 3421n^{14} + 3647n^{13} - 31197n^{12} - 239382n^{11} - 934913n^{10}-\\
&\qquad 2000682n^9 - 370347n^8 + 13644355n^7 + 57065304n^6 + 142066210n^5 + 246697388n^4+\\
&\qquad 314067852n^3 + 298498798n^2 + 188668821n + 77658017)u(n)+\\
&(-28n^{16} - 736n^{15} - 8904n^{14} - 63612n^{13} - 273662n^{12} - 519096n^{11} + 1407138n^{10}+\\
&\qquad 13023172n^9 + 40805708n^8 + 56319104n^7 - 35531800n^6 - 330428110n^5-\\
&\qquad 770318338n^4 - 1105330334n^3 - 1075406412n^2 - 670598426n - 189786146)u(n+1)+\\
&(7n^7 + 65n^6 + 163n^5 - 259n^4 - 1925n^3 - 4104n^2 - 4525n - 3979)u(n+2)
\end{aligned}$$

These examples are rare and related to particular Galois group structures of singularities. Optimal field extensions can be obtained through computation of low degree subfields of their spliting field, and can be done in polynomial time. Eventually, we present modular improvements allowing to avoid most of these field extensions problems.

19 Factoring out parameter dependence in linear differential systems

Roman Lee
Institute of Nuclear Physics, Russia.

We consider a linear first-order differential system $\frac{dy}{dx} = M(x, \epsilon)y$, where ϵ is a parameter and $M(x, \epsilon)$ is a matrix of rational functions of both x and ϵ. Such systems are of great importance for perturbative quantum field theory calculations. Recently, a remarkable observation has been made by J. Henn concerning the systems emerging in perturbative calculations. It appeared that the parameter dependence can be often reduced to a single factor ϵ in the right-hand side. We propose a method to reduce the system (when it is possible) to a Fuchsian form on the extended complex plane with factorized dependence on ϵ, i.e., to the form $\frac{dz}{dx} = \epsilon \sum_{i=1}^{k} \frac{A_i}{x-x_i} z$. Our approach can be also useful for the elimination of apparent singularities in Fuchsian systems on the extended complex plane.

20 Tracking the algebraic multiplicity of crossing imaginary roots for time-delay systems: a functional confluent Vandermonde-based framework

Islam Boussaada
IPSA & L2S, France.

It is well known that the solutions behavior for linear dynamical systems is strongly related to the associated spectrum. In particular, a standard approach in analyzing the stability of time-delay systems consists in characterizing the associated Crossing Imaginary Roots (CIRs). By characterization it is meant the identification of CIRs as well as their associated multiplicities (algebraic/geometric). Efficient approaches for CIRs identification exist. However, the multiplicity of CIRs was not deeply investigated. In this talk, we provide a functional confluent Vandermonde/Birkhoff-based framework yielding to CIRs multiplicity for retarded differential equations. Thanks to this, we show that the Polya-Szego bound can never be reached for non zero frequencies, providing a sharper bound. We emphasize also that the proposed framework can be applied to neutral systems. This is a joint work with Silviu Niculescu (L2S).

21 The EG-family of algorithms and procedures for solving linear differential and difference higher-order systems

Sergei Abramov
Computing Centre of the Russian Academy of Sciences, Russia.

This talk is prepared jointly with A. Ryabenko and D. Khmelnov.

We discuss problems of finding solutions of linear differential and difference systems of arbitrary order with variable coefficients.

For systems with polynomial coefficients, algorithms for polynomial, rational and series solutions are proposed. Besides, for the differential case we describe algorithms for finding rational-logarithmic, regular and formal exponential-logarithmic solutions.

We consider also full rank differential systems with infinite computable power series in the role of coefficients. Jointly with M. Barkatou, we propose an algorithm to construct Laurent-series solutions. We propose as well an algorithm for regular solutions. The construction of all formal exponential-logarithmic solutions of such a system is generally an algorithmically undecidable problem. We give an algorithm which allows to construct a basis for the space of such solutions, as soon as the dimension of the space is known.

All the discussed algorithms are implemented as Maple procedures, some of those procedures are incorporated into the package `LinearFunctionalSystems` of standard Maple releases. Other procedures are available on `http://www.ccas.ru/ca/doku.php/eg`

The algorithms are based on the EG-eliminations which are used, e.g., when the leading matrix of a differential system is singular (in the difference case: the leading or/and trailing matrices). An invaluable contribution to the improvement of the first version of the EG-eliminations method was made by M. Bronstein (1963–2005).

22 Quantum algorithm for phase estimation: simulation with Wolfram Mathematica

Alexander Prokopenya
Warsaw University of Life Sciences, Poland.

Quantum computer is believed to be able to perform certain computational tasks much more efficiently than classical computers. At the same time only several efficient quantum algorithms are known (see [1]) and the problem of searching new quantum algorithms is very topical. The computational capabilities of quantum computers are due to the efficient use of quantum effects, such as the existence of superposition and entangled states of quantum bits. These properties are not encountered in classical bits, and they are counterintuitive, which complicates the understanding of quantum computation and the development of new quantum algorithms. Simulation of quantum algorithms on classical computers seems to facilitate the in-depth understanding of quantum computation and help to find new problems and quantum algorithms for their solution.

In the present talk we discuss a quantum algorithm for phase estimation, which was first proposed in [2] and is a part of many quantum algorithms [3, 4]. It was shown that the use of n-qubit quantum memory register for storing the approximate value of the phase makes it possible to estimate the phase accurate to 2^{-n} with a probability greater than $8/\pi^2$. It was shown also that the decrease in the accuracy of phase estimation results in increasing the probability of the successful problem solution, and the lower bound on this probability was obtained. The validity of the results is demonstrated using concrete computations for the case of the quantum phase shift operator. The results obtained provide much better estimates than those given in [1].

Note that all calculations are done with the Wolfram Mathematica package "QuantumCircuit" that was designed for simulation of quantum computation (see [5, 6, 7, 8]). The package provides a user-friendly interface to specify a quantum circuit, to draw it, and to construct the corresponding unitary matrix for quantum computation defined by the circuit. Using this matrix, one can find the final state of the quantum memory register by its given initial state and to check the operation of the algorithm determined by the quantum circuit.

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