Frobenius Structure Conjecture and application to cluster algebras

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Ref: arXiv 1908.09861 joint w Keel
arXiv 2008.02299 joint w Hacking, Keel

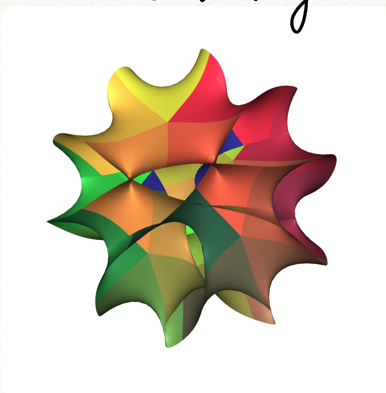
Plan:
1. The Frobenius structure conjecture
2. Structure constants of the mirror algebra
3. Application to cluster algebras

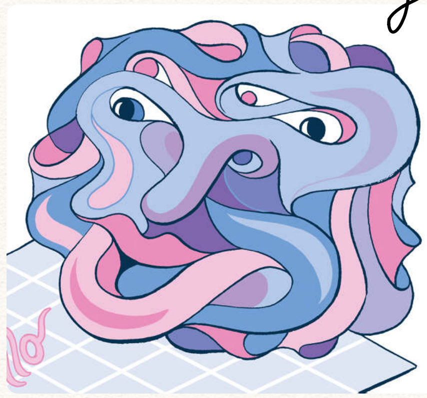
*. Application to moduli space of Calabi-Yau pairs

1. The Frobenius structure conjecture

Motivation: mirror symmetry (conjectural duality between Calabi-Yau varieties)

$\forall$ Calabi-Yau variety X mirror Calabi-Yau variety $\check{X}$





Mirror symmetry philosophy:

One may build the mirror $\check{X}$ by counting curves in X.

Frobenius structure conjecture of Gross-Hacking-Keel:

A precise yet simple formulation of this philosophy for log Calabi-Yau varieties, which boils down to intricate relations of counts of rational curves.

Question: What curves do we count?

Setup: k: any field of char 0 e.g. $\mathbb{C}$

U: affine smooth log Calabi-Yau variety $/k$

$\mathrm{Sk}(U,\mathbb{Z}) :=$ integer points in the essential skeleton of U

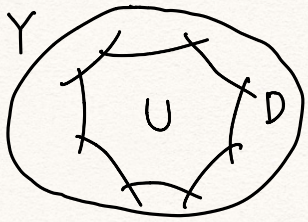
$\{0\} \sqcup \{ m\nu \mid m \in \mathbb{N}_{>0},$

ν is a divisorial valuation on $k(U)$ where ω has a pole$\}$

volume form

field of rational functions

Fix a projective normal crossing compactification $U \subset Y$ with complement



$D := Y \setminus U$

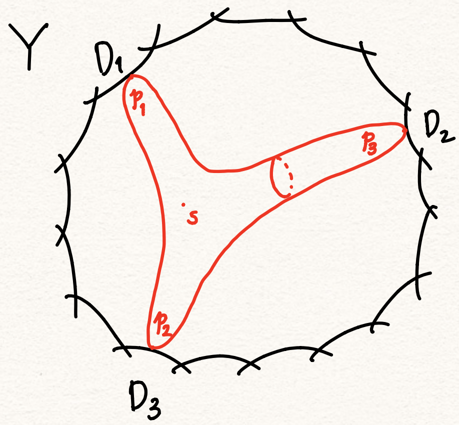
Def (counting rational curves): Given $\mathbf{P} = (P_1, \cdots, P_n)$ $n \geq 2$ $P_j \in Sk(U, \mathbb{Z})$ ← integer points in the essential skeleton

and a curve class $\beta \in NE(Y, \mathbb{Z})$ ← cone of effective curves

Write $P_j = m_j \nu_j$ for all $P_j \neq 0$.

Assume each ν_j is given by a component $D_j \subset D$ (always possible by blowup)

Define $\eta(\mathbf{P}, \beta) :=$ # closed rational curves in Y of class β that intersect D_j with order m_j for every $P_j \neq 0$.



Precise formulation: Let $H(\mathbf{P}, \beta)$ be the space of maps

$$f : [\mathbb{P}^1, (p_1, \cdots, p_n, s)] \longrightarrow Y$$

s.t.
- For every $P_j \neq 0$, $f(p_j)$ meets D_j° with order m_j. No other intersections with the boundary.
- $f_*[\mathbb{P}^1] = \beta$

Lemma: The map $\Phi := (\text{domain}, ev_s) : H(\mathbf{P}, \beta) \longrightarrow M_{0,n+1} \times U$ is finite étale over a Zariski dense open of the target.

Proof: Use the deformation theory of curves.

So we can now precisely define:

the count of rational curves $\eta(\mathbf{P}, \beta) :=$ degree of the finite étale map above.

Rem: The counts are naïve counts (no use of virtual fundamental classes).

Question: Varying $\mathbf{P} = (P_1, \cdots, P_n)$ and $\beta \in NE(Y, \mathbb{Z}) \rightsquigarrow \infty$ many numbers.
What relations between them?

Let's assemble the numbers $\eta(\mathbf{P}, \beta)$ into generating series as follows:

Let $R := \mathbb{Z}[NE(Y,\mathbb{Z})] := \bigoplus_{\beta \in NE(Y,\mathbb{Z})} \mathbb{Z} \cdot z^\beta$, the monoid ring of $NE(Y, \mathbb{Z})$ over $\mathbb{Z}$

$A := R^{(Sk(U,\mathbb{Z}))} := \bigoplus_{P \in Sk(U,\mathbb{Z})} R \cdot \theta_P$, the free R-module with basis $Sk(U, \mathbb{Z})$

Def (Frobenius form): Let $\langle , \cdots , \rangle_n : A^n \to R$ be the R-multilinear map with

$$\langle \theta_{P_1}, \cdots, \theta_{P_n} \rangle_n = \sum_{\beta \in NE(Y,\mathbb{Z})} \eta(P_1, \cdots, P_n, \beta)\, z^\beta$$

Rem:
- Nothing complicated, just putting together all possible $\eta(P_1, \cdots, P_n, \beta)$
- The sum above is finite (by the affineness of U).

Main theorem:

Frobenius structure theorem (KY): Assume U contains an open split algebraic torus. Then the following hold:

(1) The Frobenius multilinear form is non-degenerate.

(2) There exists a unique commutative associative R-algebra structure on A compatible with the Frobenius form, i.e. $\theta_0 = 1$ and $\langle a_1, \cdots, a_n \rangle_n = \mathrm{Trace}(a_1 \cdots a_n)$

taking coefficient of θ_0

(3) Consider $\mathcal{V} := \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$.

mirror family   mirror algebra

It is a flat family of affine varieties of same dimension as U, the generic fibers are log Calabi-Yau varieties (with log-canonical singularities)

Remark: • We can remove the dependence of our mirror algebra A on the compactification $U \subset Y$ by forgetting all curve classes:

$$A_U := A \otimes_R \mathbb{Z} \quad \text{where } R \to \mathbb{Z} \text{ sends every } z^\beta \text{ to } 1.$$

• The assumption that U contains a torus always holds in dim 2, not always in dim ≥ 3.

It plays two roles in our theory:

(i) It allows a degeneration of the mirror family to a toric variety, crucial for the proof of (1) and (3).

(ii) It greatly simplifies the enumerative part of the theory.
A more sophisticated enumerative theory is under development.

• This proves a (better) version of the conjecture of Gross-Hacking-Keel under the torus assumption. The original conjecture was stated via log Gromov-Witten invariants instead of naïve counts, because the smoothness of moduli space was unknown at that moment.

2. Structure constants of the mirror algebra

Question: How are products defined in the mirror algebra A ?

Given $P_1, \cdots, P_n \in Sk(U, \mathbb{Z})$, we write the product in the mirror algebra A as

$$\theta_{P_1} \cdots \theta_{P_n} = \sum_{Q \in Sk(U,\mathbb{Z})} \sum_{\gamma \in NE(Y,\mathbb{Z})} \chi(P_1, \cdots, P_n, Q, \gamma)\, z^{\gamma} \theta_Q$$

structure constants

Idea: Inspired by Kontsevich homological mirror symmetry, we would like to define the structure constants χ as counts of holomorphic disks.

Note disks do not make sense in algebraic geometry, so we go to analytic geometry.

First choice: Complex analytic geometry. It doesn't work.

counts of disks $\rightsquigarrow$ complicated curved A_∞-structures

not clear how to get well-defined numbers.

My strategy: Use non-archimedean analytic geometry.

Then we can actually have a very simple and direct definition of the structure constants χ by counting non-archimedean curves (to be explained in a second), although studying the properties of such counts requires harder work.

We equip our base field k with the trivial absolute value $|\cdot| : k \to \{0, 1\}$

$$|x| = \begin{cases} 1 & \text{for all } x \in k \setminus 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Then k becomes a non-archimedean field !

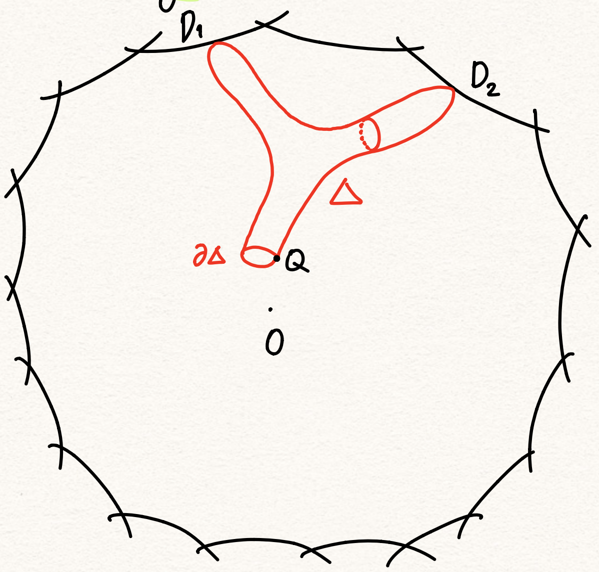
Berkovich analytification $U \rightsquigarrow U^{an}$ k-analytic space (analogous to complex analytification)

Recall: $U^{an} \overset{\text{set}}{=} \left\{ (\xi, \nu) \;\middle|\; \begin{array}{l} \xi \in U \text{ is a scheme-theoretic point} \\ \nu \text{ is an absolute value on the residue field } \kappa(\xi) \\ \text{extending the given one on } k \end{array} \right\}$

Note: $Sk(U, \mathbb{Z}) \subset U^{an}$.

Recall: Our goal is to define the structure constant $\chi(P_1, \cdots, P_n, Q, \gamma)$ by counting non-archimedean analytic disks.

Heuristically: $\chi(P_1, \cdots, P_n, Q, \gamma) = \#$ disks Δ in Y^{an} (over some field extension)

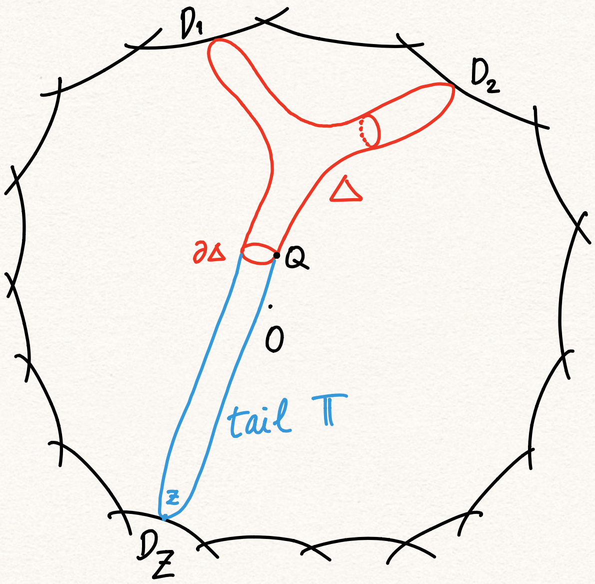


s.t.
- Δ intersects D_j with order m_j
- $\partial\Delta$ maps to $Q \in Sk(U, \mathbb{Z}) \subset U^{an}$
  a single point in non-archimedean geometry
- The derivative at $\partial\Delta$ is equal to Q
  makes sense using the theory of skeletal curves
- class of $\Delta = \gamma$
  in a limiting sense

Trouble: Unlike the situation for counting closed curves above, the space of all such disks is ∞-dimensional.

How can we extract a finite counting number from this ∞-dim space?

Idea: Reduce to finite dimension by imposing a regularity condition on the boundary $\partial\Delta$ called toric tail condition



We ask: By analytic continuation at $\partial\Delta$, our disk extends to a closed rational curve in Y^{an} s.t.

(i) The tail $\mathbb{T}$ hits divisor $D_{\mathbb{Z}}$ (opposite to Q)

(ii) The punctured tail $\mathbb{T} \setminus z \subset$ torus $\subset U$.

Adding the toric tail condition $\Longrightarrow$ finite counts of non-archimedean disks.

$\Longrightarrow$ structure constants $\chi(P_1, \cdots, P_n, Q, \gamma)$.

Theorem (KY): The sums in the multiplication rule

$$\theta_{P_1} \cdots \theta_{P_n} = \sum_{Q \in Sk(U,\mathbb{Z})} \sum_{\gamma \in NE(Y,\mathbb{Z})} \chi(P_1, \cdots, P_n, Q, \gamma)\, z^{\gamma}\, \theta_Q$$

are finite, and give the commutative associative R-algebra structure on the mirror algebra A in the Frobenius structure theorem.

3. Application to cluster algebras, comparison with Gross–Hacking–Keel–Kontsevich

Theorem (KY): Let $\mathcal{X}$ be a Fock-Goncharov skew-symmetric X-cluster variety such that

- $U := \operatorname{Spec} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is smooth
- $\mathcal{X} \longrightarrow U$ is an open immersion

Example: Double Bruhat cells in semisimple complex Lie groups.

Apply our Frobenius structure theorem to this U

$\rightsquigarrow$ obtain mirror algebra A_U (where we forget all curve classes, so it is independent of any compactification $U \subset Y$.)

We prove that A_U is isomorphic to the (combinatorially-constructed) mirror algebra of Gross-Hacking-Keel-Kontsevich.

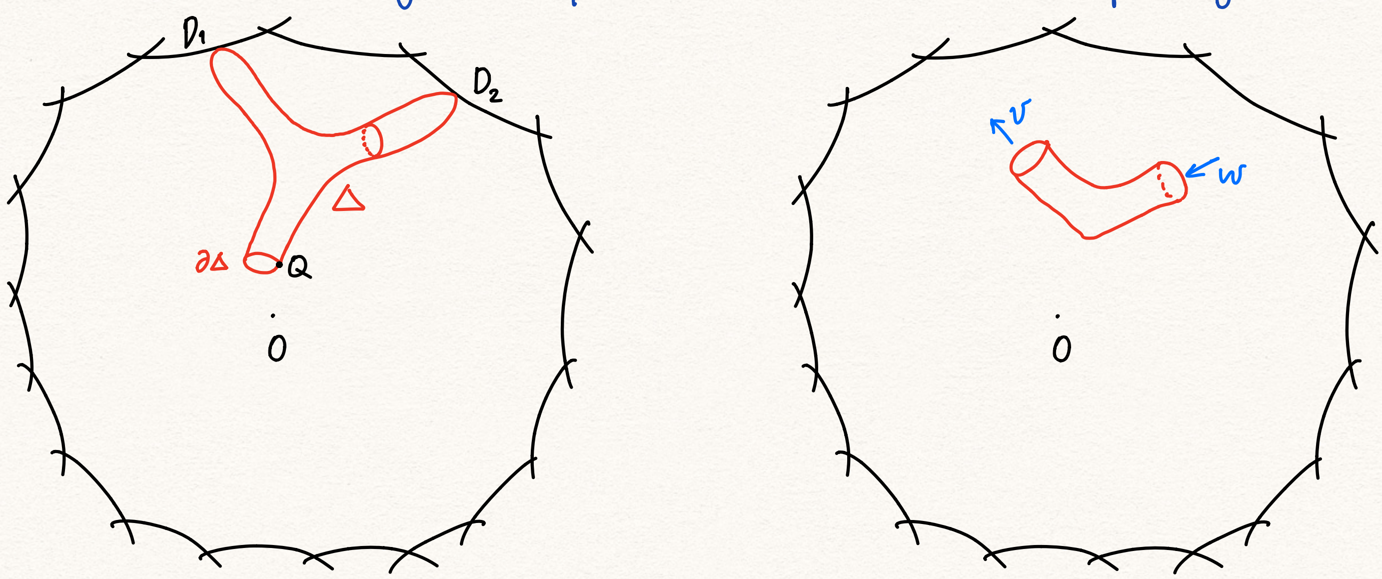
As a consequence, several conjectures of GHKK follow readily from our geometric construction, e.g.

- broken-line convexity conjecture
- independence of cluster structure conjecture
- removal of GHKK's EGM (enough global monomials) assumption

Furthermore, we obtain a much more conceptual proof of the positivity in the Laurent phenomenon for cluster algebras.

Idea of proof of the comparison theorem:

In addition to counting holomorphic disks, we count holomorphic cylinders:



Counts of cylinders $\Rightarrow$ canonical wall-crossing structure (aka scattering diagram) on the essential skeleton $Sk(U) \subset U^{an}$

Then we compare with the wall-crossing structure of Gross-Hacking-Keel-Kontsevich.

* Application to the moduli space of KSBA stable pairs

↑ Kollár - Sheperd - Barron - Alexeev

Inspired by the Frobenius structure theorem, we propose the following conjecture concerning the moduli space of polarized Calabi-Yau pairs:

Conjecture (Hacking - Keel - Yu): Any connected component Q of the moduli space of triples $(X, E = E_1 + \cdots + E_n, \Theta)$ where

- X is a smooth projective variety,
- E is a normal crossing anti-canonical divisor with a 0-stratum, every E_i smooth,
- Θ is an ample divisor not containing any 0-stratum of E

is unirational.

More precisely, note that Q has a natural embedding into the KSBA moduli space of stable pairs (via $(X, E + \varepsilon\Theta)$), we conjecture that its closure admits a finite cover by a toric variety.

Theorem (Hacking - Keel - Yu):

- We constructed the associated complete toric fan, generalizing the Gelfand - Kapranov - Zelevinsky secondary fan for reflexive polytopes.
- We extended the mirror family in the Frobenius structure theorem to the associated toric variety, generalizing the families of Kapranov - Sturmfels - Zelevinski and Alexeev in the toric case.
- In the case of del Pezzo surfaces with an anti-canonical cycle of (−1)-curves, we proved the full-conjecture.