# Quasi-isometry rigidity of lamplighter groups

Romain Tessera

CNRS, Paris Diderot

February 2, 2021

# Wreath products



**Definition**

Given two groups $F$ and $H$, the wreath product $F \wr H$ is defined as the semidirect product $\left(\bigoplus_H F\right) \rtimes H$ where $H$ acts on the direct sum by permuting the coordinates.

"Well-understood side" of wreath products:

- $Fl_{F \wr H} \approx Fl_H^{Fl_F}$ (Erschler)

# Wreath products



**Definition**

Given two groups $F$ and $H$, the wreath product $F \wr H$ is defined as the semidirect product $\left(\bigoplus_H F\right) \rtimes H$ where $H$ acts on the direct sum by permuting the coordinates.

"Well-understood side" of wreath products:

- $Fl_{F \wr H} \approx Fl_H^{Fl_F}$ (Erschler)
- Random walks: (probablity of return) (Varopoulos/Pittet-Saloff-Coste)...

# Wreath products



**Definition**

Given two groups $F$ and $H$, the wreath product $F \wr H$ is defined as the semidirect product $\left(\bigoplus_H F\right) \rtimes H$ where $H$ acts on the direct sum by permuting the coordinates.

"Well-understood side" of wreath products:

- $FI_{F \wr H} \approx FI_H^{FI_F}$ (Erschler)
- Random walks: (probablity of return) (Varopoulos/Pittet-Saloff-Coste)...
- Coarse embeddability into $L^p$-spaces (Naor-Peres).

# Wreath products

**Definition**

Given two groups $F$ and $H$, the wreath product $F \wr H$ is defined as the semidirect product $\left(\bigoplus_H F\right) \rtimes H$ where $H$ acts on the direct sum by permuting the coordinates.

"Well-understood side" of wreath products:

- $FI_{F \wr H} \approx FI_H^{FI_F}$ (Erschler)
- Random walks: (probablity of return) (Varopoulos/Pittet-Saloff-Coste)...
- Coarse embeddability into $L^p$-spaces (Naor-Peres).
- a-T-menability/ actions on CAT(0) cubical complexes: (Cornulier-Stalder-Valette/Chifan-Ioanna/Genevois).

# Lamplighters



If $F$ is finite, then $F \wr H$ is called a **Lamplighters group**.

- non stability under QI of the class of solvable groups (Erschler 00): $\mathbb{Z}_{60} \wr \mathbb{Z}$ and $\mathfrak{A}_5 \wr \mathbb{Z}$ are quasi-isometric (trivial).
- (Dymartz 15) Examples of pairs of groups that are QI but not bilpschitz equivalent (very hard).

**Problem**

*Let $F_1, F_2$ be two finite groups and $H_1, H_2$ two finitely generated groups. When are $F_1 \wr H_1$ and $F_2 \wr H_2$ quasi-isometric? When are they bilpschitz equivalent?*

# Lamplighters over general graphs

## Definition

Let $X$ be a graph and $n \geq 2$ an integer. The *lamplighter graph* $\mathcal{L}_n(X)$ is the graph

- whose vertices are the pairs $(c, x)$ with $c : V(X) \to \mathbb{Z}/n\mathbb{Z}$ is a finitely supported coloring and $x \in V(X)$ a vertex;
- and whose edges connect $(c_1, x_1)$ and $(c_2, x_2)$ either if $c_1 = c_2$ and $x_1, x_2$ are adjacent, or if $x_1 = x_2$ and $c_1, c_2$ differ only at this vertex.

## Remark

*If $X = (H, S)$ is a Cayley graph then this coincides with the Cayley graph associated to the generating subset $S \cup \delta_{1_H}$.*

# Rigidity for $H = Z$

**Theorem (Eskin-Fischer-Whyte (Annals 13'))**

*$\mathcal{L}_{n_1}(\mathbb{Z})$ and $\mathcal{L}_{n_2}(\mathbb{Z})$ are quasi-isometric if and only if $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*

# Rigidity for $H = Z$

**Theorem (Eskin-Fischer-Whyte (Annals 13'))**

*$\mathcal{L}_{n_1}(\mathbb{Z})$ and $\mathcal{L}_{n_2}(\mathbb{Z})$ are quasi-isometric if and only if $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*

- This paper is the last one of a sequence of many papers started by Farb and Mosher (Inventiones 98').

# Rigidity for $H = Z$

**Theorem (Eskin-Fischer-Whyte (Annals 13'))**

*$\mathcal{L}_{n_1}(\mathbb{Z})$ and $\mathcal{L}_{n_2}(\mathbb{Z})$ are quasi-isometric if and only if $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*

- This paper is the last one of a sequence of many papers started by Farb and Mosher (Inventiones 98').
- Extremely involved (rely on some notion of "large-scale differentiability").

# Rigidity for $H = Z$



**Theorem (Eskin-Fischer-Whyte (Annals 13'))**

*$\mathcal{L}_{n_1}(\mathbb{Z})$ and $\mathcal{L}_{n_2}(\mathbb{Z})$ are quasi-isometric if and only if $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*

- This paper is the last one of a sequence of many papers started by Farb and Mosher (Inventiones 98').
- Extremely involved (rely on some notion of "large-scale differentiability").

**Theorem (Eskin-Fischer-Whyte)**

*Given a prime $p \geq 2$, $\mathcal{L}_{p^{k_1}}(\mathbb{Z})$ and $\mathcal{L}_{p^{k_2}}(\mathbb{Z})$ are bilipschitz equivalent if and only if $k_1$ and $k_2$ are powers of $p$.*

# A general QI-rigidity result



**Theorem (Genevois-T 20)**

*Assume $H_1$ and $H_2$ are* **finitely presented** *groups and $H_1$ is* **one-ended**. *Assume $\mathcal{L}_{n_1}(H_1)$ and $\mathcal{L}_{n_2}(H_2)$ are quasi-isometric. Then $H_1$ and $H_2$ are quasi-isometric and:*

# A general QI-rigidity result



**Theorem (Genevois-T 20)**

*Assume $H_1$ and $H_2$ are* **finitely presented** *groups and $H_1$ is* **one-ended**. *Assume $\mathcal{L}_{n_1}(H_1)$ and $\mathcal{L}_{n_2}(H_2)$ are quasi-isometric. Then $H_1$ and $H_2$ are quasi-isometric and:*

1. *If $H_1$ is amenable, then $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*

# A general QI-rigidity result



**Theorem (Genevois-T 20)**

*Assume $H_1$ and $H_2$ are* **finitely presented** *groups and $H_1$ is* **one-ended**. *Assume $\mathcal{L}_{n_1}(H_1)$ and $\mathcal{L}_{n_2}(H_2)$ are quasi-isometric. Then $H_1$ and $H_2$ are quasi-isometric and:*

1. *If $H_1$ is amenable, then $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*
2. *If $H_1$ is non-amenable, then $n_1$ and $n_2$ have same prime divisors.*

# A general QI-rigidity result



**Theorem (Genevois-T 20)**

*Assume $H_1$ and $H_2$ are* **finitely presented** *groups and $H_1$ is* **one-ended**. *Assume $\mathcal{L}_{n_1}(H_1)$ and $\mathcal{L}_{n_2}(H_2)$ are quasi-isometric. Then $H_1$ and $H_2$ are quasi-isometric and:*

1. *If $H_1$ is amenable, then $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*
2. *If $H_1$ is non-amenable, then $n_1$ and $n_2$ have same prime divisors.*

- The condition is an "if and only if" in the non-amenable case, but not quite in the amenable case: not all possible QI between $H_1$ and $H_2$ are allowed!

# A general QI-rigidity result



**Theorem (Genevois-T 20)**

*Assume $H_1$ and $H_2$ are* **finitely presented** *groups and $H_1$ is* **one-ended**. *Assume $\mathcal{L}_{n_1}(H_1)$ and $\mathcal{L}_{n_2}(H_2)$ are quasi-isometric. Then $H_1$ and $H_2$ are quasi-isometric and:*

1. *If $H_1$ is amenable, then $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*
2. *If $H_1$ is non-amenable, then $n_1$ and $n_2$ have same prime divisors.*

- The condition is an "if and only if" in the non-amenable case, but not quite in the amenable case: not all possible QI between $H_1$ and $H_2$ are allowed!
- The theorem holds more generally for bounded degree graphs $X_1$ and $X_2$ (instead of $H_1$ and $H_2$), under the assumption that they are **uniformly one-ended** and **coarsely simply connected**.

# A general QI-rigidity result



**Theorem (Genevois-T 20)**

*Assume $H_1$ and $H_2$ are* **finitely presented** *groups and $H_1$ is* **one-ended**. *Assume $\mathcal{L}_{n_1}(H_1)$ and $\mathcal{L}_{n_2}(H_2)$ are quasi-isometric. Then $H_1$ and $H_2$ are quasi-isometric and:*

1. *If $H_1$ is amenable, then $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*
2. *If $H_1$ is non-amenable, then $n_1$ and $n_2$ have same prime divisors.*

- The condition is an "if and only if" in the non-amenable case, but not quite in the amenable case: not all possible QI between $H_1$ and $H_2$ are allowed!
- The theorem holds more generally for bounded degree graphs $X_1$ and $X_2$ (instead of $H_1$ and $H_2$), under the assumption that they are **uniformly one-ended** and **coarsely simply connected**.
- The theorem is true if $H_1$ is 2-ended thanks by Eskin-Fischer-Whyte. It is open if $H_1$ is infinitely ended (e.g. free groups).

# A general QI-rigidity result

**Theorem (Genevois-T 20)**

*Assume $H_1$ and $H_2$ are* **finitely presented** *groups and $H_1$ is* **one-ended**. *Assume $\mathcal{L}_{n_1}(H_1)$ and $\mathcal{L}_{n_2}(H_2)$ are quasi-isometric. Then $H_1$ and $H_2$ are quasi-isometric and:*

1. *If $H_1$ is amenable, then $n_1$ and $n_2$ are powers of a common integer $q$, i.e. $n_1 = q^{r_1}$ and $m = q^{r_2}$.*
2. *If $H_1$ is non-amenable, then $n_1$ and $n_2$ have same prime divisors.*

- The condition is an "if and only if" in the non-amenable case, but not quite in the amenable case: not all possible QI between $H_1$ and $H_2$ are allowed!
- The theorem holds more generally for bounded degree graphs $X_1$ and $X_2$ (instead of $H_1$ and $H_2$), under the assumption that they are **uniformly one-ended** and **coarsely simply connected**.
- The theorem is true if $H_1$ is 2-ended thanks by Eskin-Fischer-Whyte. It is open if $H_1$ is infinitely ended (e.g. free groups).

# Stronger version (if and only if)



All graphs are assumed to have bounded degree...

**Theorem (Genevois-T 20)**

*Let $n_1, n_2 \geq 2$ be two integers and $X_1, X_2$ two coarsely simply connected graphs. Assume that $X_1$ is uniformly one-ended.*

1. *If $X_1$ is amenable, then $\mathcal{L}_{n_1}(X_1)$ and $\mathcal{L}_{n_2}(X_2)$ are quasi-isometric if and only if $n_1$ and $n_2$ are powers of a common integer, and there exists a quasi-$(n_2/n_1)$-to-one quasi-isometry $X_1 \to X_2$.*

# Stronger version (if and only if)



All graphs are assumed to have bounded degree...

**Theorem (Genevois-T 20)**

*Let $n_1, n_2 \geq 2$ be two integers and $X_1, X_2$ two coarsely simply connected graphs. Assume that $X_1$ is uniformly one-ended.*

1. *If $X_1$ is amenable, then $\mathcal{L}_{n_1}(X_1)$ and $\mathcal{L}_{n_2}(X_2)$ are quasi-isometric if and only if $n_1$ and $n_2$ are powers of a common integer, and there exists a quasi-$(n_2/n_1)$-to-one quasi-isometry $X_1 \to X_2$.*
2. *If $H_1$ is non-amenable, then $\mathcal{L}_{n_1}(X_1)$ and $\mathcal{L}_{n_2}(X_2)$ are quasi-isometric if and only if $X_1, X_2$ are quasi-isometric and $n_1, n_2$ have the same prime divisors.*

# Amenable case

**Corollary (Lamp-rigidity)**

*Let $n_1, n_2 \geq 2$ and $X$ be an* **amenable** *a coarsely simply connected, uniformly one ended graph. Then $\mathcal{L}_{n_1}(X)$ and $\mathcal{L}_{n_2}(X)$ are biLipschitz equivalent if and only if $n_1 = n_2$.*

# Amenable case

**Corollary (Lamp-rigidity)**

*Let $n_1, n_2 \geq 2$ and $X$ be an* **amenable** *a coarsely simply connected, uniformly one ended graph. Then $\mathcal{L}_{n_1}(X)$ and $\mathcal{L}_{n_2}(X)$ are biLipschitz equivalent if and only if $n_1 = n_2$.*

**Application**

In particular, $\mathcal{L}_2(\mathbb{Z}^2)$ and $\mathcal{L}_4(\mathbb{Z}^2)$ are QI but not bilip. By contrast: $\mathcal{L}_2(\mathbb{Z})$ and $\mathcal{L}_4(\mathbb{Z})$ are bilip.

# Amenable versus non-amenable case

**Corollary (Space-rigidity)**

*Let $X$ and $Y$ be a coarsely simply connected, uniformly one ended graphs. Then for all $n \geq 2$, any quasi-isometry $\mathcal{L}_n(X) \to \mathcal{L}_n(Y)$ lies at bounded distance from a biLipschitz equivalence.*

# Amenable versus non-amenable case



**Corollary (Space-rigidity)**

*Let $X$ and $Y$ be a coarsely simply connected, uniformly one ended graphs. Then for all $n \geq 2$, any quasi-isometry $\mathcal{L}_n(X) \to \mathcal{L}_n(Y)$ lies at bounded distance from a biLipschitz equivalence.*

**Theorem (Dymarz, Peng, and Taback 15')**

*There exists two* **finitely presented** *one-ended amenable groups $H_1, H_2$ that are quasi-isometric but not bilipschitz equivalent.*

# Amenable versus non-amenable case



**Corollary (Space-rigidity)**

*Let $X$ and $Y$ be a coarsely simply connected, uniformly one ended graphs. Then for all $n \geq 2$, any quasi-isometry $\mathcal{L}_n(X) \to \mathcal{L}_n(Y)$ lies at bounded distance from a biLipschitz equivalence.*

**Theorem (Dymarz, Peng, and Taback 15')**

*There exists two* **finitely presented** *one-ended amenable groups $H_1, H_2$ that are quasi-isometric but not bilipschitz equivalent.*

**Combining these two results:**

Let $n \geq 2$. There exist two finitely presented one-ended amenable groups $H_1, H_2$ that are quasi-isometric such that $\mathcal{L}_n(H_1)$ and $\mathcal{L}_n(H_2)$ are not quasi-isometric.

# Amenable versus non-amenable case

**Theorem (Flexibility)**

*If $X$ is a* **non-amenable** *graph, and $n_1$ and $n_2$ have same prime divisors (e.g. $n_1 = 6$, $n_2 = 12$), then $\mathcal{L}_{n_1}(X)$ and $\mathcal{L}_{n_2}(X)$ are quasi-isometric.*

Note that there is no assumption on the graph here... Already surprising when $X$ is the free group!

**Corollary (Amenability criterion)**

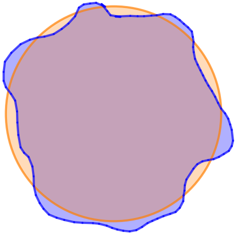
*Let $X$ be a coarsely simply connected, uniformly one ended graph. Then $X$ is amenable if and only if $\mathcal{L}_6(X)$ and $\mathcal{L}_{12}(X)$ are quasi-isometric.*

# Amenable versus non-amenable case

**Lemma**

*For every non-amenable finitely generated group $H$ and every $n \geq 2$, there exists an $n$-to-one quasi-isometry $H \to H$ at finite distance from the identity.*

Such a map never exists for amenable groups.



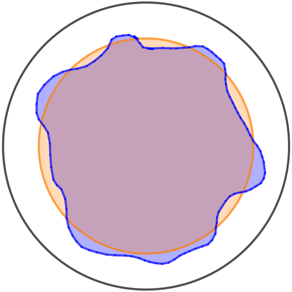
$A \subset \mathbb{Z}^2$ finite
$f^{-1}(A)$ has size $n|A|$

# Amenable versus non-amenable case

**Lemma**

*For every non-amenable finitely generated group $H$ and every $n \geq 2$, there exists an $n$-to-one quasi-isometry $H \to H$ at finite distance from the identity.*

Such a map never exists for amenable groups.



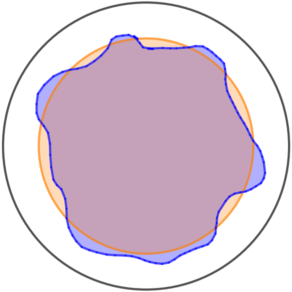
$A \subset \mathbb{Z}^2$ finite
$f^{-1}(A)$ has size $n|A|$
$f^{-1}(A) \subset A^{+C}$ for some $C \geq 0$

# Amenable versus non-amenable case

**Lemma**

*For every non-amenable finitely generated group $H$ and every $n \geq 2$, there exists an $n$-to-one quasi-isometry $H \to H$ at finite distance from the identity.*

Such a map never exists for amenable groups.



$A \subset \mathbb{Z}^2$ finite
$f^{-1}(A)$ has size $n|A|$
$f^{-1}(A) \subset A^{+C}$ for some $C \geq 0$

$$n - 1 = \frac{|f^{-1}(A)| - |A|}{|A|} \leq \frac{|A^{+C} \setminus A|}{|A|}$$

# Quasi-$\kappa$-to-one quasi-isometries

**Definition**

Let $f : X \to Y$ be a proper map between two graphs $X, Y$ and let $\kappa > 0$. Then $f$ is *quasi-$\kappa$-to-one* if there exists a constant $C > 0$ such that

$$\big|\kappa|A| - |f^{-1}(A)|\big| \leq C|\partial A|$$

for all finite subset $A \subset Y$.

# Quasi-$\kappa$-to-one quasi-isometries



**Definition**

Let $f : X \to Y$ be a proper map between two graphs $X, Y$ and let $\kappa > 0$. Then $f$ is *quasi-$\kappa$-to-one* if there exists a constant $C > 0$ such that

$$\big|\kappa|A| - |f^{-1}(A)|\big| \leq C|\partial A|$$

for all finite subset $A \subset Y$.

**Remark**

*If $X$ is non-amenable, then all quasi-isometries are quasi-$1$-to-one.*

# Quasi-$\kappa$-to-one quasi-isometries

## Definition

Let $f : X \to Y$ be a proper map between two graphs $X, Y$ and let $\kappa > 0$. Then $f$ is *quasi-$\kappa$-to-one* if there exists a constant $C > 0$ such that

$$\big|\kappa|A| - |f^{-1}(A)|\big| \leq C|\partial A|$$

for all finite subset $A \subset Y$.

## Remark

*If $X$ is non-amenable, then all quasi-isometries are quasi-1-to-one.*

## Theorem (Whyte 99')

*A quasi-isometry is quasi-one-to-one if and only if it lies at bounded distance from a biLipschitz equivalence.*

# Functoriality



**Proposition (Genevois-T)**

*Let $X, Y, Z$ be three connected graphs with bounded degree, $\kappa_1, \kappa_2 > 0$ two real numbers, and $f, h : X \to Y$ and $g : Y \to Z$ three quasi-isometries.*

(i) *If $f, h$ are at bounded distance and if $f$ is quasi-$\kappa_1$-to-one, then $h$ is also quasi-$\kappa_1$-to-one.*

# Functoriality



**Proposition (Genevois-T)**

*Let $X, Y, Z$ be three connected graphs with bounded degree, $\kappa_1, \kappa_2 > 0$ two real numbers, and $f, h : X \to Y$ and $g : Y \to Z$ three quasi-isometries.*

(i) *If $f, h$ are at bounded distance and if $f$ is quasi-$\kappa_1$-to-one, then $h$ is also quasi-$\kappa_1$-to-one.*

(ii) *If $f$ and $g$ are respectively quasi-$\kappa_1$-to-one and quasi-$\kappa_2$-to-one, then $g \circ f$ is quasi-$\kappa_1\kappa_2$-to-one.*

# Functoriality



**Proposition (Genevois-T)**

*Let $X, Y, Z$ be three connected graphs with bounded degree, $\kappa_1, \kappa_2 > 0$ two real numbers, and $f, h : X \to Y$ and $g : Y \to Z$ three quasi-isometries.*

(i) *If $f, h$ are at bounded distance and if $f$ is quasi-$\kappa_1$-to-one, then $h$ is also quasi-$\kappa_1$-to-one.*

(ii) *If $f$ and $g$ are respectively quasi-$\kappa_1$-to-one and quasi-$\kappa_2$-to-one, then $g \circ f$ is quasi-$\kappa_1\kappa_2$-to-one.*

(iii) *If $\bar{f}$ is a quasi-inverse of $f$ and if $f$ is quasi-$\kappa_1$-to-one, then $\bar{f}$ is quasi-$(1/\kappa_1)$-to-one.*

# Case where $\kappa$ is rational



Using Whyte's theorem, we can prove.

**Proposition (Genevois-T)**

*Let $m, n \geq 1$ be natural integers and $f : X \to Y$ a quasi-isometry between two graphs with bounded degree. The following statements are equivalent:*

(i) *$f$ is quasi-$(m/n)$-to-one;*

# Case where $\kappa$ is rational



Using Whyte's theorem, we can prove.

**Proposition (Genevois-T)**

*Let $m, n \geq 1$ be natural integers and $f : X \to Y$ a quasi-isometry between two graphs with bounded degree. The following statements are equivalent:*

(i) *$f$ is quasi-$(m/n)$-to-one;*

(ii) *the map $\iota \circ f \circ \pi$ is at bounded distance from a bijection, where $\pi : X \times \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow X$ is the canonical embedding and $\iota : Y \hookrightarrow Y \times \mathbb{Z}/m\mathbb{Z}$ the canonical projection.*

# Case where $\kappa$ is rational



Using Whyte's theorem, we can prove.

**Proposition (Genevois-T)**

*Let $m, n \geq 1$ be natural integers and $f : X \to Y$ a quasi-isometry between two graphs with bounded degree. The following statements are equivalent:*

(i) *$f$ is quasi-$(m/n)$-to-one;*

(ii) *the map $\iota \circ f \circ \pi$ is at bounded distance from a bijection, where $\pi : X \times \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow X$ is the canonical embedding and $\iota : Y \hookrightarrow Y \times \mathbb{Z}/m\mathbb{Z}$ the canonical projection.*

(iii) *there exist a partition $\mathcal{P}_X$ (resp. $\mathcal{P}_Y$) of $X$ (resp. of $Y$) with uniformly bounded pieces of size $m$ (resp. $n$) and a bijection $\psi : \mathcal{P}_X \to \mathcal{P}_Y$ "at bounded distance" from $f$.*

# Key rigidity results



**Theorem (Embedding result)**

*Let $X, Z$ be two graphs, $n \geq 2$ an integer, and $\rho : Z \to \mathcal{L}_n(X)$ a coarse embedding. If $Z$ is a coarsely simply connected, uniformly one ended graph, then the image of $\rho$ lies in the neighborhood of a natural copy of $X$ in $\mathcal{L}_n(X)$.*

# Key rigidity results



**Theorem (Embedding result)**

*Let $X, Z$ be two graphs, $n \geq 2$ an integer, and $\rho : Z \to \mathcal{L}_n(X)$ a coarse embedding. If $Z$ is a coarsely simply connected, uniformly one ended graph, then the image of $\rho$ lies in the neighborhood of a natural copy of $X$ in $\mathcal{L}_n(X)$.*

**Definition (Aptolic quasi-isometries)**

Let $n_1, n_2 \geq 2$ two integers and $X_1, X_2$ two graphs. A map $q : \mathcal{L}_{n_1}(X) \to \mathcal{L}_{n_2}(Y)$ is of *aptolic form* if there exist $\alpha : \mathbb{Z}_{n_1}^{(X_1)} \to \mathbb{Z}_{n_2}^{(X_2)}$ and $\beta : X_1 \to X_2$ such that $q(c, x) = (\alpha(c), \beta(x))$ for all $(c, x) \in \mathcal{L}_{n_1}(X_1)$. A quasi-isometry $\mathcal{L}_{n_1}(X) \to \mathcal{L}_{n_2}(Y)$ is *aptolic* if it is of aptolic form and if it admits a quasi-inverse of aptolic form.

# Key rigidity results



**Theorem (Embedding result)**

*Let $X, Z$ be two graphs, $n \geq 2$ an integer, and $\rho : Z \to \mathcal{L}_n(X)$ a coarse embedding. If $Z$ is a coarsely simply connected, uniformly one ended graph, then the image of $\rho$ lies in the neighborhood of a natural copy of $X$ in $\mathcal{L}_n(X)$.*

**Definition (Aptolic quasi-isometries)**

Let $n_1, n_2 \geq 2$ two integers and $X_1, X_2$ two graphs. A map $q : \mathcal{L}_{n_1}(X) \to \mathcal{L}_{n_2}(Y)$ is of *aptolic form* if there exist $\alpha : \mathbb{Z}_{n_1}^{(X_1)} \to \mathbb{Z}_{n_2}^{(X_2)}$ and $\beta : X_1 \to X_2$ such that $q(c, x) = (\alpha(c), \beta(x))$ for all $(c, x) \in \mathcal{L}_{n_1}(X_1)$. A quasi-isometry $\mathcal{L}_{n_1}(X) \to \mathcal{L}_{n_2}(Y)$ is *aptolic* if it is of aptolic form and if it admits a quasi-inverse of aptolic form.

**Theorem (Characterization of QIs)**

*Let $n_1, n_2 \geq 2$ be two integers and $X_1, X_2$ two coarsely simply connected uniformly one-ended graphs. Then every quasi-isometry $\mathcal{L}_{n_1}(X_1) \to \mathcal{L}_{n_2}(X_2)$ is at bounded distance from an aptolic quasi-isometry.*

# Key rigidity results

**Theorem (Embedding result)**

*Let $X, Z$ be two graphs, $n \geq 2$ an integer, and $\rho : Z \to \mathcal{L}_n(X)$ a coarse embedding. If $Z$ is a coarsely simply connected, uniformly one ended graph, then the image of $\rho$ lies in the neighborhood of a natural copy of $X$ in $\mathcal{L}_n(X)$.*

**Definition (Aptolic quasi-isometries)**

Let $n_1, n_2 \geq 2$ two integers and $X_1, X_2$ two graphs. A map $q : \mathcal{L}_{n_1}(X) \to \mathcal{L}_{n_2}(Y)$ is of *aptolic form* if there exist $\alpha : \mathbb{Z}_{n_1}^{(X_1)} \to \mathbb{Z}_{n_2}^{(X_2)}$ and $\beta : X_1 \to X_2$ such that $q(c, x) = (\alpha(c), \beta(x))$ for all $(c, x) \in \mathcal{L}_{n_1}(X_1)$. A quasi-isometry $\mathcal{L}_{n_1}(X) \to \mathcal{L}_{n_2}(Y)$ is *aptolic* if it is of aptolic form and if it admits a quasi-inverse of aptolic form.

**Theorem (Characterization of QIs)**

*Let $n_1, n_2 \geq 2$ be two integers and $X_1, X_2$ two coarsely simply connected uniformly one-ended graphs. Then every quasi-isometry $\mathcal{L}_{n_1}(X_1) \to \mathcal{L}_{n_2}(X_2)$ is at bounded distance from an aptolic quasi-isometry.*

The assumptions are optimal: false for $\mathbb{Z}$ (Eskin-Fisher-Whyte), or for infinitely ended groups (Genevois-T).

# Heuristic behind the embedding result



Ingredients:

1. Main observation: The Cayley graph of $F \wr H$ is foliated by left cosets of $H$, the "leaves".

# Heuristic behind the embedding result



Ingredients:

1. Main observation: The Cayley graph of $F \wr H$ is foliated by left cosets of $H$, the "leaves".
2. Leaves"diverge from one another".

# Heuristic behind the embedding result



Ingredients:

1. Main observation: The Cayley graph of $F \wr H$ is foliated by left cosets of $H$, the "leaves".
2. Leaves"diverge from one another".
3. the Cayley graph of $F \wr H$ may be thought of as a "graph of leaves": we expect a **suitable cover** $\mathcal{C}$ to behave like a "tree of leaves".

# Heuristic behind the embedding result



Ingredients:

1. Main observation: The Cayley graph of $F \wr H$ is foliated by left cosets of $H$, the "leaves".
2. Leaves"diverge from one another".
3. the Cayley graph of $F \wr H$ may be thought of as a "graph of leaves": we expect a **suitable cover** $\mathcal{C}$ to behave like a "tree of leaves".
4. Strategy: use the simple connectedness of $Z$ to **lift** its embedding to $F \wr H$ to an embedding to $\mathcal{C}$ and show there that it must lie close to a leaf (using $Z$ is one-ended!).

# Heuristic behind the embedding result

Ingredients:

1. Main observation: The Cayley graph of $F \wr H$ is foliated by left cosets of $H$, the "leaves".
2. Leaves"diverge from one another".
3. the Cayley graph of $F \wr H$ may be thought of as a "graph of leaves": we expect a **suitable cover** $\mathcal{C}$ to behave like a "tree of leaves".
4. Strategy: use the simple connectedness of $Z$ to **lift** its embedding to $F \wr H$ to an embedding to $\mathcal{C}$ and show there that it must lie close to a leaf (using $Z$ is one-ended!).
5. Problem: $\mathcal{C}$ does not really have a tree-structure. Instead it has a **median space structure**, whose hyperplanes are unbounded, but **project to bounded subsets** in $F \wr H$.

# Main steps of the proof (1)



**Presentation of the lamplighter:**

Case $X = H$ is a group. The wreath product $F \wr H$ admits

$$\langle H, F_h \ (h \in H) \mid [F_1, F_h] = 1 \ (h \in H), \ gF_hg^{-1} = F_{gh} \ (g, h \in H) \rangle$$

as a relative presentation, where each $F_h$ is a copy of $F$.

# Main steps of the proof (1)

**Presentation of the lamplighter:**

Case $X = H$ is a group. The wreath product $F \wr H$ admits

$$\langle H, F_h \ (h \in H) \mid [F_1, F_h] = 1 \ (h \in H),\ gF_hg^{-1} = F_{gh} \ (g, h \in H)\rangle$$

as a relative presentation, where each $F_h$ is a copy of $F$.

**Truncated presentation:**

Given a finite subset $S \subset H$, we define a new group $F\square_S H$ from the truncated presentation

$$\langle H, F_h \ (h \in H) \mid [F_1, F_h] = 1 \ (h \in S),\ gF_hg^{-1} = F_{gh} \ (g, h \in H)\rangle.$$

# Main steps of the proof (1)



## Presentation of the lamplighter:

Case $X = H$ is a group. The wreath product $F \wr H$ admits

$$\langle H, F_h\ (h \in H) \mid [F_1, F_h] = 1\ (h \in H),\ gF_hg^{-1} = F_{gh}\ (g, h \in H) \rangle$$

as a relative presentation, where each $F_h$ is a copy of $F$.

## Truncated presentation:

Given a finite subset $S \subset H$, we define a new group $F \square_S H$ from the truncated presentation

$$\langle H, F_h\ (h \in H) \mid [F_1, F_h] = 1\ (h \in S),\ gF_hg^{-1} = F_{gh}\ (g, h \in H) \rangle.$$

## Key observation:

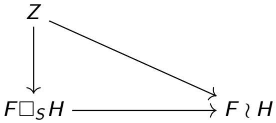
$F \square_S H$ decomposes as the semi-direct product

$$\langle F_h\ (h \in H) \mid [F_g, F_h] = 1\ (g^{-1}h \in S) \rangle \rtimes H = \Gamma F \rtimes H$$

# Main steps of the proof (2)

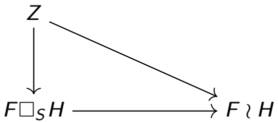


1. **(Lifting the coarse embedding)** since $Z$ is **coarsely simply connected**, its coarse embedding to $F \wr H$ lifts to a coarse embedding to $F \square_S H$ for some large enough finite $S$.



# Main steps of the proof (2)

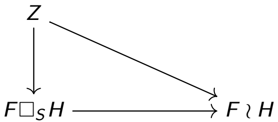
1. **(Lifting the coarse embedding)** since $Z$ is **coarsely simply connected**, its coarse embedding to $F \wr H$ lifts to a coarse embedding to $F\square_S H$ for some large enough finite $S$.




2. **(Median geometry)** $F\square_S H$ acts on a median space (1-skeleton of a CAT(0)-cubical complex).

# Main steps of the proof (2)

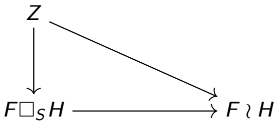
1. **(Lifting the coarse embedding)** since $Z$ is **coarsely simply connected**, its coarse embedding to $F \wr H$ lifts to a coarse embedding to $F \square_S H$ for some large enough finite $S$.




2. **(Median geometry)** $F \square_S H$ acts on a median space (1-skeleton of a CAT(0)-cubical complex).
3. **(Relative tree-like structure)** Hyperplanes project to bounded subsets in $F \wr H$.

# Main steps of the proof (2)

1. **(Lifting the coarse embedding)** since $Z$ is **coarsely simply connected**, its coarse embedding to $F \wr H$ lifts to a coarse embedding to $F \square_S H$ for some large enough finite $S$.




2. **(Median geometry)** $F \square_S H$ acts on a median space (1-skeleton of a CAT(0)-cubical complex).
3. **(Relative tree-like structure)** Hyperplanes project to bounded subsets in $F \wr H$.
4. Use that $Z$ is **uniformly one-ended** to prove the theorem.

# Towards superrigidity



Exploiting a recent result of Martínez Pedrosa and Sánchez Saldaña, we obtain

**Theorem (Algebraic constraints on a group QI to a lamplighter)**

*Let $F$ be a non-trivial finite group, $H$ a finitely presented one-ended group, and $G$ a finitely generated group. If $G$ is quasi-isometric to $F \wr H$, then there exist finitely many subgroups $H_1, \ldots, H_n \leq G$ such that:*

- *$H_1, \ldots, H_n$ are all quasi-isometric to $H$;*

# Towards superrigidity



Exploiting a recent result of Martínez Pedrosa and Sánchez Saldaña, we obtain

**Theorem (Algebraic constraints on a group QI to a lamplighter)**

*Let $F$ be a non-trivial finite group, $H$ a finitely presented one-ended group, and $G$ a finitely generated group. If $G$ is quasi-isometric to $F \wr H$, then there exist finitely many subgroups $H_1, \ldots, H_n \leq G$ such that:*

- *$H_1, \ldots, H_n$ are all quasi-isometric to $H$;*
- *the collection $\{H_1, \ldots, H_n\}$ is almost malnormal;*

# Towards superrigidity



Exploiting a recent result of Martínez Pedrosa and Sánchez Saldaña, we obtain

**Theorem (Algebraic constraints on a group QI to a lamplighter)**

*Let $F$ be a non-trivial finite group, $H$ a finitely presented one-ended group, and $G$ a finitely generated group. If $G$ is quasi-isometric to $F \wr H$, then there exist finitely many subgroups $H_1, \ldots, H_n \leq G$ such that:*

- *$H_1, \ldots, H_n$ are all quasi-isometric to $H$;*
- *the collection $\{H_1, \ldots, H_n\}$ is almost malnormal;*
- *for every finitely presented one-ended subgroup $K \leq G$, there exist $g \in G$ and $1 \leq i \leq n$ such that $K \leq gH_ig^{-1}$.*

# Towards superrigidity

Exploiting a recent result of Martínez Pedrosa and Sánchez Saldaña, we obtain

**Theorem (Algebraic constraints on a group QI to a lamplighter)**

*Let $F$ be a non-trivial finite group, $H$ a finitely presented one-ended group, and $G$ a finitely generated group. If $G$ is quasi-isometric to $F \wr H$, then there exist finitely many subgroups $H_1, \dots, H_n \leq G$ such that:*

- *$H_1, \dots, H_n$ are all quasi-isometric to $H$;*
- *the collection $\{H_1, \dots, H_n\}$ is almost malnormal;*
- *for every finitely presented one-ended subgroup $K \leq G$, there exist $g \in G$ and $1 \leq i \leq n$ such that $K \leq gH_ig^{-1}$.*

**Corollary (Rigidity of permutational lamplighters)**

*Let $F_1, F_2$ two non-trivial finite groups and $H_1, H_2$ be finitely presented one-ended groups respectively acting on two sets $X_1, X_2$ with finitely many orbits. Assume that $F_1 \wr_{X_1} H_1$ and $F_2 \wr_{X_2} H_2$ are quasi-isometric. If $H_1$ acts on $X_1$ with finite stabilisers, then $H_2$ also acts on $X_2$ with finite stabilisers.*