Introduction to Global Sensibility Analysis

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Part II

Outline

Introduction

Limits of variance based SA in the general framework

An alternative, the Shapley effects

Aggregated Shapley effects

Application: snow avalanche modeling

Introduction

In this talk, we consider

$$\mathcal{M}: \begin{cases} \mathcal{X} = \mathcal{X}_1 \times \ldots \mathcal{X}_d & \rightarrow \quad \mathcal{Y} \\ \boldsymbol{x} = (x_1, \ldots, x_d) & \mapsto \quad y = \mathcal{M}(\boldsymbol{x}) \end{cases}$$

with

- $\mathcal{M}$: mathematical or numerical model,
- $\boldsymbol{x}$: uncertain input parameters,
- y : output.

We model the uncertain input parameters by a probability distribution P on $\mathcal{X}$ and get

$$Y = \mathcal{M}(X_1, \ldots, X_d)$$

with the vector $\mathbf{X} = (X_1, \ldots, X_d)$ distributed as P.

Introduction

Independent framework: $P(d\boldsymbol{x}) = P_1(dx_1)\ldots P_d(dx_d)$

Why is the independent framework not always the right one?

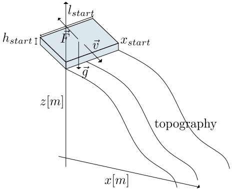
In the following, we consider an application to long-term avalanche hazard assessment. The model under consideration is:

- a snow avalanche model, joint work with INRAE (Grenoble, FRANCE).

Snow avalanche modeling

Model based on depth-averaged Saint-Venant equations (see [Heredia et al., 2020] for more details)

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial hv}{\partial x} &= 0 \\ \frac{\partial hv}{\partial t} + \frac{\partial}{\partial x}\left(hv^2 + \frac{h^2}{2}\right) &= h\left(g \sin\theta - \mathrm{F}\right) \end{aligned}$$



with $v = \|\vec{\mathbf{v}}\|$ the flow velocity, h the flow depth, θ the local angle, t the time, g the gravity constant and $\mathrm{F} = \|\vec{\mathbf{F}}\|$ a frictional force. The model uses the Voellmy frictional force $\mathrm{F} = \mu g cos\theta + g/(\xi h)v^2$, where μ and ξ are friction parameters.

The equations are solved with a finite volume scheme [Naaim, 1998]. The topography is the one of a path located in Bessans, France.

In the following, we consider two scenarii. Let us present the first scenario as an introductory example.

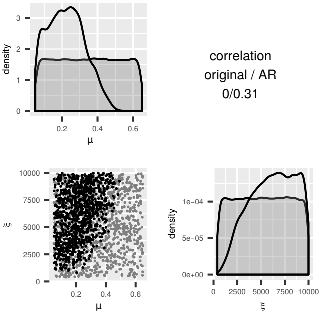
Input	Description	Distribution
μ	Static friction coefficient	$\mathcal{U}[0.05, 0.65]$
ξ	Turbulent friction [m/s^2]	$\mathcal{U}[400, 10000]$
$\mathrm{l}_{\mathrm{start}}$	Length of the release zone [m]	$\mathcal{U}[5, 300]$
$\mathrm{h}_{\mathrm{start}}$	Mean snow depth in the release zone [m]	$\mathcal{U}[0.05, 3]$
$\mathrm{x}_{\mathrm{start}}$	Release abscissa [m]	$\mathcal{U}[0, 1600]$

Let's $\mathrm{vol}_{\mathrm{start}} = l_{start} \times h_{start} \times 72.3/\cos(35^\circ)$ instead of $\mathrm{h}_{\mathrm{start}}$ and $\mathrm{l}_{\mathrm{start}}$.

AR rules:

- avalanche simulation is flowing in $[1600m, 2412m]$,
- $vol > 7000m^3$,
- runout distance $< 2500m$ (end of the path).

From $n_0 = 100\,000$ initial runs, we keep $n_1 = 6152$ constrained ones.



Outline

Variance based SA in the general framework

We still consider $\mathcal{M}$: $\left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R} \\ \boldsymbol{x} = (x_1, \dots, x_d) & \mapsto & y = \mathcal{M}(\boldsymbol{x}) \end{array} \right.$

Uncertain parameters are no longer assumed independent, thus $P(d\boldsymbol{x})$ is not necessarily equal to $P_1(dx_1) \dots P_d(dx_d)$. We have $F_{\mathbf{X}}(\boldsymbol{x}) = C\left(F_{X_1}(x_1), \dots, F_{X_d}(x_d)\right)$ (Sklar's Theorem) with $F_{X_i}(\cdot)$ and $F_{\mathbf{X}}(\cdot)$ the cdf of X_i, $\boldsymbol{X}$. If the F_{X_i} are continuous, then the copula C is unique.

We still define, for any $i \in \{1, \dots, d\}$: $S_i = \dfrac{V\left[\mathbb{E}\left[Y|X_i\right]\right]}{V\left[Y\right]}$ and

$S_i^{\text{tot}} = \dfrac{\mathbb{E}\left[V\left[Y|\boldsymbol{X}_{-i}\right]\right]}{V\left[Y\right]}$.

However, nice properties due to orthogonality are lost.

Outline

An alternative, the Shapley effects

Let $\mathcal{D} = \{1, \ldots, d\}$. Let team $u \subseteq \mathcal{D}$ create value $\mathbf{val}(u)$. Total value is $\mathbf{val}(\mathcal{D})$. We attribute ϕ_i of this to $i \in \mathcal{D}$.

Shapley axioms [Shapley, 1953]

- Efficiency $\sum_{i=1}^{d} \phi_i = \mathbf{val}(\mathcal{D})$
- Dummy If $\mathbf{val}(u \cup \{i\}) = \mathbf{val}(u)$ for all $u \subseteq \mathcal{D}$, then $\phi_i = 0$
- Symmetry If $\mathbf{val}(u \cup \{i\}) = \mathbf{val}(u \cup \{j\})$ for all $u \cap \{i, j\} = \emptyset$, then $\phi_i = \phi_j$
- Additivity If games **val**, **val'** have values ϕ, ϕ', then **val** + **val'** has value $\phi + \phi'$

Unique solution

$$\phi_i = \frac{1}{d} \sum_{u \subseteq -\{i\}} \binom{d-1}{|u|}^{-1} \left(\mathbf{val}(u+i) - \mathbf{val}(u)\right)$$

Let $X_1, \ldots, X_d$ be the team members trying to explain the variability of $\mathcal{M}$. The value of any $u \in \mathcal{D}$ is how much can be explained by $\boldsymbol{X}_u$.

We choose $\mathbf{val}(u) = \dfrac{V\left[\mathbb{E}\left[Y|\boldsymbol{X}_u\right]\right]}{V\left[Y\right]}$ which leads to the definition of Shapley effects [Owen, 2014]:

$$\phi_i = \frac{1}{d} \sum_{u \subseteq -\{i\}} \binom{d-1}{|u|}^{-1} \left(\frac{V\left[\mathbb{E}\left[Y|\boldsymbol{X}_u, X_i\right]\right]}{V\left[Y\right]} - \frac{V\left[\mathbb{E}\left[Y|\boldsymbol{X}_u\right]\right]}{V\left[Y\right]} \right)$$

It is equivalent to consider to choose $\widetilde{\mathbf{val}}(u) = \dfrac{\mathbb{E}\left[V\left[Y|\boldsymbol{X}_{-u}\right]\right]}{V\left[Y\right]}$ [Song et al., 2016].

Main properties

Independent framework: $\forall\, i = 1, \ldots, d$, $\phi_i = \sum_{\mathbf{u}: i \in \mathbf{u}} \frac{1}{|\mathbf{u}|} S_{\mathbf{u}}$

We also have: $\forall\, i = 1, \ldots, d$, $0 \leq S_i \leq \phi_i \leq S_i^{\text{tot}} \leq 1$ and $\sum_{i=1}^{d} \phi_i = 1$.

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Dependent framework:

In this framework, we still have $0 \leq \phi_i \leq 1$ and $\sum_{i=1}^{d} \phi_i = 1$

We do not necessarily have $S_i \leq \phi_i \leq S_i^{\text{tot}}$

The Shapley allocation rule is based on an equitable principle, which ensures that $\phi_i \approx 0 \Rightarrow X_i$ has no significant contribution to $\text{Var}[Y]$, neither by its interactions nor by its dependencies with other inputs.

What happens on simple models?

Ex. 1: Gaussian framework, affine model, $d = 2$

[Owen and Prieur, 2017, Iooss and Prieur, 2019]

$\mathbf{X} \sim \mathcal{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $Y = \beta_0 + \boldsymbol{\beta}^{\mathsf{T}}\mathbf{X}$, with

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \rho \in [-1, 1], \sigma_i > 0.$$

We have $\sigma^2 = V[Y] = \beta_1^2\sigma_1^2 + 2\rho\beta_1\beta_2\sigma_1\sigma_2 + \beta_2^2\sigma_2^2$. Then

$$\begin{cases} \sigma^2\phi_1 = \beta_1^2\sigma_1^2(1-\frac{\rho^2}{2}) + \rho\beta_1\beta_2\sigma_1\sigma_2 + \beta_2^2\sigma_2^2\frac{\rho^2}{2}, \\ \sigma^2 S_1 = \beta_1^2\sigma_1^2 + 2\rho\beta_1\beta_2\sigma_1\sigma_2 + \rho^2\beta_2^2\sigma_2^2 \text{ and } \sigma^2 S_1^{\text{tot}} = \beta_1^2\sigma_1^2(1-\rho^2). \end{cases}$$

(i) $\phi_i \le S_i^{\text{tot}} \Leftrightarrow$ (ii) $S_i \le \phi_i \Leftrightarrow$ (iii) $\rho\,(\rho\frac{\beta_1^2\sigma_1^2 + \beta_2^2\sigma_2^2}{2} + \beta_1\beta_2\sigma_1\sigma_2) \le 0$.

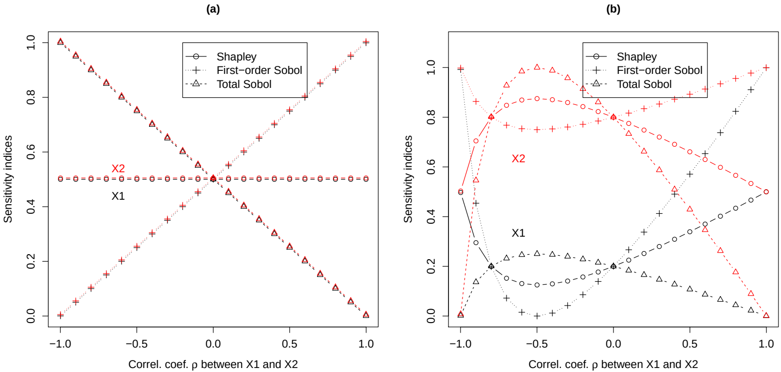

Figure: Sensitivity indices on the linear model ($\beta_1 = 1$, $\beta_2 = 1$) with two Gaussian inputs. (a): $(\sigma_1, \sigma_2) = (1, 1)$. (b): $(\sigma_1, \sigma_2) = (1, 2)$.

Ex. 2: correlated input not included in the model

$Y = \mathcal{M}(X_1, X_2) = X_1$ with (X_1, X_2) two dependent standard Gaussian variables with a correlation coefficient ρ.

Shapley effects: $\phi_1 = 1 - \frac{\rho^2}{2}$ and $\phi_2 = \frac{\rho^2}{2}$.

If ρ is close to zero, ϕ_2 is small and X_2 can be fixed without changing the output variance (Factor Fixing Setting).

Sobol' indices: $S_1 = 1\,,\ S_1^{\text{tot}} = 1 - \rho^2\,,\ S_2 = \rho^2\,,\ S_2^{\text{tot}} = 0.$

X_2 is only important because of its correlation with X_1. One should be able to evaluate the uncertainty of Y accurately by only accounting for the uncertainty in X_1.

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For a black box model, if $S_i^{\text{tot}} = 0$, the model output is a measurable function of $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d)$ only. Then, if then $S_i > 0$, X_i is correlated to $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d)$.

What about algorithms?

Algorithms to compute Shapley effects [Castro et al., 2009] are based on the value function $u \mapsto \frac{\mathbb{E}[V[Y|\mathbf{X}_{-u}]]}{V[Y]}$. Note that

$$\phi_i = \frac{1}{d\,!} \sum_{\pi \in \Pi(\{1,\ldots,d\})} \left(\widetilde{\mathbf{val}}(P_i(\pi) \cup \{i\})) - \widetilde{\mathbf{val}}(P_i(\pi)) \right)$$

with $\Pi(\{1,\ldots,d\})$ the set of all possible permutations of the inputs and for a permutation $\pi \in \Pi(\{1,\ldots,d\})$, the set $P_i(\pi)$ is defined as the inputs that precede input i in π.

Exact permutation algo. (moderate d) all possible permutations are covered.

Random permutation algo. ($d >> 1$) it randomly sample permutations of the inputs.

In [Song et al., 2016], $\widetilde{\mathbf{val}}(u) \to \widehat{\widetilde{\mathbf{val}}}(u)$.

For each iteration of the loop on the inputs' permutations, the expectation of a conditional variance must be computed.

The cost C of these algorithms is the following:

$$C = N_v + m(d-1)N_0 N_i$$

with N_v the sample size for the variance computation, N_0 the outer loop size for the expectation, N_i the inner loop size for the conditional variance and m the number of permutations according to the selected method.

Bootstrap confidence intervals can be computed. A costly model can be replaced by a metamodel. [Iooss and Prieur, 2019, Benoumechiara and Elie-Dit-Cosaque, 2019]

Outline

If output is multivariate or the discretization of a functional output $\mathbf{Y} = (Y_1, \ldots, Y_p)$, we define aggregated Shapley effects as:

$$\forall\, 1 \leq j \leq p\,,\ \forall\, 1 \leq i \leq d\,,\ \phi_i^{\text{agg}} = \frac{\sum_{j=1}^{p} V[Y_j]\phi_i^j}{\sum_{j=1}^{p} V[Y_j]}$$

with ϕ_i^j defined as the Shapley effect of Y_j associated to input X_i [Heredia et al., 2020] (see also [Lamboni et al., 2011]).

Proposition [Heredia et al., 2020, Prop. 2.1]

The set of aggregated Shapley effects $\left(\phi_i^{\text{agg}}\,,\ i \in \{1, \ldots, d\}\right)$ *correspond to the set of Shapley values with characteristic function:*

$$u \subseteq \{1, \ldots, d\} \mapsto \boldsymbol{val}(u) = \frac{\sum_{j=1}^{p} V[Y_j]\boldsymbol{val}_j(u)}{\sum_{j=1}^{p} V[Y_j]}$$

with $\boldsymbol{val}_j(u) = \dfrac{V\left[\mathbb{E}\left[Y_j|\mathbf{X}_u\right]\right]}{V\left[Y_j\right]}$ *or* $\boldsymbol{val}_j(u) = \dfrac{\mathbb{E}\left[V\left[Y_j|\mathbf{X}_{-u}\right]\right]}{V\left[Y_j\right]}$.

Estimation procedure

It is possible to plug algorithms presented in [Castro et al., 2009, Song et al., 2016] in the estimation of aggregated Shapley effects. Those algorithms require the ability to sample from the distribution of $\mathbf{X}_u \mid \mathbf{X}_{-u}$, $\forall u \subsetneq \{1, \ldots, d\}$.

We rather present here a procedure based on nearest neighbors [Broto et al., 2020, Heredia et al., 2020].

As already mentioned, a first step is to estimate:

$$V[Y_j]\,\mathbf{val}_j(u) = \mathbb{E}[V[Y_j|\mathbf{X}_{-u}]] = \mathbb{E}\left[V^j_{-u}\right]$$

for all $u \subseteq \{1, \ldots, d\}$, with $-u = \{1, \ldots, d\} \setminus u$.

Algorithm introduced in [Broto et al., 2020] depends on:

- a n-sample $\left(\mathbf{x}^k, \mathbf{y}^k\right)_{1 \leq k \leq n}$ (*given data*),
- N_{tot} the estimation cost,
- $(N_u)_{u \subsetneq \{1,\ldots,d\}}$ integers such that $\sum_{u \subsetneq \{1,\ldots,d\}} N_u = N_{\text{tot}}$,
- N_I number of neighbors.

Let us describe the three main steps of the algorithm we propose to estimate aggregated Shapley effects.

Step 1

Let $u \subsetneq \{1, \ldots, d\}$. Let $(s_\ell)_{1 \leq \ell \leq N_u}$ a sample of uniformly distributed integers in $[1, n]$. Then, for any $1 \leq j \leq p$, compute:

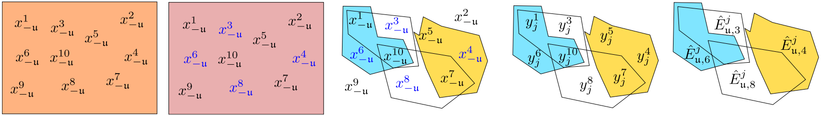
$$\widehat{V}^j{}_{-u,s_\ell} = \frac{1}{N_I - 1} \sum_{k : \mathbf{x}^k_{-\mathbf{u}} \in \mathcal{B}_{-u,\ell}} \left(y_j^k - \bar{y}_{s_\ell} \right)^2 \text{ with } \bar{y}_{s_\ell} = \frac{1}{N_I} \sum_{v : \mathbf{x}^k_{-u} \in \mathcal{B}_{u,\ell}} y_j^k \, .$$

with $\mathcal{B}_{-u,\ell}$ the set of N_I closest neighbors of $x_{-u}^{s_\ell}$.

$$\widehat{V}^{j}{}_{-u,s_\ell} = \frac{1}{N_I - 1} \sum_{k:\mathbf{x}^k_{-\mathbf{u}} \in \mathcal{B}_{-u,\ell}} \left(y_j^k - \bar{y}_{s_\ell} \right)^2 \text{ with } \bar{y}_{s_\ell} = \frac{1}{N_I} \sum_{v:\mathbf{x}^k_{-u} \in \mathcal{B}_{u,\ell}} y_j^k$$

with $\mathcal{B}_{-u,\ell}$ the set of N_I closest neighbors of $x_{-u}^{s_\ell}$.

Illustration with $n = 10$, $N_{\mathbf{u}} = 4$ and $N_I = 3$



@M.B.Heredia

Step 2

2.1 Compute, for all $u \subsetneq \{1, \dots, d\}$,

$$V[Y_j]\,\widehat{\mathbf{val}_j(u)} = \frac{1}{N_u}\sum_{\ell=1}^{N_u} \widehat{V}^j{}_{-u,s_\ell}.$$

2.2 Compute a bootstrap sample $\left(V[Y_j]\,\widehat{\mathbf{val}_j(u)}\right)^{(b)}$, $1 \leq b \leq B$ by sampling uniformly with replacement in $\widehat{V}^j{}_{-u,s_\ell}$.

2.3 Compute a bootstrap sample $\hat{\sigma}_j^{2,(b)}$, $1 \leq b \leq B$ of the empirical variance

$$\hat{\sigma}_j^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(y_j{}^k - \frac{1}{n}\sum_{k=1}^{n} y_j{}^k\right)^2$$

by sampling uniformly with replacement in $\left(y_j{}^k\right)_{1 \leq k \leq n}$.

Step 3

3.1 For any $i \in \{1, \ldots, d\}$, compute $\widehat{\phi_i{}^{\text{agg}}}$ (and a bootstrap sample $\widehat{\phi_i{}^{\text{agg}}}^{(b)}$, $1 \leq b \leq B$) as:

$$\frac{\sum_{j=1}^{p} \sum_{u \subseteq -i} \binom{d-1}{|u|}^{-1} \left(V[Y_j]\, \widehat{\mathbf{val}_j(u \cup \{i\})} - V[Y_j]\, \widehat{\mathbf{val}_j(u)} \right)}{d \sum_{j=1}^{p} \hat{\sigma}_j^2}.$$

3.2 Compute bootstrap confidence intervals from the sample $\widehat{\phi_i{}^{\text{agg}}}^{(b)}$, $1 \leq b \leq B$.

How to choose N_I, N_{tot} and $N_{\mathbf{u}}$?

Following [Song et al., 2016], we fix $N_I = 3$.

The cost of the estimation procedure is related to the number of nearest neighbors we seek for. It is proportional to $\sum_{\emptyset \subsetneq u \subsetneq \{1,\dots,d\}} N_u$. Choosing $N_u = n$ for all u would lead to a cost $N_{\text{tot}} = n(2^d - 2)$ which explodes exponentially with the dimension.

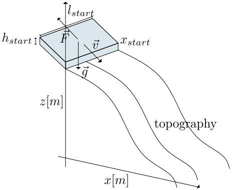
The authors in [Broto et al., 2020] suggest to choose $N_u = N_u^* = \text{Round}\left(N_{\text{tot}} \binom{d}{|u|}^{-1} (d-1)^{-1}\right)$, thus taking into account the weights in the definition of Shapley effects.

Outline

Application: snow avalanche modeling

Model based on depth-averaged Saint-Venant equations (see [Heredia et al., 2020] for more details)

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial hv}{\partial x} &= 0 \\ \frac{\partial hv}{\partial t} + \frac{\partial}{\partial x}\left(hv^2 + \frac{h^2}{2}\right) &= h\left(g \sin\theta - \mathsf{F}\right) \end{aligned}$$



with $v = \|\vec{\mathbf{v}}\|$ the flow velocity, h the flow depth, θ the local angle, t the time, g the gravity constant and $\mathsf{F} = \|\vec{\mathbf{F}}\|$ a frictional force. The model uses the Voellmy frictional force $\mathsf{F} = \mu g cos\theta + g/(\xi h)v^2$, where μ and ξ are friction parameters.

The equations are solved with a finite volumes scheme [Naaim, 1998]. The topography is the one of a path located in Bessans, France.

Objective: better understanding the numerical model.

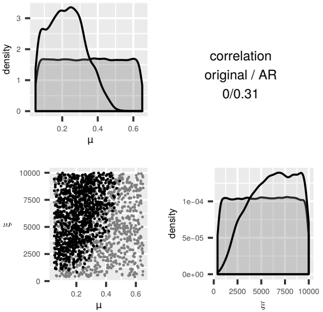
Input	Description	Distribution
μ	Static friction coefficient	$\mathcal{U}[0.05, 0.65]$
ξ	Turbulent friction [m/s^2]	$\mathcal{U}[400, 10000]$
l_{start}	Length of the release zone [m]	$\mathcal{U}[5, 300]$
h_{start}	Mean snow depth in the release zone [m]	$\mathcal{U}[0.05, 3]$
x_{start}	Release abscissa [m]	$\mathcal{U}[0, 1600]$

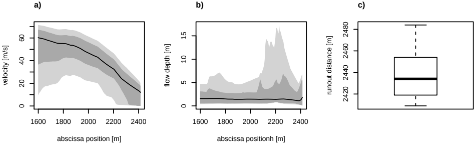
Let's $\mathrm{vol}_{start} = l_{start} \times h_{start} \times 72.3/\cos(35°)$ instead of h_{start} and l_{start}.

AR rules:

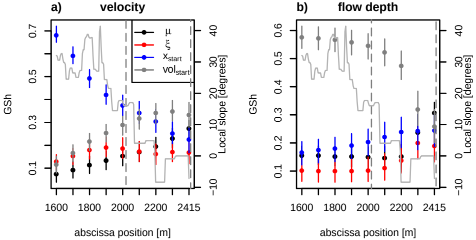
- avalanche simulation is flowing in $[1600m, 2412m]$,
- $vol > 7000m^3$,
- runout distance $< 2500m$ (end of the path).

From $n_0 = 100\,000$ initial runs, we keep $n_1 = 6152$ constrained ones.





Aggregated Shapley effects of velocity and flow depth curves calculated over space intervals $[x, 2412m]$ where $x \in \{1600m, 1700m, \ldots, 2412m\}$



We have $n = 6152$, $N_{\text{tot}} = 2002$, $B = 500$. Effects are estimated using the first (2, *resp.* 4) fPCs [Yao et al., 2005, Ramsay and Silverman, 2005] explaining more than 95% of the variance. Local slope is drawn with a gray line. A gray dotted rectangle is drawn at $[2017m, 2412m]$ where avalanche return periods vary from 10 to 10 000 years.

Objective: long-term avalanche hazard assessment to address related risk for buildings and people inside.

Input	Distribution
$x_{nstart} = \frac{x_{start}}{1600}$	Beta$(1.38, 2.49)$
$h_{start} \mid x_{nstart}$	Gamma$\left(\frac{1}{0.45^2}(1.52 + 0.03x_{nstart})^2, \frac{1}{0.45^2}(1.52 + 0.03x_{nstart})\right)$
l_{start}	$31.25{+}87.5h_{start}$
$\mu \mid h_{start}, x_{nstart}$	$\mathcal{N}(0.449 - 0.013x_{nstart} + 0.025h_{start}, 0.11^2)$

AR rules:

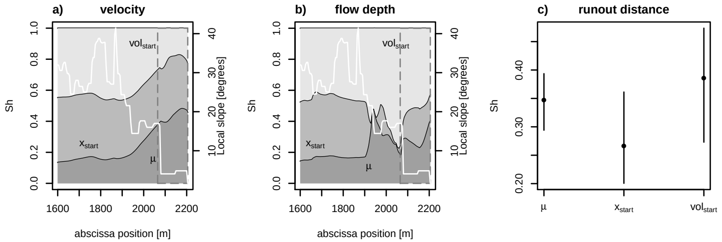
- avalanche simulation flowing in $[1600\text{m}, 2204\text{m}]$ (return periods from 10 to 300 years),
- avalanche volume $\geq 7\,000$ m^3,
- $\mu \leq 0.39$ (dry snow avalanches).

From $n_0 = 100\,000$ initial runs, we keep $n_2 = 1284$ constrained ones.

Input distribitions in the table are obtained from a Bayesian inference [Eckert et al., 2010].

ξ is fixed to 1300.

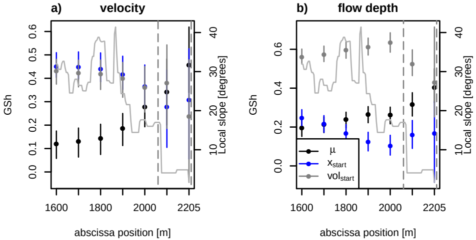
We have $l_{start} = 31.25 + 87.5h_{start}$. Once more we consider vol$_{start}$ instead of l_{start} and h_{start}.



Ubiquitous Shapley effects of velocity (a), flow depth (b) and runout distance (c). Shapley effects. Shapley effects are estimated with $n = 1284$ and $N_{\text{tot}} = 800$. The local slope is displayed with a white line. A gray dotted rectangle shows the interval $[2064m, 2204m]$ where return periods vary from 10 to 300 years. The bootstrap sample size is fixed to $B = 500$.

We note that:

- ubiquitous effects show fluctuations corresponding to changes in local slope;
- concerning runout distance, all inputs appear as relevant.



Aggregated Shapley effects of velocity and flow depth curves calculated over space intervals $[x, 2204]$ where $x \in \{1600, 1700, \ldots, 2204\}$ and using the first fPCs which have 95% of output variance. Shapley effects are estimated with $n = 1284$ and $N_{\text{tot}} = 800$. The local slope is displayed with a gray line. A gray dotted rectangle is displayed at $[2017m, 2204m]$ where return periods vary from 10 to 300 years. The bootstrap sample size is fixed to $B = 500$.

In summary,

- it is fundamental to have a good approximation of the released volume and abscissa for velocity forecasting, while for flow depth forecasting, a good approximation of released volume is desirable;
- nevertheless, none of the other inputs are negligible.

To outperform the estimation accuracy at the end of the path generating a larger initial sample of avalanches is possible, but the computational burden is prohibitive.

Conclusion, perspectives

Conclusion: Shapley effects present an alternative to allocate parts of variance in the correlated framework. It is possible to define aggregated Shapley indices. There exist algorithms to estimate these indices, see Jupyter notebook Part II.

Open questions

- What about goal-oriented Shapley effects? (see recent work in [Da Veiga, 2021])
- Nearest neighbor algorithm depends on many parameters to tune (number of neighbors, total cost, $N_{\mathbf{u}}$)? Is it possible to propose an adaptive choice of these parameters?
- How can Shapley effects be related to gradient-based measures of sensitivity?
- ...

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